Applied Time-Series Analysis
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Lecture - 60
Lecture 26 B - Fourier Transforms for Deterministic Signals 2

Before we get into Fourier transforms again for futuristic reasons it is important to define the cross-covariance functions for deterministic signals also. So that when we learn a very classical unifying result in the form of Wiener-Khinchin theorem we will understand how it applies to both the random signal and the deterministics signal world. We have already defined covariance functions for the random world.

Now, we are defining this cross-covariance function for the deterministic world. The interpretations remain the same it measures the linear dependence and normalized version is known as cross correlation all of that applies, it is only that the definitions are different. When we dealt with random signals we defined using the cross-covariance functions using expectation operators, because for a random process we have to look at all possible realizations and we have to come up with the definition that is collective in nature, whereas with the deterministic signal I need to focus only on the signal.

So, you will see that expectations are gone and expressions look somewhat similar to what you have seen before. Now having noted some of the similarities and the differences at least qualitatively in what I have stated, I should also tell you that the cross-covariance function definitions in the deterministic world are different for periodic and aperiodic signals, and we will see that shortly.

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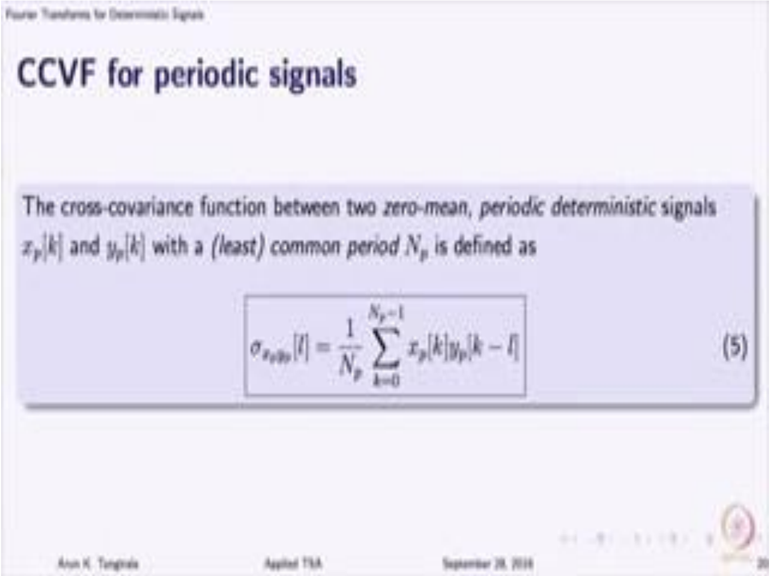

So, let us begin with periodic signals, and the definition of cross-covariance function is over one period. For understandable reasons if you actually do not limit it to a single period and if you try to evaluate over the entire existence the signal exists forever and the right hand side summation may not converge. So, we define the cross-covariance function over one period, that is the first difference that we notice. Of course, if you generally we tend to compare with the definition of the CCVF for a random signal.

And what you must see that the expectation is missing here, that is number one. Secondly, there is no mean subtraction mean centering here. We assume that the periodic signal as a mean 0, if it does not then it is it will just a come up as an additional term there. But, generally we assume that these periodic signals have a 0 mean, and therefore we do not even subtract the mean. And the third is that the summation is limited only to the period. What is this period N P? It is the common period for x and y, x and y are two periodic signals 0 mean periodic deterministic signals with the least common period in N P. If it does not exists then this cross-covariance function itself does not exist. So, there is no point in even computing a cross-covariance function; so that is the important thing.

Very soon we will specialize to autocovariance function and you will see that there is utility to this autocovariance functions. Now, in a way that we have define cross correlation for random signals here also the same definition applies, the same reasons carry forward; the cross-covariance is unbounded, it is sensitive to units and so on. So, to take care of all such short comings we define cross correlation in exactly the same way as we define cross correlation for random signals. Essentially it is a normalized cross-covariance function.

