Applied Time-Series Analysis
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Lecture - 39
Lecture 17B - Models for Linear Stationary Processes-3

Now this is where we would like to slightly digress and briefly review the concept of linear systems theory for the deterministic world, so that you understand the parallels.

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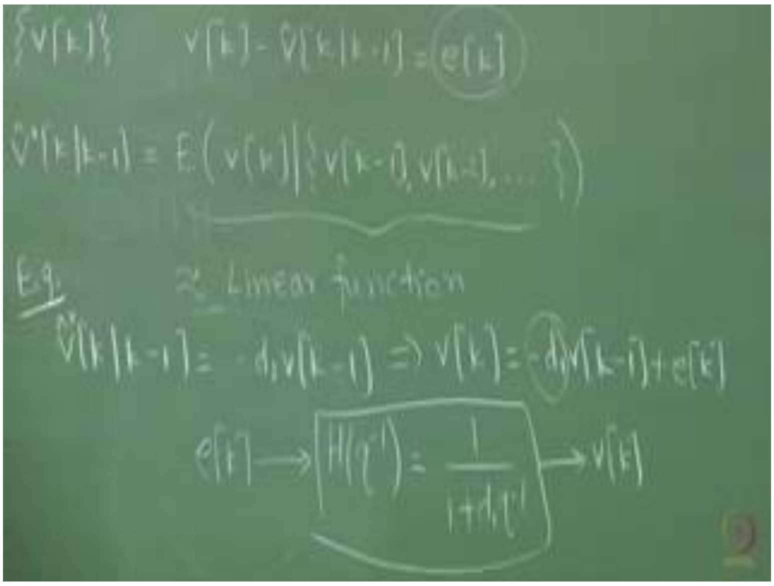

Now before we do that we have already done one step, another step here where we introduce the transfer function operator for this process, it happens to be 1 over 1 plus d 1 q inverse and essentially now I can think of v k being driven by this H, which is a first order autoregressive process right. So, straight away you get this interpretation that white noise is saying input that is driving v k, it is only an interpretation and most importantly you should remember that these white noises endogenous remember right. So, that is the thing.

Let me also write here. So, that you tell yourself once again that this is (Refer Time: 01:22) external signal it is an internal part of v that is driving itself, we have no other choice if you look at it intuitively we are saying I do not know what caused v and I am still supposed to build a model; then the only way I can explain a person who is talking to himself or who is just enjoying, who is excited is self excited there is no other person who is telling something to this person or causing this person to be excited and so on. So, the only way I can explain is something internally is exciting this process. So, therefore, this kind of representation does make sense in a qualitatively.

Now, this representation itself of some signal passing through a linear filter to produce a response is not new in the linear systems theory world this is fairly well known and it is important to draw some connections with the deterministic world, so that some of the terminology, some of the requirements, some of the concepts such as spectral density, frequency response function, impulse response and so on, straight away can be borrowed and applied to the stochastic world.

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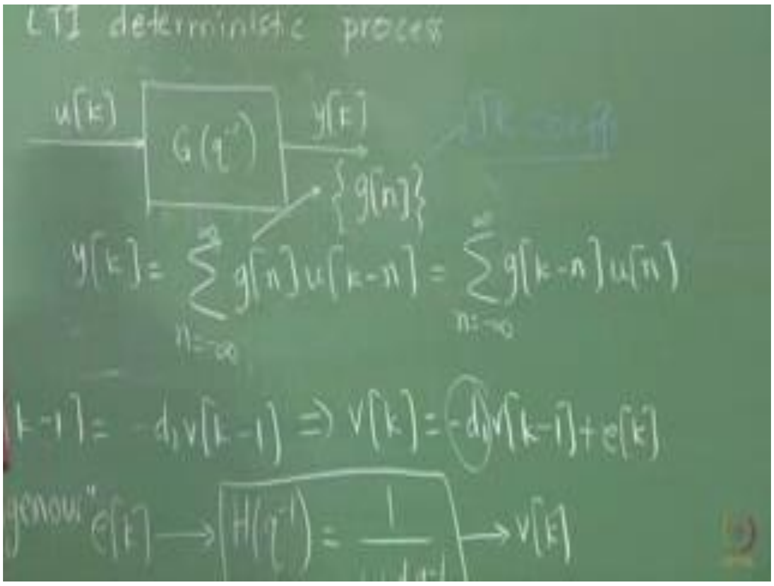

