Applied Time-Series Analysis
Prof. Arun K. Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture – 48
Lecture 21A - Models for Linear Stationary Processes 12

Very good morning, what we will do today is conclude our discussion on auto regressive models and then briefly talk about ARMA models and hopefully we will have the time to move on to non stationary models where we look at how to handle trends and integrating effects, where eventually we will be led to ARIMA models, but today let us begin with from where we left off yesterday. So, we said that the ACF of the auto regressive models can be obtained by setting of the Yule Walker equations and then solving accordingly for the p up to lag p and then lag p onwards you use the difference equation form if needed and this Yule Walker equations can also be used to estimate the model parameters, as we have discussed yesterday.

In practice what you do is; you use this Yule Walker equations even on the estimates of the ACFs and later on we will study how good these estimates are you are; both for moving average and auto regressive models when we get into estimation. So, significant part of modelling exercise the effort goes into determining the order of the model whether it is auto regressive or moving average or any other model that you are fitting to data this. So, called order really is very important and we know already with respect to auto regressive models the ACF does not provide any clear information on what the order is and to that effect we have looked at PACF.

So, we will just dwell a bit more on that and talk about the connections between PACF and the auto regressive model of order p we have mean; I mentioned this before that the PACF at the lag l is the last coefficient of the auto regressive model that you fit of order l. We will re trade that point today, so in general if you look at keeping aside PACF concept. If you were to determine the order of auto regressive processes, there are 2 ways of approaching this problem 1 is of course, to try out AR models of different orders.

So, you begin with AR 1 in general how do you determine for example, if a model order is suited to the data or not, what is a generic procedure for the test of a model; it could be

M A it could a ARMA it does not matter any model that I fit; how do I know if I have reached or I have guessed appropriate order for the data.

Student: (Refer Time: 03:22).

Sorry.

Student: Residual.

Residual should be right correct. So, what you do is here also you fit AR models of successive orders right and then as you move as you fit an AR model of a certain order examine the residuals, if they are satisfactory then you stop if they are not then you successively increase; until you reach a point where you say I have kind of reach the so called true order or the appropriate order.

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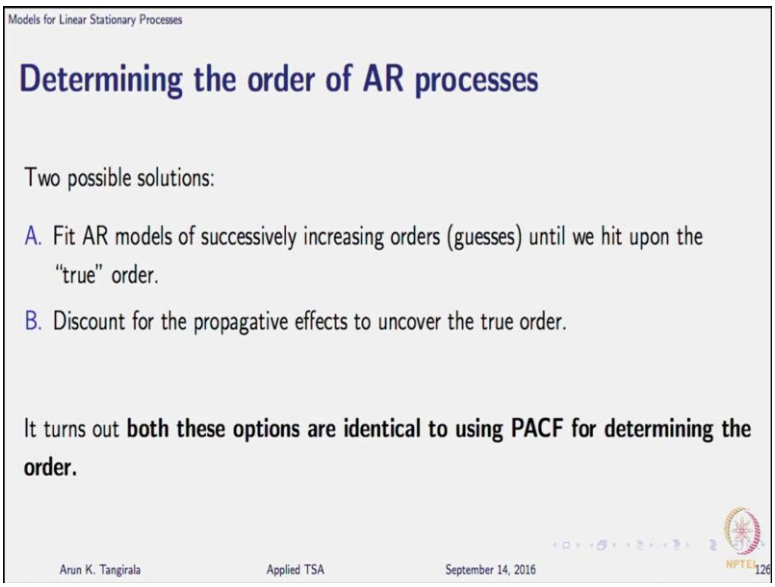

Now, that is the basic idea also that you will see essentially in the PACF, the other approach is to of course, discount for the propogative effects in ACF which leads us to PACF, it turns out that both of them are actually equivalent to using PACF formally although on the face of it they appear to be different, computing PACF essentially amounts to fitting AR models of successively increasing orders. Intuitively if you look at the PACF how are we estimating the PACF, we can discuss from lag to onwards because at lag 1 anyway PACF is same as c ACF.