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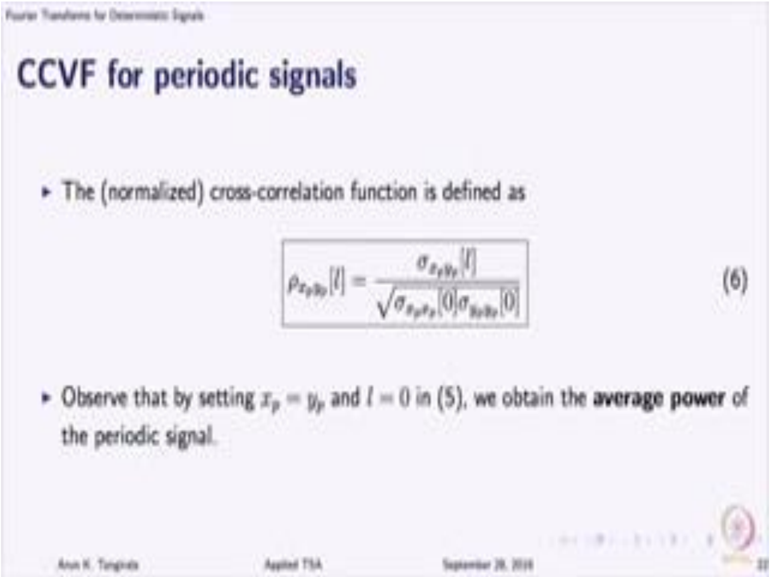

And what you can see is in this equation that we have for cross correlation; when we set x equals y; that means, we do not look at two different signals it is the same signal and evaluate the cross-covariance function at lag 0 what you would recover is the average power, am I right? Over one period so set x equals y and set l equals 0, so the cross correlation function cross-covariance function at lag 0. In fact, since you are setting x equals y we can say autocovariance function for the deterministic signal at lag 0 is the average power of that signal.

So, we are able to now derive some connections between covariance functions and power. Remember there is a difference between the two power is directly define in terms of the signal, of course autocovariance function is also define using signals, but the utilities are different. The autocovariance function for example as we will see can detect periodicities, but the autocovariance at lag 0 for a periodic signal is the average power. And we will see if such a connection exists for periodic signals also, we have not defined what is power density and so on, but we have defined what is the power for deterministic signal; we need to define what is meant by power for a random signal, we have not defined that.

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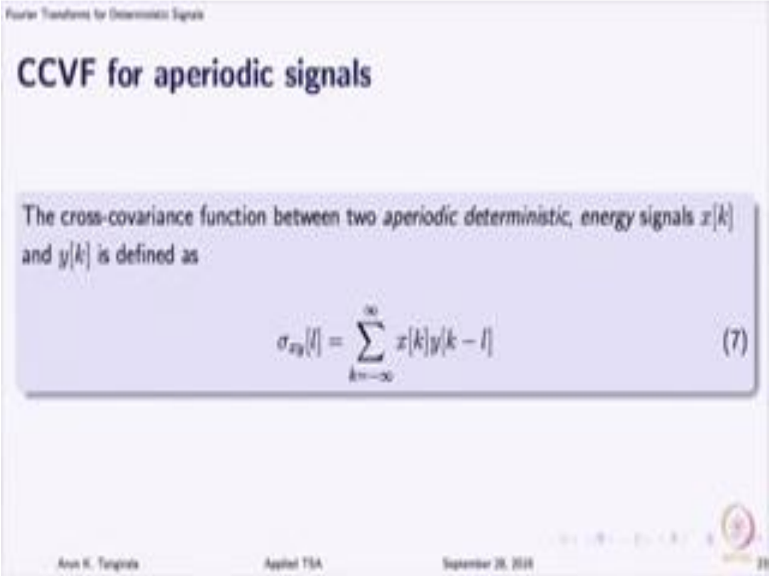

Now, we move on to the cross-covariance function definition for aperiodic signals. Now you can see when it comes to two aperiodic signals the definition is modified. Now I can afford to take the summation over the entire existence of the signal provided this right hand summation exists. And it exists for all signals that are energy signals; that means both x and y should have finite energy, then the cross-covariance function exists. And once again here you can observe by setting y equals x that is the autocovariance function for a deterministic energy signal at lag 0 gives me the energy.

So, this lag 0 has some important information embedded in it. At other lags it has some other informations; so that is why the called covariance functions are very useful in data analysis.