So, if you take an LTI deterministic process, one finds the same kind of representation. To distinguish the deterministic process from the stochastic process we will use the notation G; and in a deterministic world there is an input that is I know the cause and I know the response. So, Y is the response and U is the input or the probe signal or whatever and LTI stands for linearity and time in variance; for those of you who are familiar with the linear systems theory, you should recall that any LTI system almost all just bearing very few peculiar ones, you could by virtue of the property of linearity and time in variance, you could write a mathematical relation between the input and output as a convolution equation, which can also be written this way where n runs from minus infinity to infinity.

Now, this is in some sense of fundamental equation that is central to linear systems theory, it is you can say the mother of all descriptions of linear time in variance systems that you would encounter, although there is another form that is very popular which is a difference equation form, but will talk about it a bit later. In fact, the form that you see the autoregressive form is not straight away in this frame work, it is actually in the difference equation world, but very soon we will see that a difference equation form is born out of the convolution form under some conditions, many text books may not present the linear systems theory in this way, but it is a very very important prospective that one should have of the difference equation form.

So, let us look at this convolution equation a bit more closely, this is called the convolution form because it is not a direct product you can see that there is a convolve convolution of g with u or u with g whichever way you look at it, but another way of looking at this equation is that the output is a weighted sum of the past inputs; past present and future to be more exact of course, you can ask why are we including future inputs we do not have to we can restrict the summation from zero to infinity, but we will retain this non causal nature because it is useful one for theoretical analysis, two there are systems that are non causal; may not be the a man made processes that we see, but human beings for example, the behavior is non causal. We imagine something to be happening in future and react now right. So, that the non causal behavior in the traffic lights we know that the signal is going to turn from red to green already you start honking 10 seconds before that is a non causal behavior. So, there are many processes that are non causal therefore, we will retain the future inputs as well.

Now, always remember the convolution equation as being a linear sum of the past present and future inputs and it really helps and the other point to keep in mind is which is very important this sequence g the coefficients of g they are called impulse response coefficients, they are called the impulse response coefficients IR stands for impulse response; what do you mean by impulse response? If I were to give or excite the system with an impulse signal discrete time impulse we are not talking of dirac delta functions a kronecker delta function which is a we assume it to be unit impulse.

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So, this delta n or delta k whatever you want to denote that k is just dummy variable, this is the kronecker delta it is non zero in fact unity, if it is a unity impulse function at k is equal 0 and as well, like the acf that we have seen for a white noise. When I excite the system with this impulse I get this response g, and you can see that straight away by substituting here replace u k minus n with delta or even here does not matter in both equations you see straight away that why k is nothing, but g k .

Now, this impulse response based description of a system is very very widely used in filter design, in stability analysis, causality analysis everywhere and the all the properties including the prediction of the linear system can be derived by just knowing the impulse response coefficients that is the beauty of an LTI system. So, you see how convenient life is the moment we assume linearity and time in variance. When we talked about stationary processes we only talked about linearity, but we dint talk about time in variance or did we. The linearity here how is linearity defined here in deterministic world? So, now, we are drawing parallels lowly between the deterministic world and the stochastic world, the linearity here is defined in a mathematical sense that is the if I have two inputs producing two outputs, then a super position of the inputs should produce the same super position of the respective outputs that is very clear.

But did we define linearity that way or have we not at defined linearity in the stochastic world, did we define linearity yet? We have just talked about linear predictors, but we have not talked about linear models here, but we can do so. How can we do that now? Earlier it was not possible if you consider the signal as is it is not possible to think of linearity, when we talk of a system and the linearity typically we are talking of the mapping between the input and output and when I look at it is signal alone in the stochastic world, there is no input.

However, we have said that we have we can give this interpretation, when I am con when I am living in the linear predictor world I can give this interpretation that e k is driven by an input fictitious input whereas, this is not fictitious and this is deterministic whereas, this is stochastic. I know exactly the values of the input at a very instant in time in the deterministic world whereas, I do not know and maybe I do not even bother so much as much as I worry about it is statistical properties. So, there are parallels and they and get there are very strong differences, but we can use some of the concepts.

