Applied Time-Series Analysis
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Lecture – 59
Lecture 26A - Fourier Transforms for Deterministic Signals 1

What we are stepping into is world of frequency domain analysis and basically if you recall what we have learnt until now, we have learnt that the correlation structure of a random process can be understood by examining the auto covariance functions and so on and what we learnt importantly is that a linear stochastic process; its stationary, can be given a linear representation if it satisfies so called Wiener Paley condition.

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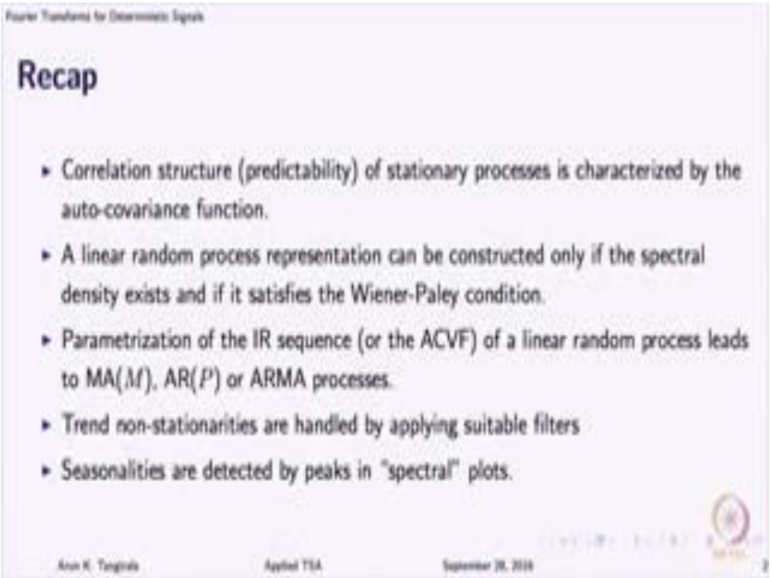

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Well the Wiener Paley condition says that the logarithm of the spectral density should be greater than minus infinity, one of these. So, this is the condition that is called a Wiener Paley condition, I am not stated that earlier, but remember I said that the spectral density should satisfy certain conditions and this is one of the conditions; main conditions and also we have noted the parameterization of the impulse response which leads to either AR or MA models implicitly amounts to parameterizing the spectral density and thirdly when it comes to dealing with nonstationarities, we have trends and we have seasonal components and so on and we have learnt how to use the so called periodogram to figure out, what is a cyclicity in a given series and so on. So, we have now given enough previews on to into what is known as the spectral representation or at least we have enough motivation to look at the frequency domain analysis. You must treat the frequency domain analysis and spectral analysis as more or less anonymous.

Now, the question that we want to ask now is for example, what is meant by spectral density? What is its theoretical definition? That is question number 1.

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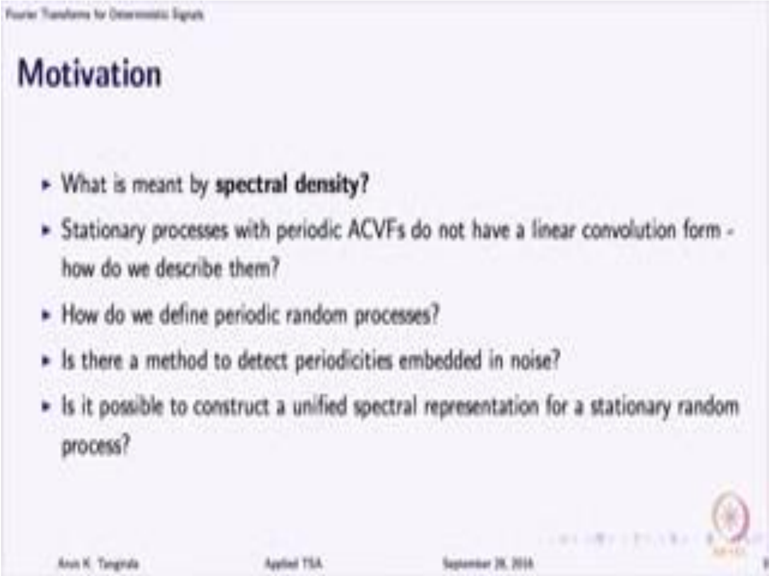