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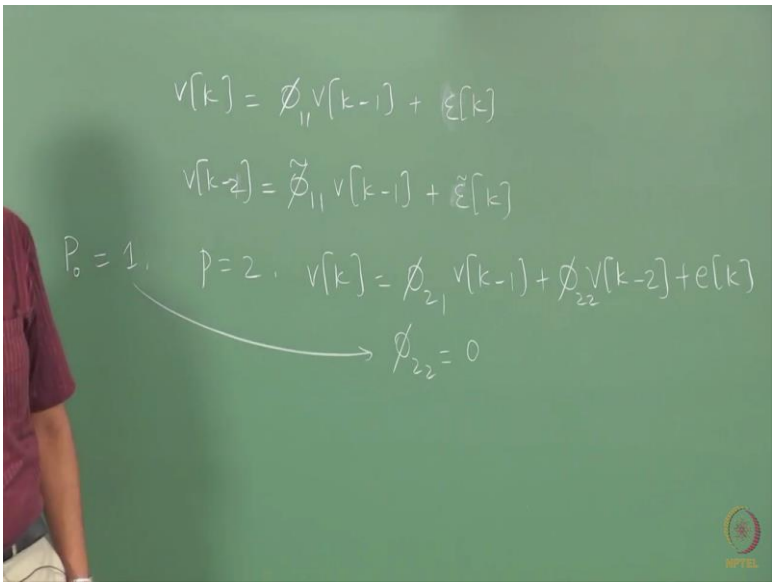

So, at lag 2 what I am doing; I am actually discounting; I fit a model first order model AR for v k right. So, let us call this as phi 1 1 and plus some epsilon k and we hope if you force it we kind of pretend that the residuals will have white noise characteristics, 1 has to test later on whether indeed this is the case. So, we fit an AR 1 model; forward model and we also pretend AR 1 model backwards. Let us call this as may be tilde 5 phi 1 1; v k minus 1 plus e tilde k. This is not necessarily going to be correct we do know; we will have to figure out, but to compute PACF at lag 2; whether you are doing this explicitly or not, this is what you are actually doing implicitly. You are fitting an AR 1 model for v k, forward model and a backward model and then you are going to work with this residuals.

So, in place of v k suppose I had epsilon just to be more generic. So, I have epsilon and epsilon tilde, after having fit the forward and backward models then I am going to look at the correlation between epsilon k and epsilon tilde of k. So, what you are actually looking at is also the residuals of auto regressive models, when you move on to the PACF at lag 3; again you are going to fit an AR 2 model; a forward AR 2 model and a backward AR 2 model and then you are examining the residuals. At some point if you are hit the so called true order then there will be nothing in the residuals and what you are managed to do is between the PACF at lag l and lag l plus 1; let us say your theta at l plus 1; what you are managed to achieve is explain the process perfectly by an AR and l

minus 1 in module in the since here of course, we are looking at lag 2, so you are fitting AR 1, when you move on to lag 3; you will fit an AR 2 and so on.

So, what you manage when you reach the true order, you are managed to explain the process by an AR l model. So that is the basic idea; anyway let us look at it more in detail now.

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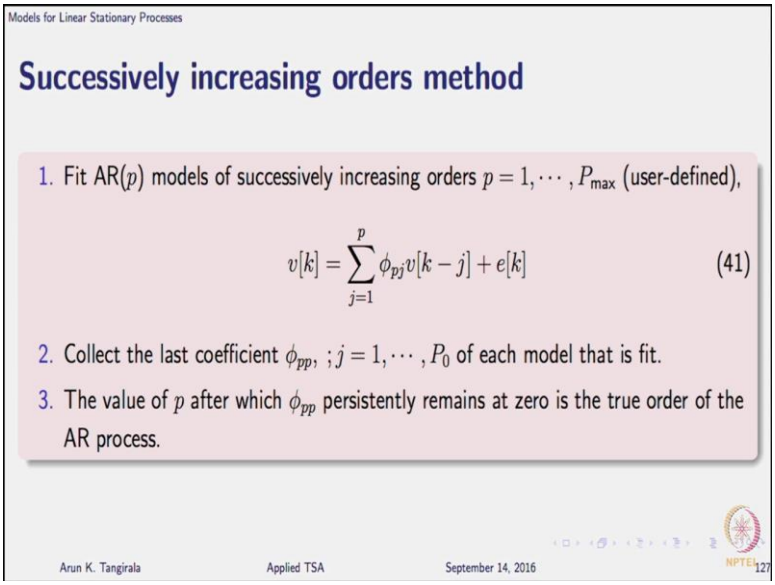

So, this is how you would fit this is a procedure that you would follow when you have to fit successively increasing orders AR of that is order in the sense the auto regressive models where you do not have to worry about the PACF concept itself, you just say well I am going to do a trial and error I am going to fit an AR model of successively increasing orders and keep track of the last coefficient and essentially the value of t after which you start seeing the last coefficient that you have included; turning out to be negligible you say I kind of hit the true order because the inclusion of any further order has not helped in any significant way that is the idea.