So, now we can say that all linear models that I think of, may has the same can we given the same linear definition as we see here in the deterministic world, but what about time in variance?

So, the time in variance in the deterministic world translates to stationarity. So, you see what we are doing is whatever we learn in the deterministic world most of it now we translate that to the statistical properties. So, here e k is a an input, but it is white noise has an impulse like behavior and v k and of course, e k is stationary and because h is linear under some conditions on this coefficient d 1 v k is stationary that is more or less the idea of time in variance.

So, for a linear stationary process we may be able to write a model like this, we will not write that write now, but we may be able to and we will in fact, eventually write a model of that form, when we do write a model of that form then we can give similar interpretations to the coefficients of such a convolution equation, we may call that as the impulse response sequence, although it does not make so much sense because what do you mean by e k impulse response in the stochastic world right because e k can be an impulse.

However, we can give some interpretation of those coefficients as impulse response coefficients will come that and remember whatever I am saying right now is based on analogy, but the actual formal way of arriving at a linear random process the conditions and so on were worked out independently of this, people did not actually developed models for linear random processes by starting from deterministic world do not think that way, you just turned out that they started off in a very rigorous way and it ended up being so strikingly similar to what you see in the deterministic world and it make sense.

I just now said for the deterministic world knowing the impulse response sequence amounts to knowing everything about the system, what I mean by everything is you can predict the responses system to any arbitrary input, you can determine stability property of the system, stabilities have very very important property right what does stability mean in the deterministic world at least in the linear world deterministic world? Are you familiar with the definitions of stability?

Student: (Refer Time: 13:28).

One is bounded input bounded output what about the other one.

Student: The system is (Refer Time: 13:35).

can you extend that; there are essentially two forms of stability that one encounters in the linear deterministic world, one is the asymptotic stability which is when you perturb the system that is the system is perturb by some non zero initial condition, then you leave it on it is own it should return to it is equilibrium, this we call as a stability based on the free response natural response, you just pull the pendulum for example, or some for some reason the pendulum has a non zero displacement and that is it you leave it, it should return to it is equilibrium. If it does we say it I s a asymptotically stable, it is based on the free response of the system then there is another stability which is called the BIBO stability; bounded input bounded output stability not the BIBO that you see on water bottles ok.

BIBO stability which demands that if I apply a bounded input then I should get a bounded output; that notion is based on the force response to one stability concept is based on the free response and other stability concept is based on the force response. Now without going to too much into the theory generally speaking for a large classes of linear systems both BIBO stability and asymptotic stability are equivalent, except under some conditions they will not go into that, so we can work with BIBO stability; for now there are systems for which asympto there are systems that are not asymptotically stable, but are BIBO stable; it is possible, but we will not worry about such systems as of now, we will assume that the system is both asymptotically and BIBO stable.

Now, we turns out that stability basically what stability usually means is, the output should not run away with time right more or less that is the notion that stability means and in fact that is exactly what BIBO stability implies. When the input itself is bounded why the output not run away with time and it turns out that this requirement translates to necessary and sufficient condition on the impulse response sequence that it should be a absolutely convergent, do not relate this absolute convergent to the that of the ACVF that is a different condition, but we will see a similar condition in the linear random world as well; when we write the convolution equation for the linear random process, we will see that this is a requirement and this requirement in return in for the random process guarantees that the linear that the autocovariance is absolutely convergent.

So, there is a connection, but the connection is somewhat indirect; for now you should remember that stability for the deterministic world amounts to saying that the output should remain bounded, which is one of the features of a stationary process right if I have a random process whose values just grow with time, there is no way you can think of it has being stationary. So, it is a necessary condition not fully sufficient, you remember this stock market index the series was running away with time, but then there are other series which were bounded. Not all bounded once are stationary remember that, but if the series is unbounded if the signal is unbounded then clearly it is not stationary therefore, this necessary condition; in the case of deterministic world this both necessary and sufficient for stability because it I s a deterministic world.