What is the interpretation? How do I interpret spectral density? There was one such questions long ago, what is meant by non negative definiteness and so on and what do we mean by a periodic random process? For example and in general how do you define periodicity for a random signal and how does one arrives at this condition? For example, we will not prove that, but where does its stem from and it is possible in fact, ones we have understood the definitions of spectral density and so on, and by invoking the non negative definiteness of AVCF as a requirement, it is possible to derive the linear representation which is a convolution form that we have learnt until now. So that compel as to look at frequency domain analysis of signals, of course, if you look at the practicality of frequency domain analysism essentially it is used to detect periodicities.

But in theory, there is another use to frequency domain analysis or frequency domain representations which is for example, to arrive at this conditions, but more importantly for understanding a process from the filtering view point and the frequency domain analysis is perhaps the most powerful way of interpreting or obtaining a filtering perspective of any linear operation linear time in variant operation and that is what we will step into.

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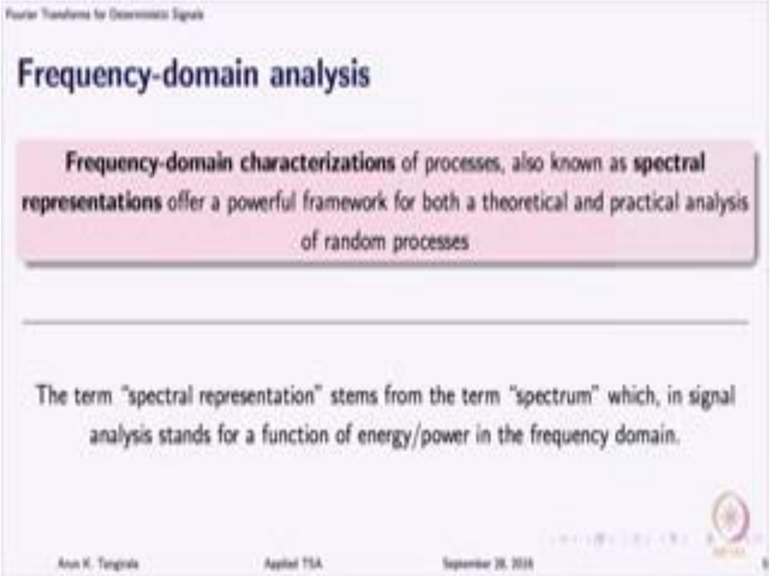

Now when as I am just going to skip this because I have already said this before the frequency domain characterization is nothing but spectral representation and this term itself, the spectral representation term itself stands from the word spectrum and usually spectrum is associated with frequencies in mathematic spectrum is also associated with Eigen values. Here we will assume that the spectrum term always convertates frequency domain analysis and occasionally I must have mention at some point in time power or energy and so on. We will first learn what are these? What are the definitions of energy and power of a signal in a formal way and then move on to frequency domain analysis? So, there are several questions that we are going to look at for example, we will ask what is the definition of a spectrum, what is the difference between energy and power of a signal.

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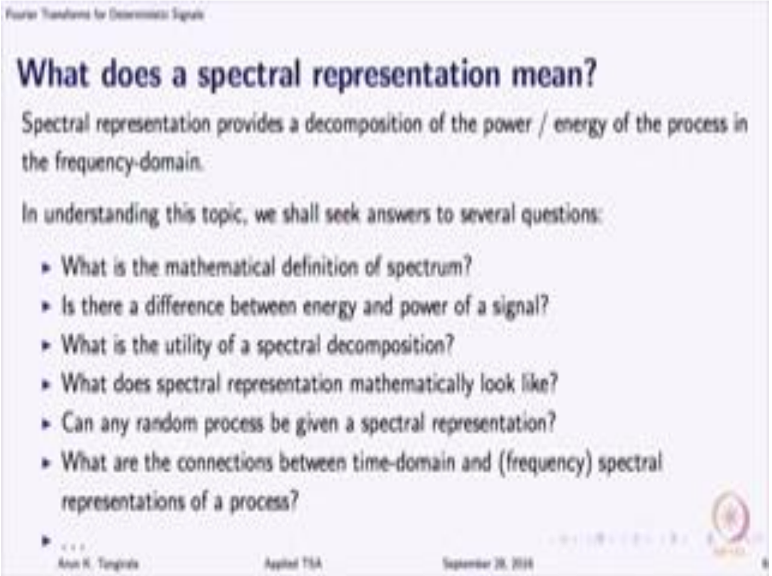

What is the utility of a spectral decomposition and so on, I am not going to read all the questions, but these are some of the burning questions that we want to obtain answers to this.