But there is a premise underlying this; we need a theoretical backing to this is at what we are claiming here is; if the model or if the process is of some order let us say third order and I fit a forth order AR then the extra coefficient that I have included should theoretically turn out to be 0; that is a premise on which this method works the successively increasing order method works, it claiming that as I successively increase the order and when I hit the correct order and I go beyond that then the extra terms that I

have included do not contribute at least theoretically do not contribute or you can say practically do not contribute in a significant way.

So, we need to prove that theoretically although I am not going to prove that here, it is so difficult to prove, but I will state the result; the result is when an AR p model is fit to a process of order p naught.

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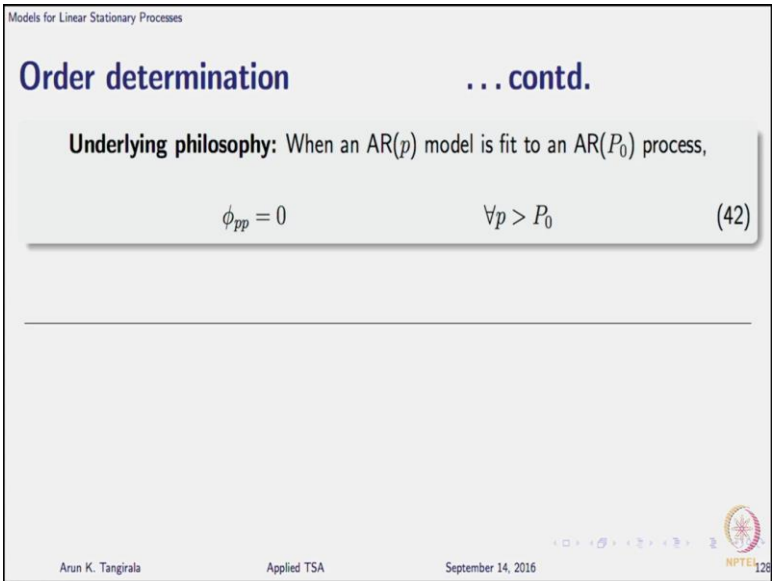

Then the additional coefficients that you have included; additional meaning in excess of the true order, they all turn out to be 0; how do you prove this, well you can in one way you can prove by setting up the Yule Walker equations and show that they turn out to be 0 taking into account the fact that underlying the process is AR of order P naught right when we say in the for example, if I say P naught is 1 that is the underlying process is the first order AR and I fit let us say a second order and so this premise here the underlying philosophy or the result says that the second coefficient of this AR to model is going to be 0.

What is the models symbolically v k is phi 2 1; v k minus 1 plus phi 2 2; here we are not using the coefficients ds because generally we reserve the notation d for a given AR process, here we are trying out different orders so as to keep track of the different orders we introduced this phi notation, this is fairly conventional in time series literature to use this notation phi. So, the first subscript here denotes the order the time fitting and a second subscript here denotes the coefficient.

So, I am fitting a model of this form, when I say I fit a second order AR and what this result claims is if the process is first order AR then phi 2 2 is 0, if this is how do you see that; what you do is you set up the Yule Walker equations right for phi 2 1 and phi 2 2 correct, so, what would be the Yule Walker equations. So, we can go back to the Yule Walker equations that we had here for example, in this here we have d 1 and d 2 in place of d 1 and d 2 you can have minus phi 2 1 and minus phi 2 2 right.

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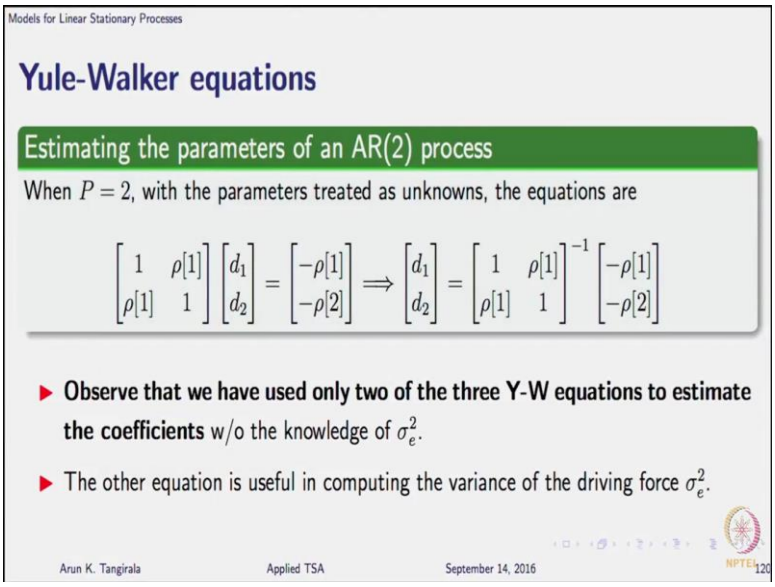

What would be the value of d 2; let us say if d 2 is 0 then phi 2 2 is also 0 for an AR 1, the equations that you setup in Yule Walker equations are fairly generic; for a generic AR process, but if you are given that the underlying process is AR 1.