In the random linear random world we will come and we. In fact, we will come across exactly an identical condition on the impulse response coefficients of the model, which will then guaranty that the series stationary, then under the autocovariance function is absolutely convergent. So, you should understand then the parallels also and then of course, there is also this requirement if you want to restrict yourself to causal processes, then it is necessary that the impulse response one of the ways of stating that the processes causal is to require that the impulse response is zero at negative times; that means, even before I excite the system at an impulse if it response then obviously it is non causal; children are like that they anticipate that parents are going to scold starts screaming upfront itself, that is a non causal behavior right? Causal behavior is after the scolding has been and then you starts screaming or just at that moment I do not know about students though, but that is what causality here translates to that g k should be 0, for all k less than 0 this is what causality means and one can impose a similar condition in the linear random world as well.

Then there are two important sub classify I mean children of this model; this is you can say the model for the linear time in variance system; from here take both two different classes of models, this is by the way called in general and infinite impulse response model because you assume that the impulse response exist at all times and this itself tells you that you are restricting yourself if you are if you are restricting yourself to stable LTI system, then you are restricting yourself to systems whose impulse response decays with time. So, coming back to the point form this model you can see two different classes of models taking birth known as the finite impulse response model and the difference equation model call the FIR model standing for finite impulse response model and the other one being the difference equation model.

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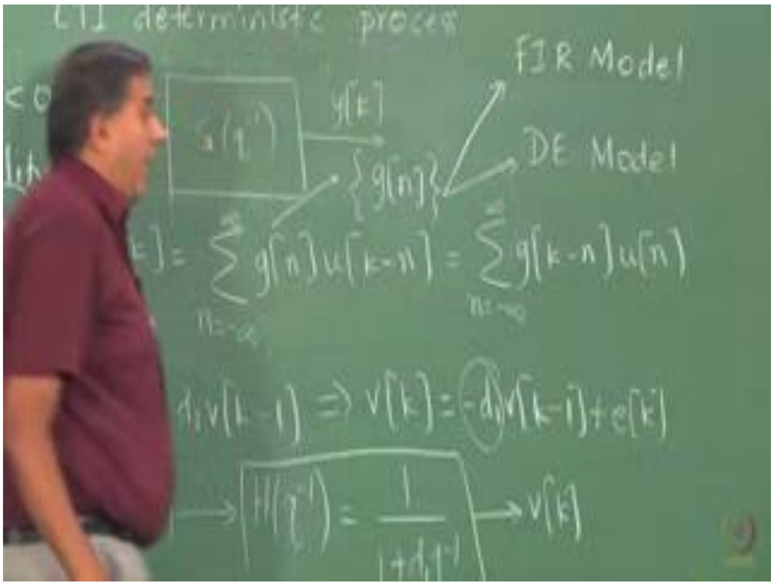

They straight away do not they are not born just like that, you have to make some assumption on the impulse response. Now this is the thing that you will find somewhat missing in many texts on statistical signal processing or stochastic processes; to point out that whatever we call as moving average and autoregressive models are not just born like that, you do make some assumptions on the impulse response coefficients of the linear random model.

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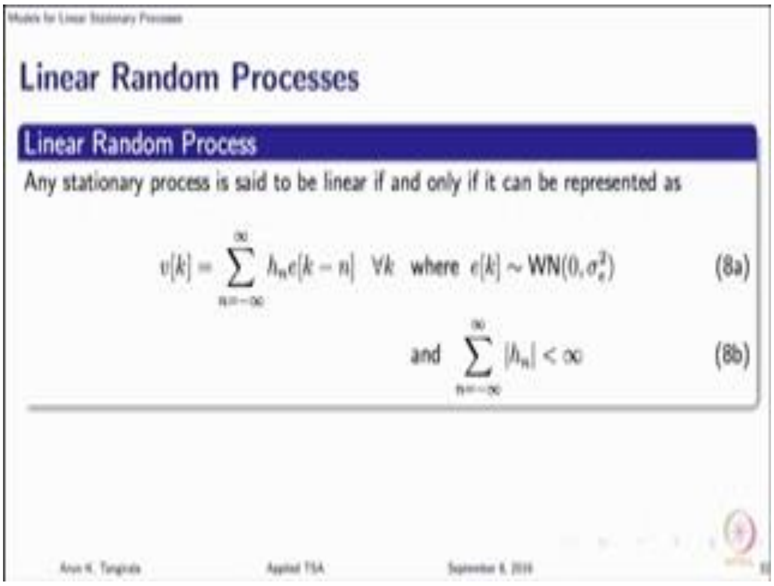

Now, let me actually show you the linear random process model; this is the model as you see on the screen, you see there is a striking similarity for the definition of a linear random process, we will talk about this F I R and difference equation models when we talk of moving average and autoregressive models respectively.