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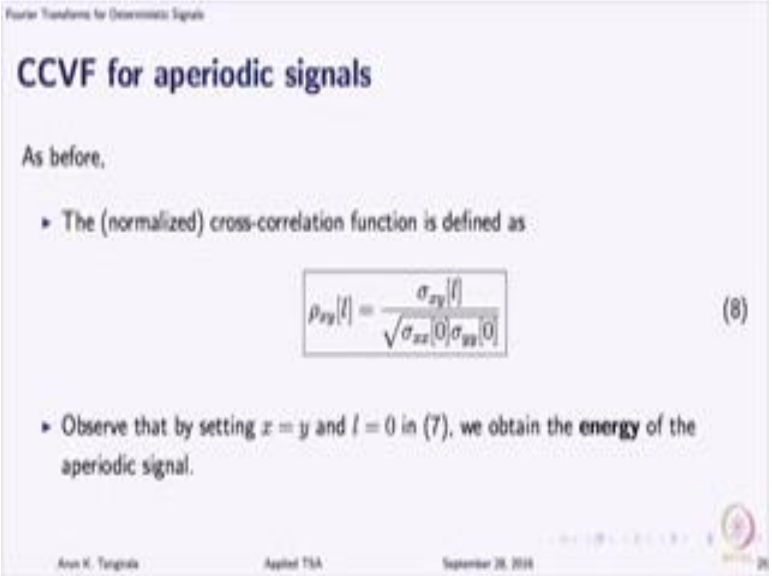

And once again we have the cross correlation function as a normalized version of the cross-covariance function, and a point that we have already observed.

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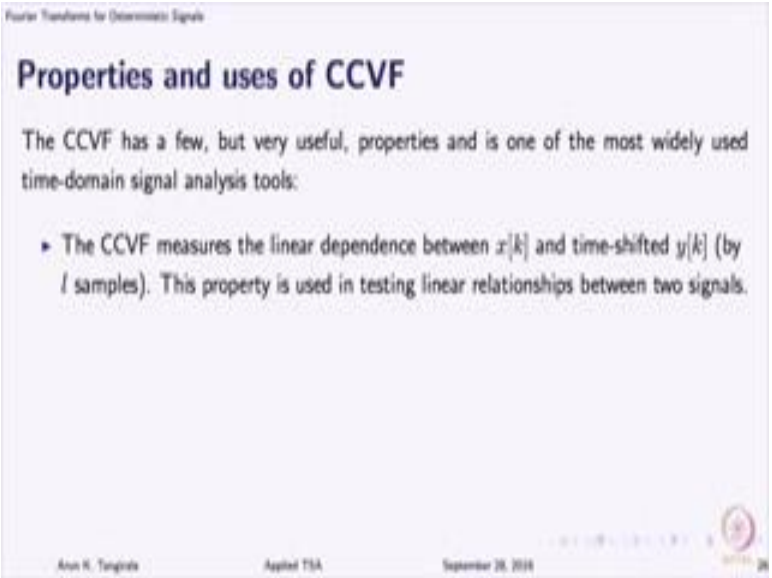

As I have said the autocovariance function is useful in detecting periodicities or at lag 0 it gives you the power or energy as the case may be, the cross-covariance function has a lot of uses and one of the most prominent uses is in estimating delays. Suppose x is a transmitted signal and y is the received signal in radar signal processing the cross-covariance function is extremely useful in knowing what is the delay lag between the two and using that lag I can actually determine the position of the object moving in the air. Or I can use the cross-covariance function to see if there are linear influences, and if I can use one to predict the other using a linear model and so on. Pretty much is the same story, what we have learnt earlier.

And again we know that the cross-covariance function is asymmetric in the random signals world and so is the case in the deterministic signals as well. And that is what allows us to estimate the delay in linear time invariant systems. So, very quickly we review autocovariance functions: nothing new all we have done is we have set y equals x in the respective definitions for periodic, and aperiodic energy signals you have to be very clear the cross-covariance functions we have defined were for periodic signals and not any p aperiodic signal, and a periodic signal which as finite energy. You cannot apply the cross-covariance function definition that you saw earlier here for example in equation 7. To any pair of aperiodic deterministic signals they have to be finite energy.