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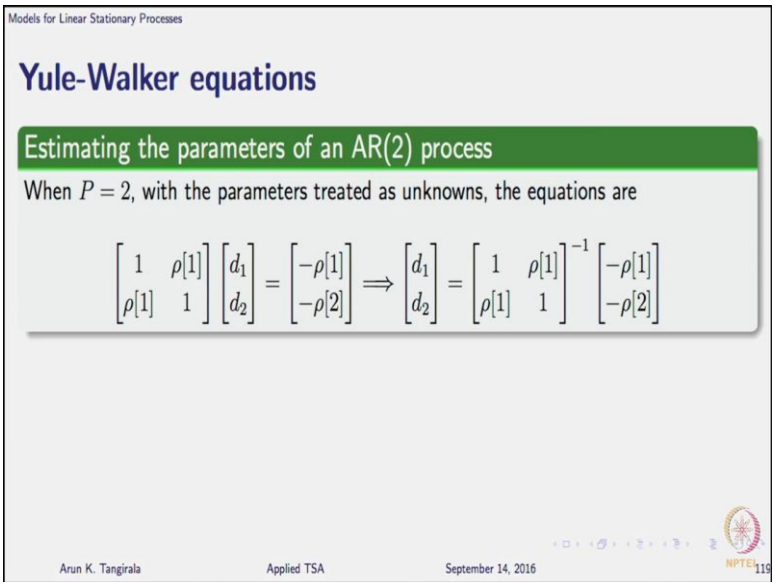

So, from this equations here what is a value of d 2; this generic solutions symbolic solution. So, d 2 would be you; you remember Cramer's rule when you want to selectively estimate coefficients you can Cramer's rule right, what is Cramer's rule by the way, to replace it; one of the columns the corresponding column with a right hand side and then of course, then you always have the determinant in hand.

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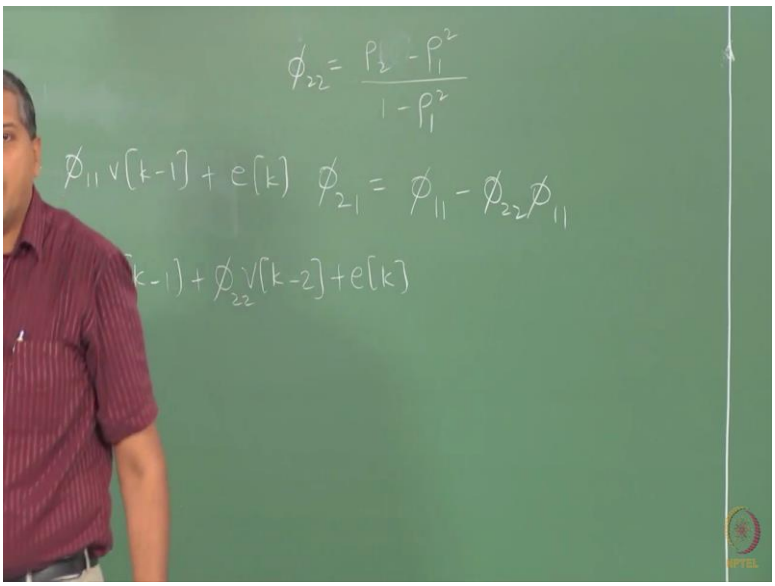

So, the determinant is 1 minus rho 1 square which appears in the denominator for d 2; what about the numerator.

Student: Rho 1.

Rho 1 square minus rho 2 or we can in order to connect it with the PACF, we can say it is a rho 2 minus rho 1 square; I am using a different notation here; it is a subscript it should be at lag 2, but you should understand. Now this is a generic solution for fitting an AR 2 model 2 any process; if I fit an AR 2 model to any process this is the value of d 2, until this point we have not made any assumptions on underlying process.

If the underlying process is AR 1 then you would want to ask what is the value of d 2 which is nothing, but your minus phi 2 2; they are the same except with a difference of sign. So, if the underlying processes is AR 1, what is a value of d 2 why; because if the underlying process let us say is a first order with some coefficient d 1 naught right.