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And in all of this, the Fourier transform is the main tool of course, will define energy and power independent of the Fourier transform, but by enlarge to be to be able to carry out a spectral decomposition of the signal or to detect periodicities to obtain a filtering perspective or to do all of this we will use Fourier transform as the main vehicle.

But that is not the only transform in which you can do things, but this is by enlarge the most popular transform and even Fourier never imagined the impact of his proposition right such powerful is the Fourier transform. Now before we jump into the random world that is understanding how Fourier transforms are applied in the random world it is very important to understand how this transform is applied to deterministic signals because things are a lot easier Fourier transforms exist in the deterministic world for signals whereas, when we move into the random world will realize that the Fourier transforms of random signals do not exist in the way, it defines for the deterministic world. So, what do we do? I have just now said that Fourier transform is a main tool and now I am giving you kind of a kind of a anti climax statement that look the Fourier transform does not exist for random signals is that the end of the road or is at some other path. And that is where we will take the Weiner Kirstein root, but that can wait what will begin with is understanding the Fourier analysis of continuous time and discrete time deterministic periodic and aperiodic signals.

And see how they are useful you will notice that some of these transforms or series and so on are useful in theory while some others. In fact, may be one or 2 are useful in practice. So, it depends on what you want to do we will use this tools for both for a theoretical understanding as well as particle analysis of signals and I will of course, point out as we learn each of this tools I will point out which of this is useful in theory and which of it is useful in practice. But I should also tell you that in general it is assume that you have this background, I am not going to make that assumption, but in a general time series analysis course it is assume that you have some basic background and since you have learned Fourier transform four years ago some of you; you must have conveniently forgotten. So, will assume that you do not know much, but some of this expression should allow you to tap into your memory and recall these expressions.

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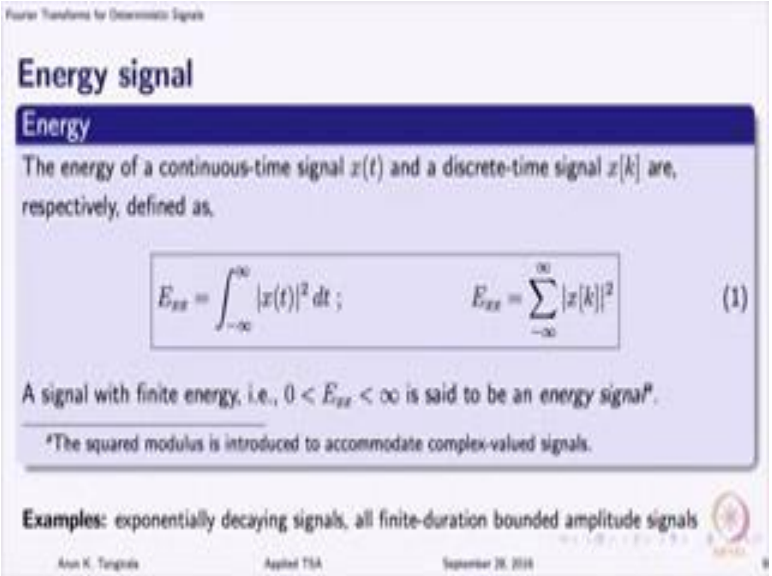

Let us begin with the definition of the energy of a signal we keep using energy and power left and right sometimes interchangeably and we learn that we should not be doing that.