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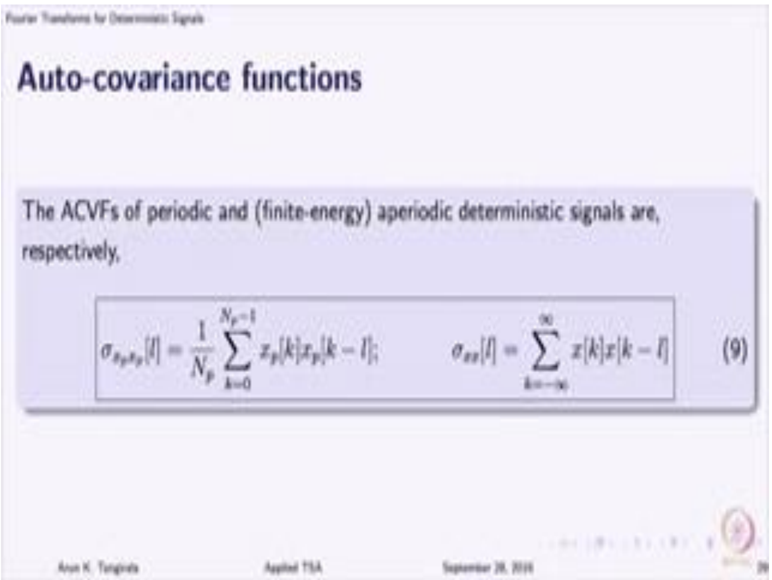

So, by setting x equals y in these respective definitions we obtain the autocovariance functions for the periodic and aperiodic functions with finite energy.

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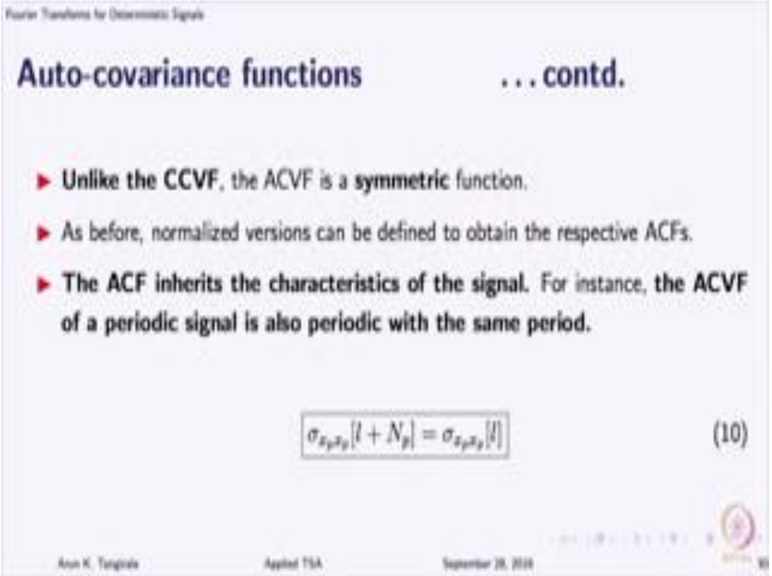

And as usual your autocovariance function is a symmetric function and you can define autocorrelation function and so on. And most importantly what you should remember is the autocorrelation function inherits all the basic features of the signal. For example, if the signal is periodic then autocorrelation or covariance is periodic.

So, if the signal is periodic the autocorrelation function is periodic, if the signal as an exponential decay then the autocorrelation function has an exponential decay and so on.

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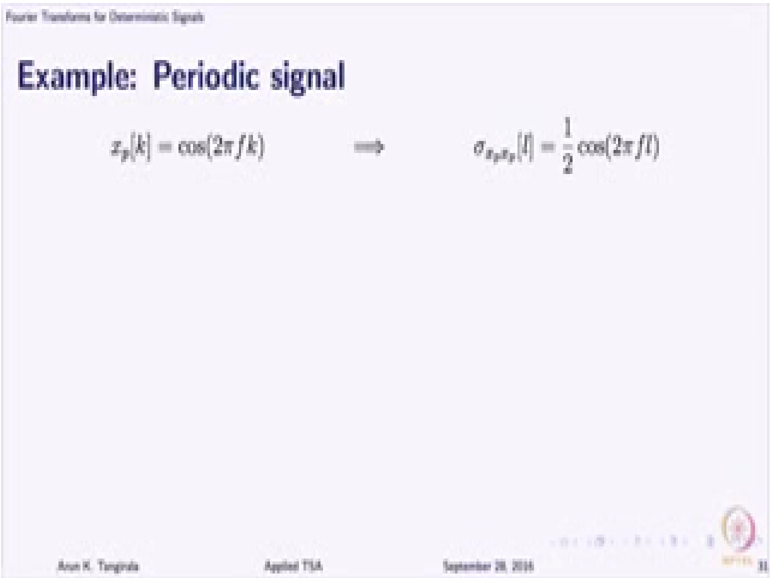

Now let us look at couple of examples: I have a periodic signal here and I will leave it to you to show that for this periodic signal you have to apply the autocovariance function definition that you saw earlier and show that the autocovariance function is what I have shown here. That is the very simple homework and we just for the 5 minutes you can do it when you go back.