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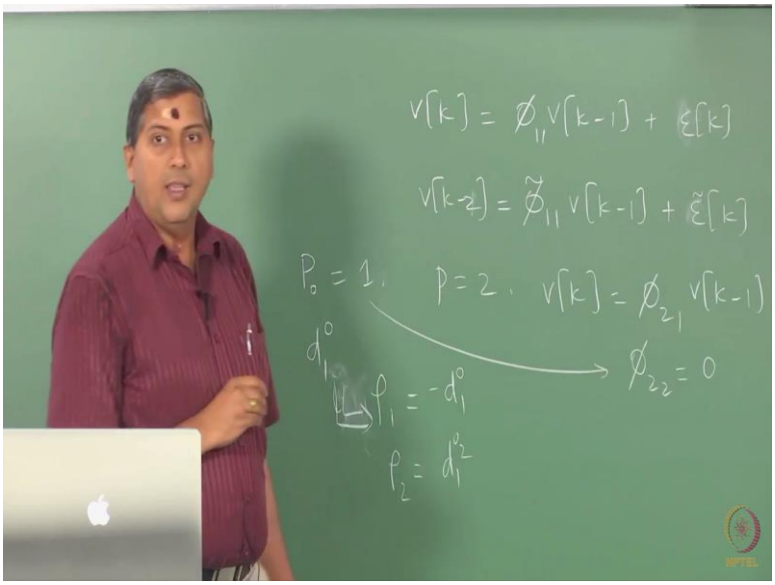

So we have said P naught is 1 let us say the coefficient is d 1 naught; this is how the underlying process involves. Therefore, what would be rho 1, so this implies rho 1 is minus d 1 and rho 2 would be d 1 square; sorry d 1 naught here d 1 naught square. So, you plugin this values; theoretical values into your d 2 and you can straight away see that this extra coefficient that you have included in excess of the true order as turned out to be 0. This is not co incidental, you can show this now for any AR process of order P naught; if you fit a model of order p to a process that is being generated by P naught then the excess coefficients are going to be 0; that is the base idea and that is also the basic idea for using PACF.

So, hopefully now at least you are convinced by way of example that the excess coefficients turn out to be 0. In fact, what you should go back as a simple excise and do is set up the Yule Walker equations for an AR 3 model; model would be AR 3 fix the process to AR 1 for the same process first order AR; fit an AR 3 model and see if the third coefficient also that you have included it turns out to be 0. Now of course, as I said you can actually estimate AR models of any order using Yule Walker equations, but there is an additional benefit of this approach of fitting AR models of successive order, which is that you can come up with the recursive way of estimating the auto regressive models of increasing orders; you do not have to solve the Yule Walker equations all over again.

So, if I have for example, worked out the estimate of a first order AR; for a given process and I want to now fit an AR 2 and I want to see if AR 2 is better suited than AR 1 then I do not have to necessarily solve those 2 equations that we saw earlier all over again, I can use a recursive relation and this supplies to any order, if I know the coefficients of the AR model of an order p then the coefficients of an AR model of order p plus 1 can be computed recursively and that was a algorithm given by Durbin and Levinson and today it is known as the Durbin Levinson's algorithm.

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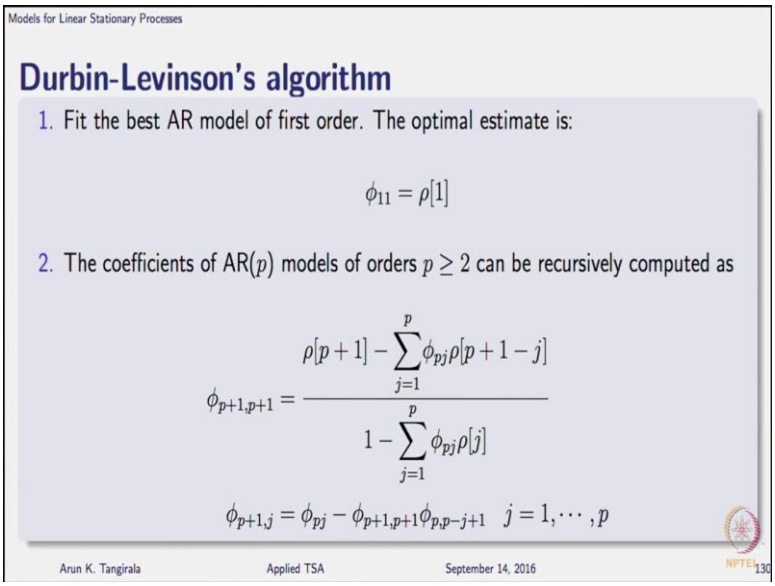

So, what you do is you start of by fitting an AR model of first order and that that is no big deal it is nothing, but the optimal estimate even from a least square few point or a

Yule Walker equations view point approach; the optimal estimate of the coefficient is rho at 1 the ACF at lag 1. Now the coefficients of AR models of successive orders that you have that is increasing now second order third order and so on; it can be computed recursively using the equation that is given on the screen; that is the recursive relation due to the due to Durbin and Levinson.