The energy of a continuous time signal is define as have given in the box there its simply integral mod x of t square d t. Now for the first time we dealing with continuous time signals, you should note that and therefore, we have an integral there and why do I have an modules to take in to account the fact that a signal can have a complex value representation the signal may not be complex value, but I can always give a complex value representation will not go into that. For real valued signals you can through away the modulus there in the definition and for a discrete time signal which have also define of for which have define the energy is simply a summation. Now these definitions are inspired partly from physics and electrical engineering will not go into the origins, but you should simply remember that this is the energy. From a functional analysis view point, you can think of this as a square 2 norm of a function if you think of x of t or x of k as a function this is a squared 2 norm if it is exist for functions existing in normed spaces; so called you know Hill Bert spaces.

Alright and there are several examples you can think of for which these integrals or summations exist, the moment we perform an operation, we always have mathematical operation we ask the question thus this operation yield anything, we also think of this in a regular surgical operation will this operation yield anything we should not be like a operation successful patient died right here we are evaluating integrals or summations for the discrete time signal what is the guarantee that it converges. Now all signals that have finite 1 norm or 2 norm will be will qualify for the energy calculations.

As an example exponentially decaying signal right or all finite duration bounded signal it does not matter what does signal is if it is finite duration its energy will exist, on the other hand if I take a sinusoidal signal can I compute the energy, I mean will it will this integral turn out to be finite or even the summation, what do you think? For a sinusoidal signal the notion of energy does not exist as define this way you can define energy in whichever way you want, but this by enlarge the definition of energy that is used and accepted.

In this sense, the energy of a sinusoidal signal does not exist over the infinite time you can; however, compute the energy of a sign way over one period this that is a not an issue at all. So, we say that all periodic signals in general are not energy signals because say energy infinite when that is what now leads us to the definition of an energy signal any signal with finite energy is set to be an energy signal.

In general, if a signal exists forever, now they can be signals that are not periodic, but can exist forever.

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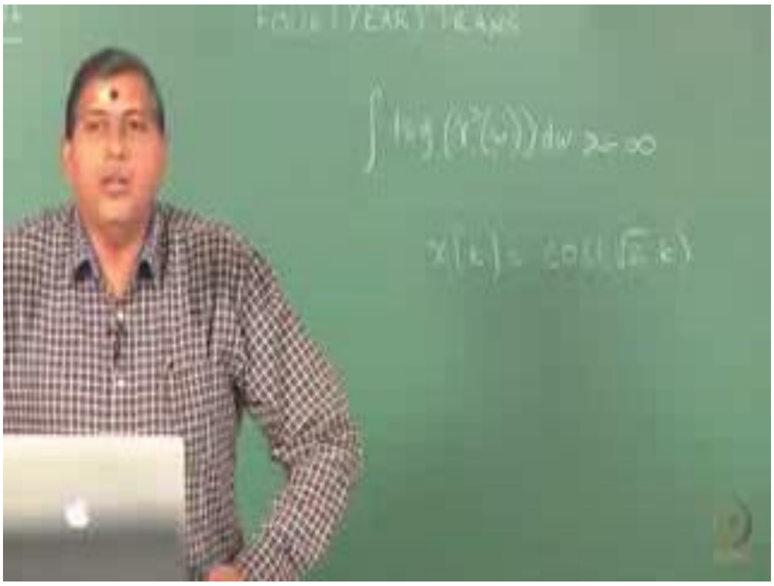

For example, you could have a signal; discrete time signal, is this a periodic signal?

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Why?

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The frequency is irrational and therefore, it is not possible to find, this is a discrete time signal, you have to be careful if I replace this with a corresponding continuous time signal then it is a periodic signal, discrete time signals are periodic if and only if I can find an integer number of observations after which the reputation occurs and I cannot find such an observation for this signal. On the other hand the signal exists forever, now if you ask whether this is an energy signal or not, what would be your answer? What about a random signal? Do you think random signals are energy signals, why?

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Ok, Any other answer? The answer is it does not converge, yes Priyan.

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No, what about random signals I am asking.

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Can I call it as an energy signal? No. So, it is kind of obvious know because its exist for ever there is no way this integral is going to yield me a finite value therefore, the simple lesson that we learn is random signals are not energy signals. Now this definition of energy also is complemented by the energy definition of a power then will talk of energy densities and power densities. So, power signal is defined.