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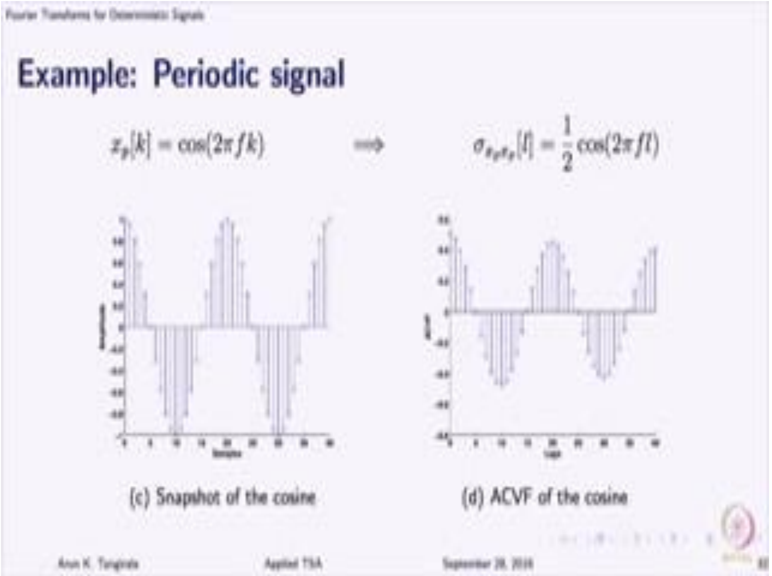

Now, here is a pictorial relation of this result I just took a signal. You can figure out what is N P here. Can you tell me what is the period, if you are able to see clearly I do not know, let me zoom in; 20 very good. So, your eyes are actually doing a good job. So, I have used a N P as 20 and I have generated the signal, on the right hand side I have the autocovariance function. In r if you want to compute the autocovariance function for the deterministics signals you should have noticed by now, sorry more or less this definition here that we have seen tallies with the definition that we use for sample autocorrelation function with some differences. For example, there is no mean centering, and i if you were to compute the autocovariance function let us say of a periodic signal in r using the sample ACF routine that you have, then you have to make adjustments to that factor 1 over n there. In the sample ACF it uses the length of the signal, whereas here it uses the period of the signal.

Suppose I do not know the period and I want to use it, then what can I do is I can throw away this 1 over N P by first computing this sample ACF and multiplying by the length of the signal. In other words I only compute this part here, only the summation I do not compute 1 over N P. So, I turn of the mean centering and then multiply the result from ACF with the length of the signal, so I get the summation. And then that will give me the period of the signal. Of course, the magnitude would be different here, but that is the periodicity is not affected by that. So, you can see that the autocovariance function theoretically as well as pictorially is periodic with the same period.

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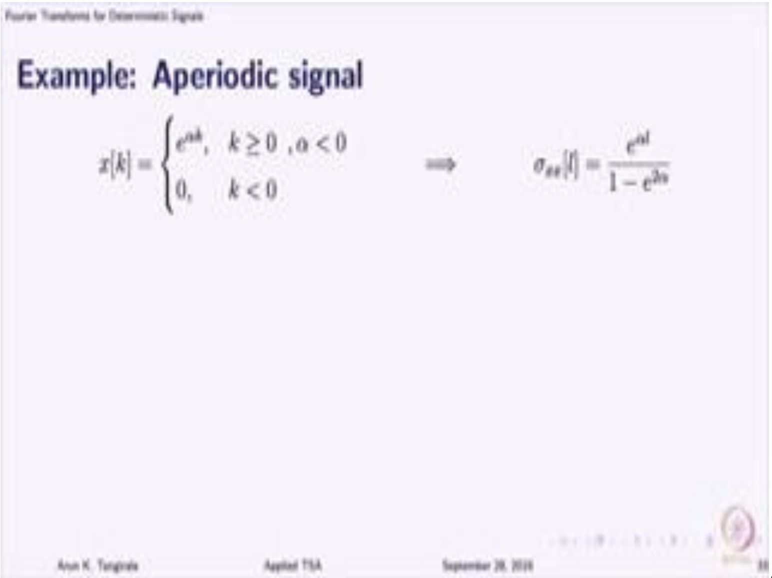

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Now, on the other hand if I take an exponentially decaying signal remember it has to be a finite energy, therefore I have here a discrete time exponential signal e to the alpha k and alpha is less than 0.