So how do you work this out; what it says is if you have an AR model of order p; first you estimate the last coefficient of the next higher order model; do not get confused here, this is an AR model of order 2.

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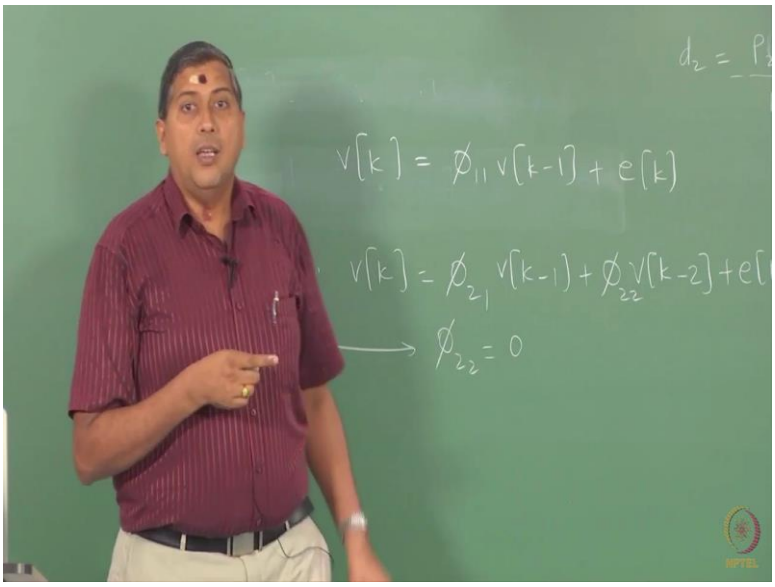

What you would have estimated is an AR model of order 1, you would have estimated this phi 1 1; v k minus 1 plus some v k do not get confused between these 2 e's; they are actually different.

So focus on the coefficients here; you would have estimated this and now you want to estimates phi 2 1 and phi 2; 2 and the algorithms says you can do it a recursive manner without having to solve the Yule Walker equations all over again and what the algorithm says is start off by estimating phi 2 2; once you are estimated phi 2 2, get the estimate of phi 2 1; do I need phi 2 1; yes if you want to estimate if I want to estimate an AR model of order 3 and so on, but if I have decided to stop and I just want to know the order then I do not need those other coefficient, but typically we would need because once we have hit the true order then you may want to actually know the remaining coefficients.

But typically we want to know the remaining coefficients of the previous model for example, here for this process we have just now shown that if the underlying process is AR 1 phi 2 2 will turn out to be 0 then what is the correct model order 1. So, I have to go back and use phi 1 1; I should not use phi 2 1, that is a very important subtle point to remember. Once I have hit the true order; I should work with the previous AR model because it is says you have crossed the line and realize that know that there was the line.

So, you go back to that line and then use the coefficients there, in other words here if you discover the phi 2 2 is 0. In fact, you cannot stop at phi 2 2 being 0, it may be just that one coefficient can work out to be 0; for some processes. You have to make sure that persistently phi 2 2 and then phi 3 3; all when you have to move on to AR 3 and check that phi 3 3 is also 0 and so on; at least until the few lags; then to come back and say what is a first time I cross the line and I cross the line here in this example at 2.

So, I go back to first order and use phi 11 for my model; you cannot use phi 2 1 because that is assuming that you are working going to work with AR 2. Theoretically phi 2 1 may be same as phi 1 1, but practically it would not be because what happens is in estimation depending on how many parameters you are including in your model; estimates will change and so will the errors. So, therefore, you have to go back to the previous model and then work with that, so that is the idea.

So, if you look at the equations carefully what it does it estimates phi 2 2 first and then the second relation there allows you to estimates phi 2 1; when you are done with that and you want to move on 3; third order AR then you estimate phi 3 3 first with the help of phi 3 3; you would estimate phi 3 1 and phi 3 2 and so on. So, as an example; I do not know if I have; so let us I will come back to the PACFs. So, as a keep a side PACF fact; let us look at how would you estimate given phi 1 1, how do you estimate phi 2 2.