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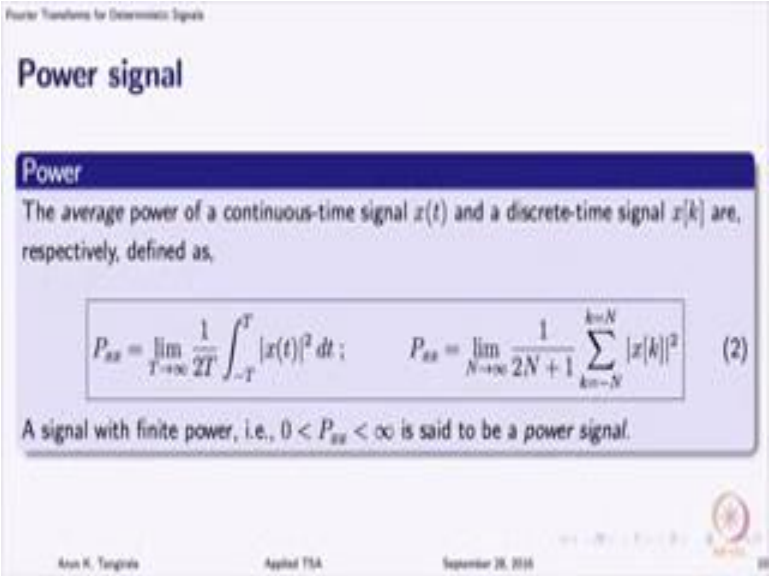

Based on the definition of power itself, in fact, we define average power. So, if I am given a continuous time signal or a discrete time signal, the power is defined in a limiting sense and in an average sense. So, you can see here that we are evaluating the power in the limit as T goes infinity and what we are doing is we are computing the energy over that period 2T and then dividing it by 2T because power is the rate at which energy changes with time that we already know we have essentially use that definition in coming up with this one.

The interval is 1 over 2T integral minus T to T x of t square d t is a energy of the signal over the interval 2T divided by 2T will give me the average power over that is signal over that interval in time and then we let we evaluate this in the limit as T goes to infinity that is what gives us the overall power, power for the overall duration of the signal and likewise we have here the definition of power for the discrete time signal again a signal with finite power is set to be a power signal.

Now, if you take a sinusoidal signal for example, do you think these limits exist? It will exist. So, that is a point here or any periodic signal for that matter it is not an energy signal, but it is a power signal and a signal cannot be both an energy and a power signal which means if its energy is finite; obviously, the average power over the duration will be zero all right and we say that the signal is power signal if the power is nonzero and finite. And likewise you cannot has signals if it is analysis energy signal it cannot be a power signal it has to be a only one of them there are some pathological cases for its neither power or energy signal types will not worry about it but.

Now we will ask a question is a random signal a power signal what you think? Can we assert that say with confidence at a random signal is a power signal or no not necessarily what if it a stationary, what do you think? Stationarity would mean that that it cannot go unbounded right. So, you can expect a station is signal to have finite power on the other hand, an non stationary signal we are not, so sure. So, in general, we will concern ourselves with random signals that are power signals because we deal by enlarge with stationary signals we can hope that the signal that we are looking at has finite power we have already concluded at that random signals are not energy signals. So, as I said examples as I said periodic signals stationary random signals in general all finite duration and amplitude signals will have zero power because their energy signals alright.

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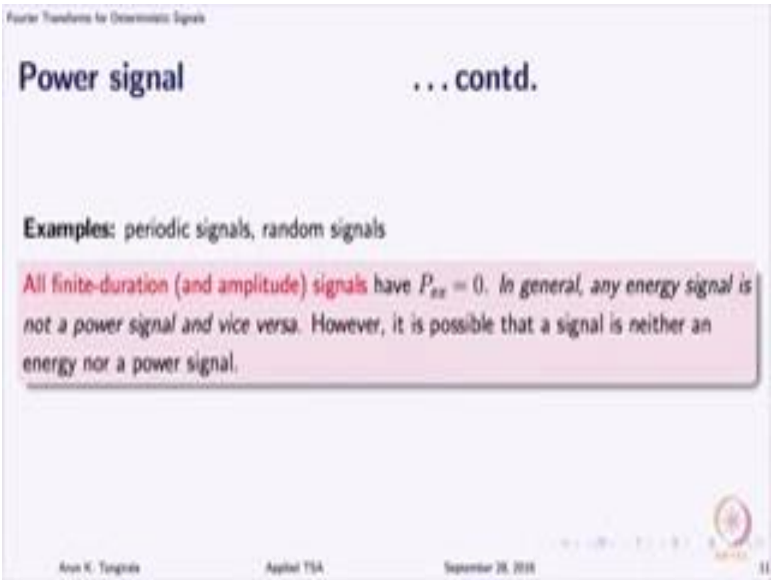