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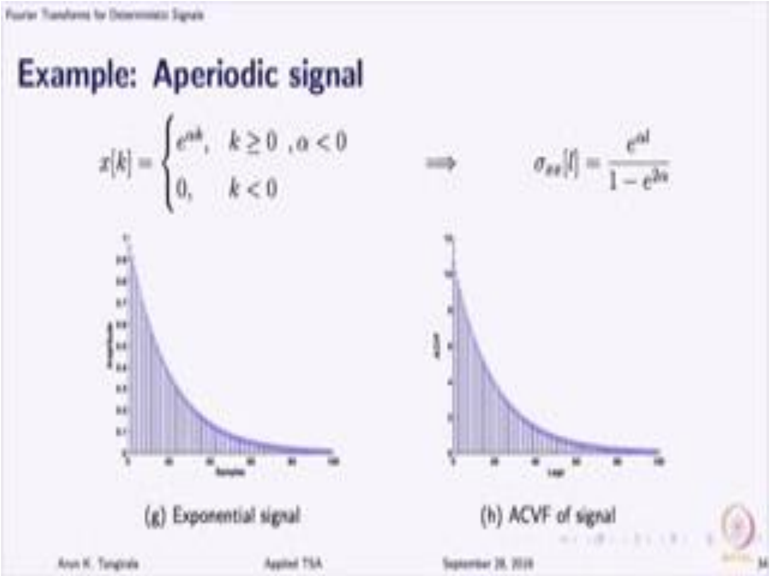

You can show again by applying the definition of autocovariance function that it is what I have given you on the slide here. And you can see pictorially that the shape the nature of the ACVF is a same as that of the signals, so it inherits. That is the advantage of working with autocovariance functions. Of course, you have seen more than this you have actually worked with deterministic signals corrupted with noise, but there also it would be the same case. Remember, what we are working with is more or less of version of sample ACF. And if the signal were to be corrupted by white noise we know already that the effect of noise is felt in the lag domain only at lag 0, whereas in the signal the effect of noise is felt at every instant in time.

So, it is hurting you at every time instant, this noise. Whereas, the moment you move to lag domain all that effects from minus infinity to infinity effect of noise is collected at lag 0, if it is white noise. If it is correlated then may be of few more lags, but then you have kind of contained the effects of noise to a very finite number of points in lag domain; that is the beauty of moving from time domain to lag domain.

Already now you are in the transform world. The moment you move from one domain to another domain you have transform except there it is not a Fourier transform here, but you are transform the signal from time domain to a lag domain. We do not call it as a transform per say, but for all practical purposes you have moved to a new domain and the reason you want to move to a new domain is something certain thing, certain analysis are easy, much more visible certain features are highlighted in a much better fashion than in the time domain that is the motivation for every transform; any questions.

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And now with this motivation we move to the Fourier world, where we look at Fourier series and transforms. And as I have said we look at both continuous time periodic and aperiodic and discrete time periodic and aperiodic signals. And at the end of all of this I guaranty that you will be confused. But, I will also point out the common thread which will allow you to remember things. And there are some very subtle things that you should remember and I will point them out as we proceed.

So, when you enter when you step into the world of transforms not just Fourier transforms it could be Laplace, it could be z, it could be wavelet, whatever transform in today there are so many transforms that you see in signal analysis.

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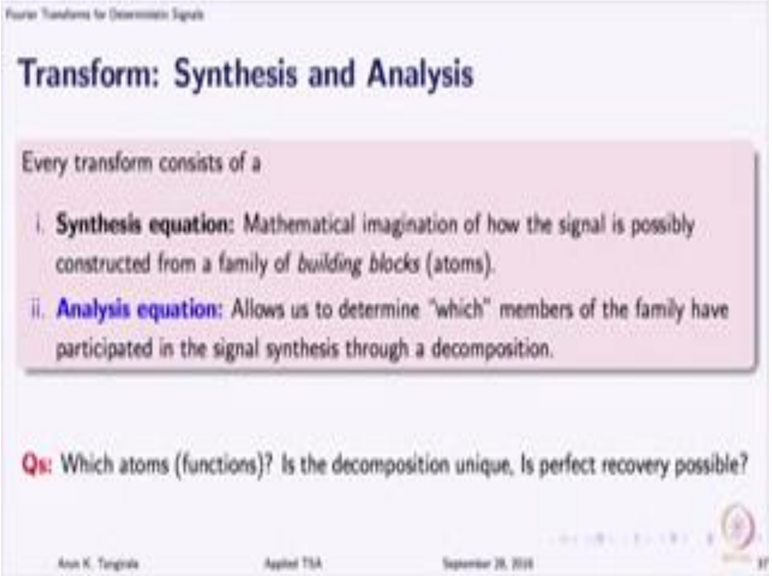