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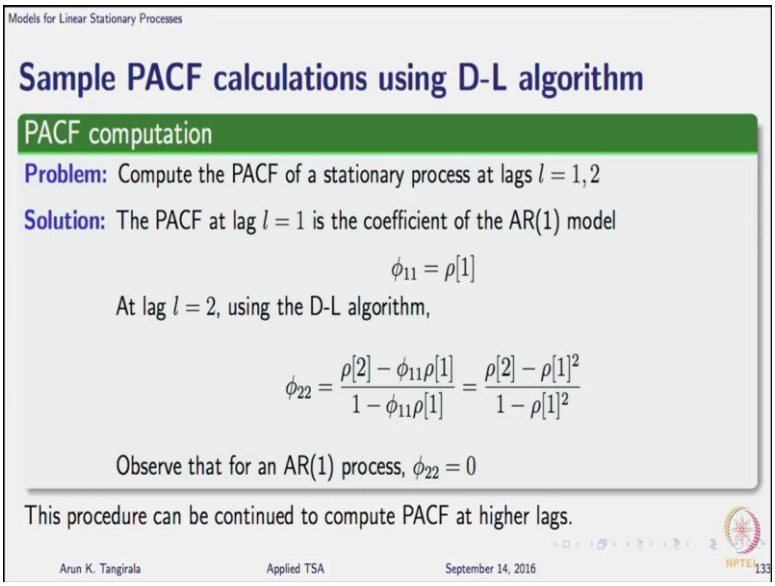

So, go back to your Durbin Levinson's algorithm; if I am given phi 1 1; I already know from the previous discussion that phi 2 2 is essentially this. We have already solved for it, we did not use the recursive relation at that time; we solved a Yule Walker equations all over. Now let us see if the Durbin Levinson's is actually given me that. So, if you look at the relation there, it says 5 f e at p plus 1 comma p plus 1 that is phi 2 2; p is 1 would be rho at 2 in a numerator, the first time is rho at 2 minus how many terms do you have in the summation 1 and that happens to be phi 1 1; times rho 1, but phi 1 is already rho 1 right. So, you have rho 1 square and in the denominator you can straight away see you have 1 minus rho 1 square.

So, you do get that right of course, this algorithm has been proposed and verified many times within theoretically you derive and you can also see that it allows you to derive phi 2 1, what would be phi 2 1; it says phi 2 1 is. So, let us writes for phi 2 1 using the b l algorithm without solving the Yule Walker equations; what do you get for phi 2 1; phi 1 1 minus.

What you get, why this so difficult.

Student: (Refer Time: 23:16).

Sorry phi 2 2 times phi 1 and you should verify, if indeed you get this as the answer we are just left it at in terms of phi's, you can substitute for phi 1 1 and phi 2 2 and see if

you indeed get the same from the Yule Walker equations right, it is a very simple exercise and this is now the algorithm also that is used to estimate the PACF coefficients.

How, because there is a strong connection with this approach and the PACF and what is that connection as I said earlier and even previously; the last coefficient if I want to estimate PACF at lag l then I fit a AR model of order l and plug out the last coefficient. So, which means phi 2 2 is nothing, but the PACF at lag 2 phi 1 1 is; obviously, the PACF at lag 1; PACF at lag 3 would phi 3 3 and so on right. Now if I want to compute, so this proof itself I am avoiding, but I think you would have seen by now in the sense when you setup those equations for PACF; computing PACF and you compare with the successively increasing orders methods, you will be able to prove very easily that the last coefficient is nothing, but PACF where inventively you can look at it this way, when you look at this model what is phi 2 2 measuring.

What is actually phi 2 2 measuring, it is measuring the impact or influence; linear influence of v k minus 2 on v k in presence of v k minus and that is what PACF is also trying to do; I mean I just giving you a very strong qualitative proof here that this phi 2 2 that you are looking at in an AR 2 model is actually measuring the linear influence of v k minus 2 in presence of v k minus 1; phi 2 1 is not measuring the linear influence of v k minus 1 on v k directly it is directly doing, but it is not PACF at lag 1. It is definitely measuring the soul influence of v k minus 1; on v k given the conditioning variable being v k minus 2, but that is not PACF at lag 1 PACF at lag 1 is nothing, but ACF at lag 1.

So, you have to understand the notion of PACF clearly the PACF is nothing, but the co relation or the measure of linear influence between v k minus l and v k; conditioned on all intermediate observations not any other future observation.