As I said it is also possible that a signal is neither energy, not a power signal, but will not worry about the signals.

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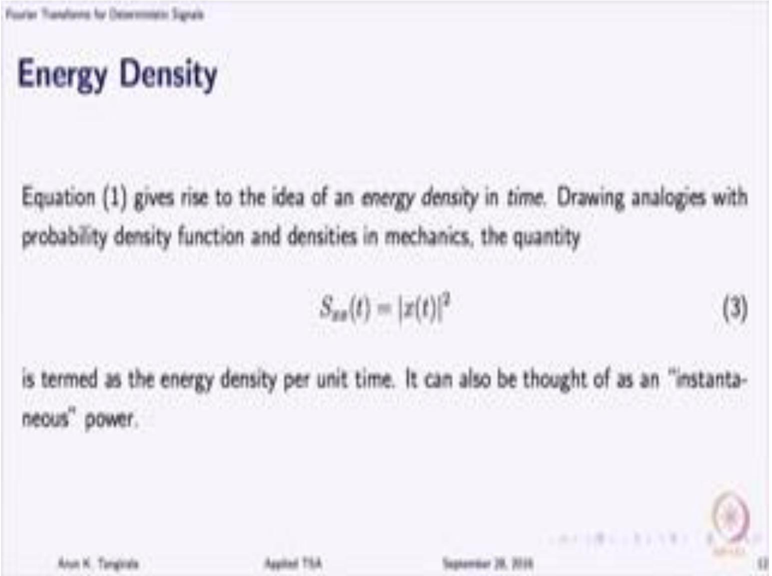

Now, we move on to the notion of energy density and this is important because we have used the term spectral density before power spectral density before of course, without defining what it is now slowly we are getting into the definitions of what is density. Now this is not a complicated definition it stem from the classical notion of density something per unit something else, some quantity per unit something else. So, here if I take you back to the definition of energy right you see that the energy is integral mod x of t square d t.

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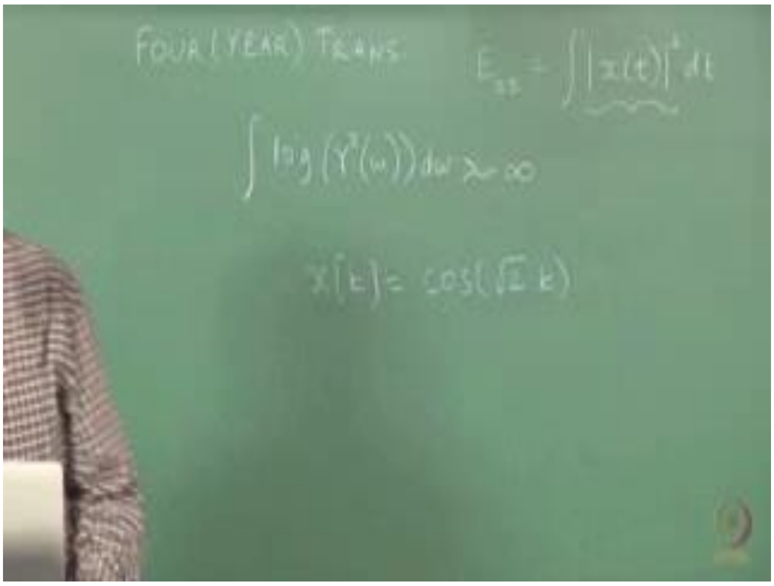