Almost every transform will have two parts to it: one is call the synthesis equation other is call the analysis equation. I have already talked about why we want to transform a signal; because of ease of analysis. Fourier introduced his transform so as to solve the differential equations that arise in heat conduction. We have learned in differential equations power series expansions and so on. So, it makes things easy, it allow why do I have use Laplace transform in for solving differential equations because it is easy; it allows it converts the differential equation in time domain to an algebraic equation in the Laplace domain.

Fourier transform does something else for you, therefore you should be clear what you want from the signal, what is a kind of information you are looking for, what is a kind of application that you are looking at. Then choose the transform not the other way round. You cannot say I like this so far and I want to now use it wherever I go. It is it cannot be that you have to look at the utility and then buy the appliance or furniture and so on. Likewise here, you have to be clear in what you are seeking. Here we are interested in detecting periodicities, we want to obtain a filtering prospective, and we want to define spectral densities so as to understand why the linear random process is define that way and so on.

Therefore, we are studying Fourier transforms, but if your purpose is something else then you should look at some other transform. In other words no transform is the panacea for all your problems.

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Now, when you as I said when you step into any world, any transform world you would encounter a synthesis equation and an analysis equation. Very often when you refer to a book and so on you are presented with the analysis equations first, and that is kind of confusing because you wonder; for example if you take the Fourier transform of an aperiodic signal you would be presented with this expression x of t, I will give you this expression later on. This is what you would notice. This is the Fourier transform definition for a continuous time aperiodic signal with finite energy.

Now, you wonder how somebody could have imagined all of a sudden such a complicated integral, that to at the beginning of 1800s, how could people even think of this, why I mean was it just in a dream that people imagine this: no. The analysis equation does not is not the beginning of any transform, the synthesis equation is a beginning of a transform.

In other words first you imagine the signal to be made up of some building blocks, some atoms; and that is what is your synthesis equation, that is you assume ho the signal must have been constructed this way. And then you ask how much of those atoms are present in that signal, and that is what is your analysis equation. Since we search for Fourier transform when you go to the net it will present to you with the analysis equation, this is what is Fourier transform; you ask for it I gave you, but you did not ask where this came from. And if you ask where this comes from it comes from the synthesis equation. So, it is always a good practice to start with the synthesis equation and understand why such an equation has been postulated, what was the motivation for it, then you will understand what are the building blocks?

For example, if I ask you what is the wall made up of, any wall you pick. Why do I want to do that, why do I want to imagine what the wall is made up of; maybe I want to understand the properties of the wall. You know whether it is inflammable or what are it is mechanical properties chemical properties and so on. How would I know the mechanical or chemical properties? Only when I know the elements that make up the wall. In this case I can go and touch the wall there are ways of figuring out, but this is the signal it is bunch of numbers, I cannot go and touch the number, I cannot go and really apply some instrument to figure out what the properties are.

What are the properties that we are looking for? Periodicity and something else when it comes to systems I am actually interested in filtering nature and so on. Here is where imagination is therefore required. Fourier proposed that all periodic signals for example can be imagined to be constructed from sines and cosines. Now it turns out the it is a beautiful choice for describing all periodic functions, never imagine the periodic functions would mean always sines and cosines. I can have a triangular periodic aperiodic function, I can have a very complicated behavior within a period and it can be periodic.

So, not all periodic functions implies sines and cosines, but what his result says- at least when he initially proposed and it was taken with a truck of salt lot of people criticized at there were not so sure whether that preposition was correct; that all periodic signals can be a imagined to be constructed from sines and cosines. Whether it is actually the case we do not worry. For mathematical purposes I imagine that way; whether that imagination is suitable it is correct or not is secondary question. If it helps me imagining that way in my analysis I will imagine it. Then came about you know people asking when is it possible, when is it a valid thing to imagine for what class of periodic signals can I imagine, this and all of that followed later on. But it was Fourier who came up with this proposition and that is where the Fourier series and Fourier transform was born. And then nearly 100 years later was this notion of periodogram that was born; I am not sure Fourier lift that long to see.