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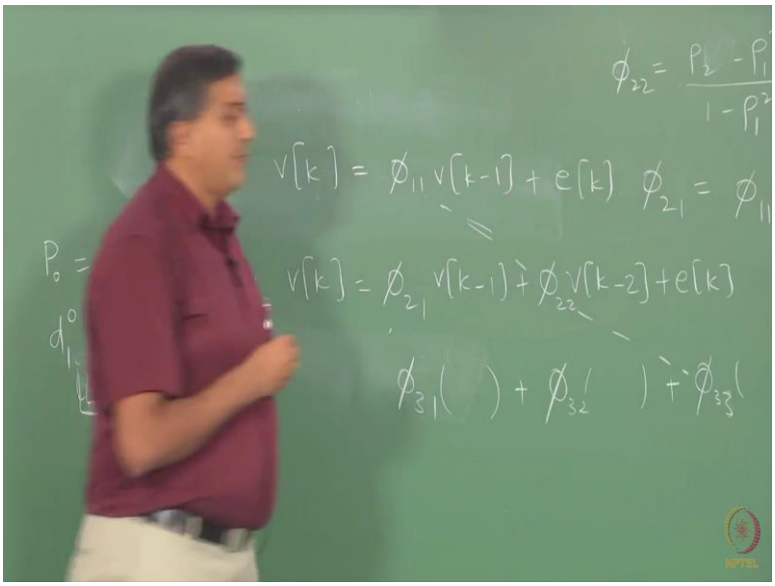

So, in other words, if I move to a third order; I would have phi 3 1 and phi 3 2 and a phi 3 3 right plus e k. So, when I move to the third order model you cannot say phi 3 2 is a PACF at lag 2. Obviously because PACF at lag 2 is only looking at discounting for the effects of v k minus 1, it is not discounting for the effects of the other sample; only the intermediate once because it is looking at the time chain. So, phi 3; 3 is now measuring how much v k minus 3 is directly influencing v k; in presence of v k minus 1 and v k minus 2. So, that kind of convinces you that indeed the last coefficient of the AR model has to be the PACF. So, using this connection and the Durbin Levinson's algorithm; now we can compute PACF in a very recursive manner computationally efficient manner yes.

Student: (Refer Time: 27:18).

Student: (Refer Time: 27:22).

Ok.

Student: (Refer Time: 27:26); that means, p is 2.

P is p is 1 it depends on where you are; if you are fitting if see in the first step you assume that you are fit in AR 1 right. So, the coefficients of the AR p model of orders; I see p greater than or equal that is the confusion that you have. So, you look at it this way disregard the statement p greater than or equal to their; sorry for the confusion, the way you use the recursive relations is; it says if you have coefficients of AR p model then

how do you estimate AR p plus 1. So, I will make that small correction I understand the source of confusion; is it clear.

So, I will take away the p greater or I will just say of orders greater than or equal to 2; any other question, glad you are watching the notes quite closely right. So, now with these connection, we have a recursive or a computationally efficient algorithm for computing PACF and that is what you are PACF routine in r or any other package ideally use as it use as a d l algorithm, not only this does this d l algorithm give me computationally efficient way of computing PACF, but even for estimating the AR models, the d l algorithm was devised in general for estimating any AR model of a given order by knowing the AR model of the previous order.

So, it is does not bind itself to PACF, PACF happens to be only a small piece of information that, but an important piece of information that you are extracting out of the AR model. So, one keeps track when you want to compute PACF, one keep track of this these chain of coefficients.

So, that is the generally and widely used PACF algorithm for recursively computing PACF; if you have gone through this I am going to skip. So, that kind of brings us to the conclusion of the discussion on auto regressive models, we have discussed what an auto regressive model is that is how does it take birth from the linear random process and it takes birth whenever we assume a certain kind of parameterization for the impulse response coefficient and that parameterization is nothing, but an exponentially decaying one and parameterization of I r coefficients is a same as parameterization of the ACF as well.

So, you might as well say an auto regressive model is that whose ACF is parameterized to d k exponentially and of course, another way of looking at auto regressive model of orders p is that any information that you give beyond p lags is not going to make any difference in my forecast; that is another way of looking that is from prediction view point and there are several advantages of working with AR models it gives raise to linear predictors unlike M A models, where I have to go through this additional step of figuring out what the shock wave is in order to complete the forecast.

But then order determination requires a computation of PACF, although you can say it is PACF what you are actually doing is as we have discussed just now; you are fitting AR

models of successive orders and you are computing it in an efficient way that is all you are doing, but conceptually PACF is a conditioned co relation; conceptually this approach of fitting successive orders and PACF are quite different, the notion of conditioning there appears, but now you can see the notion of conditioning amounts to actually fitting linear model itself.

So, for example, here and I am actual when I am computing PACF at lag 2; what I am actually doing is multiple linear regressions. I have many regressors but some specific regresses when I am looking at PACF and then we discuss the issues of stationarity which we said that even though the underlying process as a non stationary causal representation, I can give a stationary non causal representation, but we will stick ourselves to the causal models. And hence forth when I say causal AR model, you should understand it is stationary and then we talked about the Yule Walker equations which are central to the estimation of AR models and the computations of the PACF.