**Applied Time-Series Analysis**
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**Lecture – 33**
**Lecture 14B - Autocovariance & Autocorrelation Functions-8**

We have now understood that not any sequence symmetric sequence qualifies to be the ACF of a stationary process, it has to be non negative definite and we have learnt a way of testing for non negative definiteness and we have also learnt that the ACF of a general MA M process has a sharp cut off after lag m; although we have not formally proved it we have just gone by induction 1, 2 and so on. So, we are able to see intuitively we can we will now prove it more formally with the introduction of what is known as an Autocovariance generating function.

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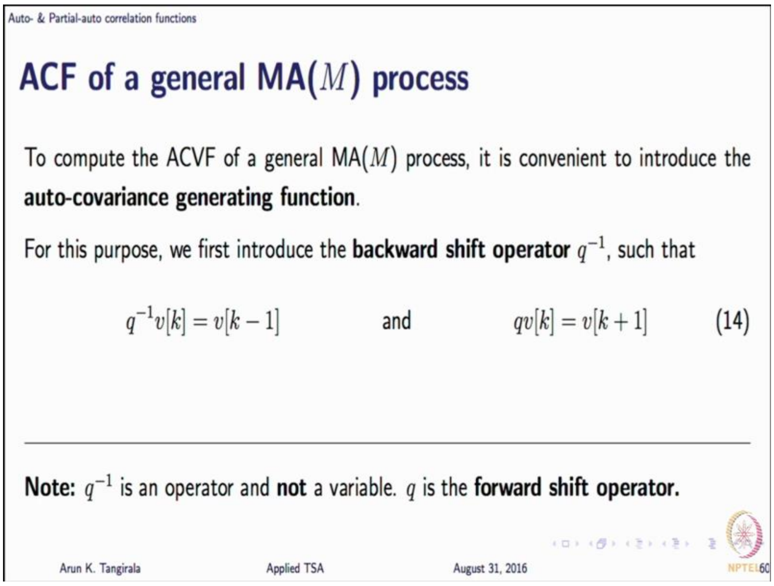


I do not know you must have heard in probability theory movement generating functions right and characteristic function. So, this auto-covariance generating function is nothing, but a movement generating function, but of the joined pdf, but particularly of the second order movement.

Now, in order to understand the auto-covariance generating function and even our time series models later on, we do work with what are known as shift operators and I have given this notations earlier as well; the role of the shift operator is to shift backward shift operator is to shift the observation by one observation in the past and likewise for forward shift operator fine.

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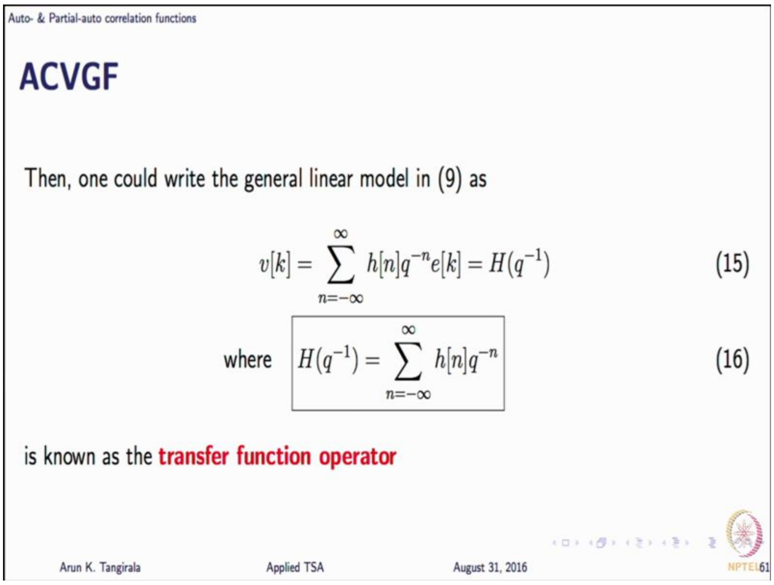


So, let us first define this auto-covariance generating function as follows: In some textbooks you will see in place of the operator, you will see the complex variable z, but the result is no different finally, the way the auto-covariance generating function is applied to determine the ACVF of any stationary process, it is the same any linear random process you can apply this auto-covariance generating function and I will show you with an example how to use this.

So, the primary reason for introducing auto-covariance generating function is to allow us to compute the theoretical ACVF for any linear random process; when we derive the ACVF for MA 1 we went through certain procedure right we note down the expectations, we started from the definition and then we wrote down the results using the properties of white noise. On the other hand this ACVF sorry generating function will allow us to compute in a much more easy way. So, let us now first introduce this transfer function operator. So, we have noticed earlier this definition of linear random process, but now let us introduce what is known as a transfer function operator.