Now, if you think of an energy density and you think of this as that energy density, in time when we talk about densities will have to specify in which domain, for examples, in physics or in mechanics, we talk of densities we say mass per unit volume or mass per unit length, mass per unit area and so and so, we are specifying in which dimension, I am looking at the density. Likewise here, I am specifying here energy density in time that is what the function that we are interested in. If we think of such a function any density function first of all that domain has to be a continuum that automatically tells us that in discrete time I cannot think of energy densities, because as the definition shows energy is a summation we can speak of densities even if you recall the notion of random variable probability theory we have spoken of probability density functions only for continuous valued random variables not for discrete value random variables because then the outcome space is not a continuum for discrete value random variable.

Likewise here for continuous time signals I can think of an energy density in time and that energy density is simply mod of x of t square. Typically when we talk of densities we say area under the density should give me the quantity itself. Here we are looking at 1 dimensional time, but you can think of energy density in space, you can think of joint energy densities in space and time and so on, this is only the beginning. So, we have now coined this term energy density in time and its defined as mod of x of t square simply them magnitude square, but you should remember in which domain we are talking of energy densities here it is time. You can also think of this an energy density as an instantaneous power, but I do not want you to think more on those lines it is just for your understanding. So, here is a simple example.

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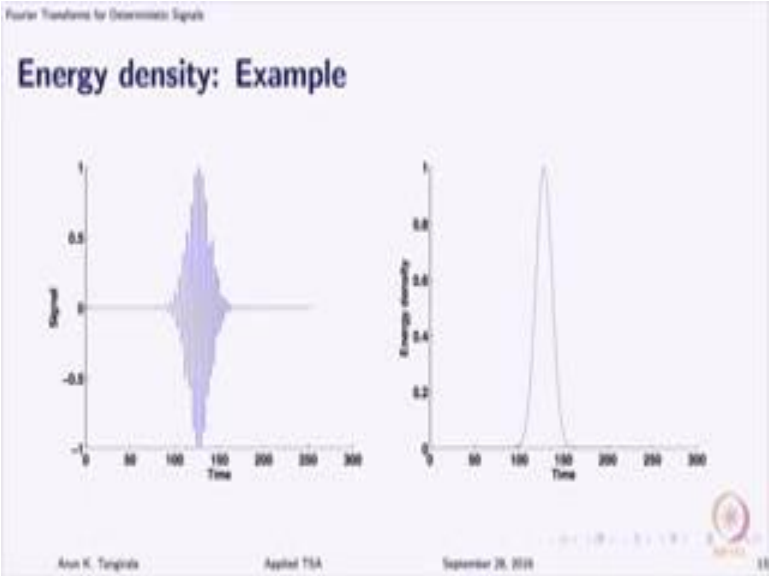

I have an amplitude modulated signal, here you can see it is a finite duration, it dies down. So, it should have finite energy on the left you have the signal and on the right you have the energy density in time.

Why are these densities useful? Although we do not get into the utilities of this energy density in time here, in a more advance course like a joint time frequency analysis course or multi scale analysis and so on, we use this density functions to define what is known as a duration that is how long the signal must have existed to get an idea of what how long the activity existed where in time is energy concentrated such pieces of information are useful in fault detection and so on, suppose this was a feature of a fault I would like to know how long the fault persisted and when it occurred in time and so on for all particle purposes. Now this is a density function and you can define mean variance all higher order moments and so on, the only difference is the interpretation we are looking at we are not looking at random signals we are looking at deterministic signals and we are talking of energy densities just like probability density is allowed us to calculate probabilities and then moments and so on.

Energy densities allow us to calculate energies and then moments of the energy density we do not use the moments of energy density in time in this course at all, but I am just telling you that there is a lot more utility to this energy density than what you can imagine. So, on similar lines, we can define power density nothing great about it, it just is inspired from the same approach as a energy density go back to the definition of power density in time absorb that big T in the denominator into the integral there bring that into the integral. So, you get half of minus T to T integral mod of x of t square by t d t of course, you can say have a left out a factor of 2, but that is ok.