So, the some other questions that we want to see ask or look out for is which building blocks do we use, in general, in any transform. And is that imagination the decomposition of the signal unique in the new world is perfect recovery possible. So suppose, I want to know what the wall all is made up of, I take a hammer whatever sledge everything I break it down into atoms- oh yeah it is made up of this material, this is not flammable. You should be able to put back the wall together. Can you do that? Not for a physical wall, but you have this lego toys building logs and so on. And if I give you the constructed one made by one child then you can break it down into pieces and then you will be able to put it back. That is what we mean by perfect recovery.

That is very important, perfect recovery condition is extremely important for estimation purposes. The basic idea and signal estimation is to break up the signal into elements and then throw away what I think should not be present in the signal that is essentially performing some kind of operation in the domain, in the transform domain and then reconstructing. That is the basic idea of filtering.

So, I will just conclude with this slide. If you look at the transform world by and large there are two applications: one is analysis other is filtering. Analysis involves taking the signal into a new domain mathematically; you start with the signal decomposition and proceed to energy or power decomposition.

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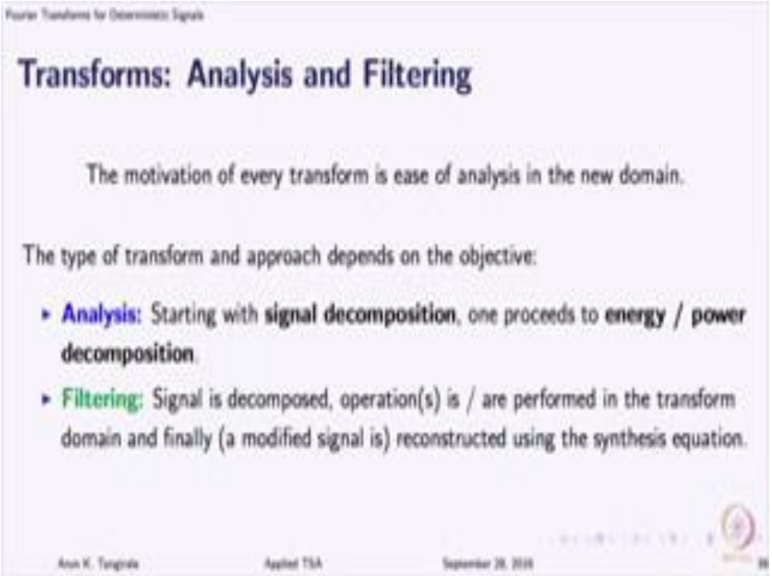

Sometimes you can bypass the signal decomposition to go straight to the signal and energy decomposition; I will talk about it later on, but that is the goal in analysis. Whereas, in filtering as I just said the signal is broken up into parts, which part depends on what transform I am using and then I perform some operation in the new domain, it is easy to do that and then I come back. That is what is filtering. So, that is what you are doing in laundry, what are you doing in you know cloths laundry; the cloths are dirty and I want to get rid of the noise, the dirt. Here I have signals, measurements, signal corrupted with noise I cannot really take out the noise just like that, I take it into a new domain. What do I do with my clothes? I cannot really get rid of that dirt I take into a new domain from air medium to water medium.

Why do I do that? Because this separability between the cloth and the dirt is improved by leaps and bounds; that is the main reason why I soak. Why cannot I soak get an oil, I do not want to do that. So, I soak it in water because the separability is improved, now to further enhance that separability I add some detergent that is some kind of an operation. The detergent will determine- these are noise they have to be taken away, but it will also eat away a bit of the cloth if it is a very aggressive one. That is the story even in signal estimation. When you transform it into a new domain and you separate the signal from the noise there using some mathematical operation you may lose a bit of the signal. If you are very aggressive you will lose a lot of signal, if you are very conservative you will retain some noise.

If you do not add any detergent or if you add some mild detergent because that is what you can afford then you will actually get a cloth that is not so clean as they you get from Ariel or Surf or whatever. Anyway, so there are strong parallels between what we do in daily life and signal estimation, but you can see how the transform world at least qualitatively helps you.

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What we will do tomorrow is we will start with Fourier continuous time Fourier series and transform and before that we will actually have a small prelude on fixed and adaptive basis.