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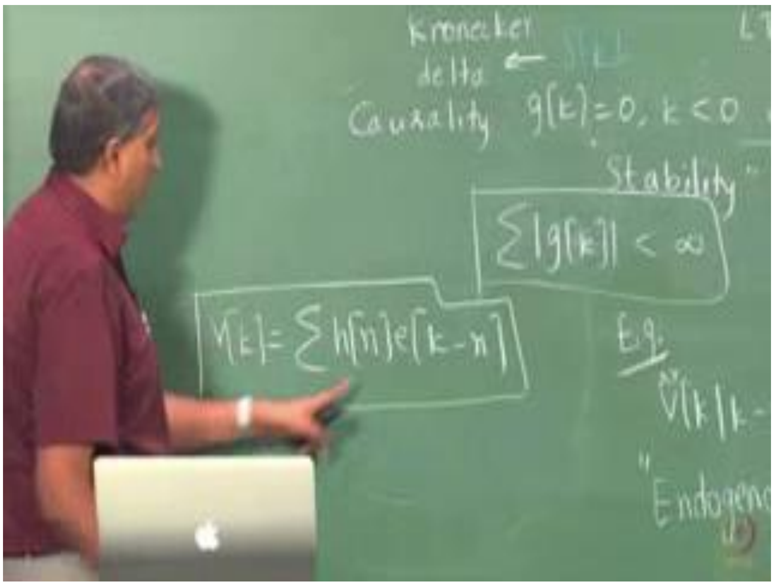

But it is useful to at least before we adjourn be familiar I have used an h subscript n, but we will use this notation as well h index with an; both h subscript and then h in extend mean the same thing. So, what you see on the screen is the definition of a linear random process. Look at the difference in the deterministic world we stated or we assume the process to be LTI, and you can show all the way of not proved, you proved you can arrive at this equations starting from linearity and time invariance whereas, in the random signal world, we are not defining linearity and time in variance upfront; we are, but even with that we are no we will not be able to necessarily come with this straight away, you have to go through some formulization and then arrive at this.

So, in general what you will see in texts a linear random process is itself defined this way; what is the requirement? Any process that can be represented as white noise passing through a filter and LTI filter is essentially a linear random process, subject to two conditions: one condition is that this impulse response coefficients h that you see now should be absolutely convergent; in the deterministic world that was required for stability, in the random world you can say that this requirement of absolute convergence of h is you is necessary in for many reasons, one of the reasons is that the summation that you see on the right hand side should converge it should converge to what to a random variable, what kind of if you were to look at all times it should converge to a stationary random process, more over this requirement that h should be absolutely convergent also guarantees that the autocovariance is absolutely convergent; which is the condition the foremost condition that is required for us to develop a model like this. We will talk about this conditions and the moving average model tomorrow and then we will continue with the discussion on autoregressive models subsequently, but you should see the this condition here on absolute convergence and the other condition that I mention there is a two conditions: one is on absolute convergence and the other one is that the driving force is a Gaussian white noise.

Now, that you will find variations across text; we write this we require that e k is Gaussian white noise, which is assumption made in many text books. In many other text books the requirement on e k is that it should be IID, independent and identically distributed; remember white noise only requires only focuses on the second moment up to the second moment, it says it should be uncorrelated whereas, IID requires that the driving force should be independent now; however, a Gaussian white noise is also an

IID. So, we are not really in deviation from the general requirement, so we will assume that the driving force is actually a Gaussian white noise.

So, although I have written white noise we will assume it to be Gaussian white noise and the stationarity that we are talking about is still the second order stationarity only; that means, the mean should be bounded and should be invariant with time, variant should be bounded and invariant with time and autocovariance should only be a function of the lag l.

So, those are the conditions that we imply by a linear by stationarity here, we will talk more about this in particular the moving average model and it is connections with the F L R model and so on then talk about an important concept known as inevitability.