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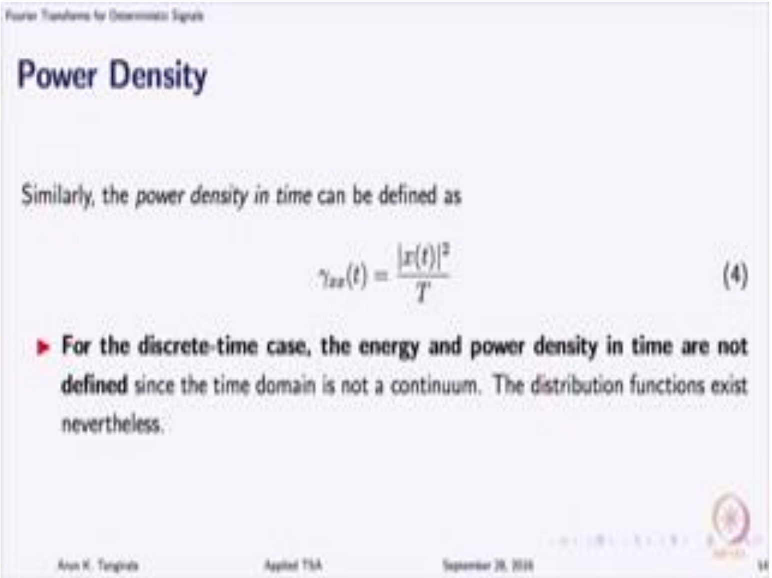

You can say ideal it should be mod of x of t square over 2T, it is correct. So, strictly it should have been that it is so, we will we missing the factor of 2, there as long as I factor that into my integral it should recover the total power, but the difference between the energy density and the power density is only these factor; obviously, right because energy and power differ by factor of time. Now for the discrete case the energy and power densities are not defined we have already observed that energy density does not exist in time for discrete time signals. So, does the power. So, does not the power density because the time is not a continuum, it is as simple as it.

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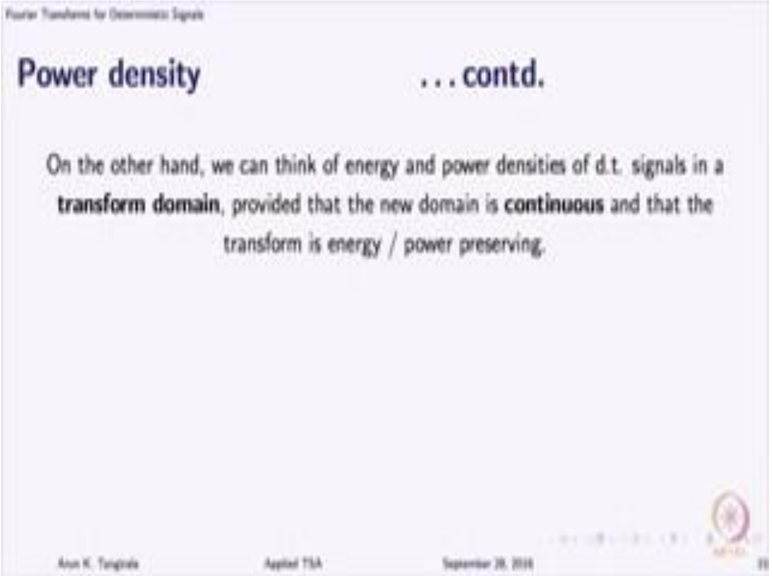

Now, having said that we can think of densities for this discrete time signals slowly, we are emphasizing on discrete time signals because that is what we work with for discrete time signals, we can actually turn to another domain in which the density is exist for a and the classical domain is frequency domain. We will see how that is done.

At this moment we will not define anything more in terms of densities, we have just learnt how to define densities in time at the suitable time, we will learn how densities are defined in frequency domain, in that domain the densities may exist because the in the transform domain may be the signal representations are continuous functions just because I have a discrete time signal it does not mean that in a new domain the signal does not have a continuous representation; that means, in a new domain it can be a continuum where as in the time domain the discrete times signal leaves in the discrete domain. So, we will that is a (Refer Time: 23:25) of our Fourier transform where we move from densities in time to densities in frequency, what allows us to move from densities in time to densities in frequency is what we are going to learn shortly.