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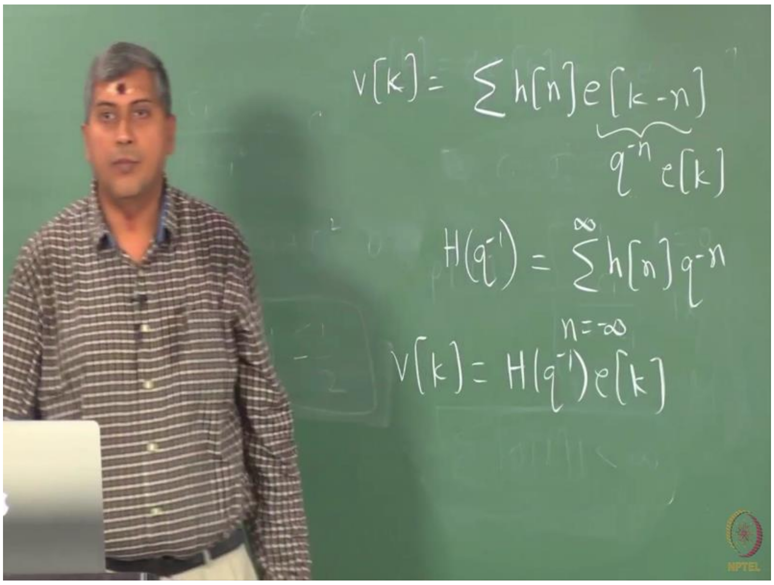


So, we have the general linear random process and we could write e to the sorry e of k minus n as q to the minus n operating on e k.

Remember q is an operator it is or q inverse is an operator it is not a multiplier. So, do not think of it as a multiplication operation. Now with this change we can now introduce this transfer function operator which is a polynomial operator, in general of infinite degree, we can restrict the summation to 0 it does not matter, so that I can write in general any linear random process as some operation on e k.

We have already stated that schematically yesterday we said that we could think of this random process v k as some operation on white noise and keep telling yourself that this e k is not exogenous it is an internal part of v k. So, it is a self exciting sorry some excitation that you cannot predict and that excitations being operated upon by h to produce what you observe that is the imagination. Now with this notation of course, you will also see transfer functions in time series literature, but we will stick to the transfer function operator for now.

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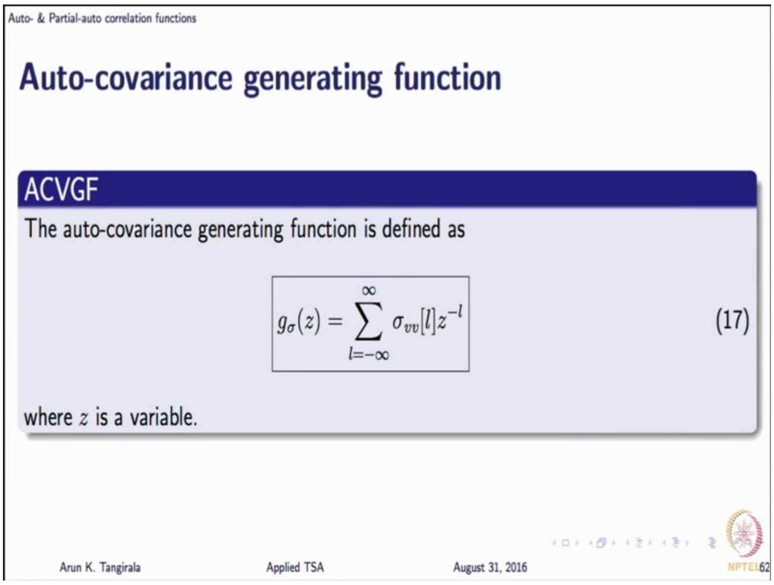


With this notation first also now and then we can move forward to the definition of ACVGF, it is auto covariance generating function which is defined as the z transform, the h of q inverse is a notation that is going to be not only useful now, but also later on. Now we are talking about the definition of the generating function.

The auto covariance generating function is simply the two sided z transform of the auto covariance function, do not search for physical meanings of the generating function, remember generate the role of generating function is only to facilitate easy computations when you turn to probability theory, there are moment generating functions; what is a role of a moment generating function? Facilitating easy computation of the movement's, likewise this generating function will allow you to easily compute the ACVGF.

So, given an auto covariance sequence, the auto covariance generating function is simply the two sided z transform of the auto covariance, you could write the left hand side is a function of z inverse or z it really not a does not matter so much. So, what is a connection now? This definition alone does not allow me to compute, it says given an ACVF I will construct the generating function this way, what I need to establish is a connection between this operator or we can say this transfer function operator and the generating function and that is the result that given in this slide on the screen.

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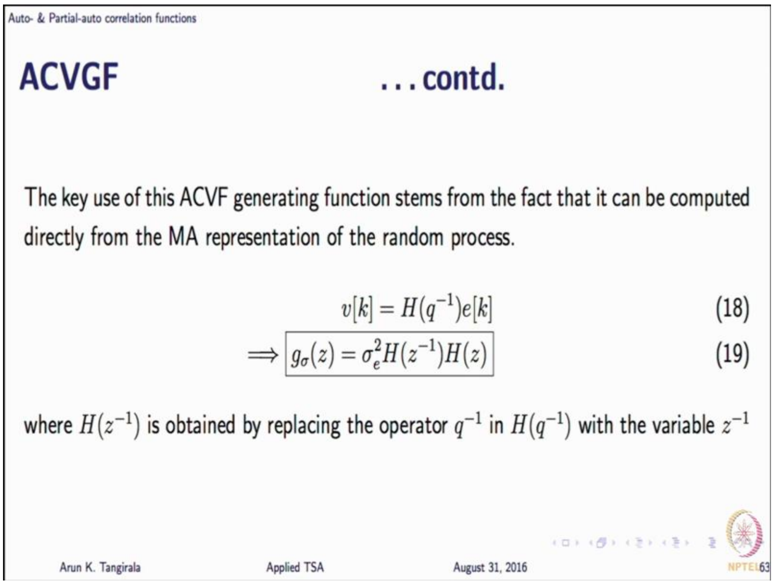


It says that if v k is coming out of a linear random process with a transfer function operator h of q inverse, then the auto covariance generating function can be written as a product of sigma square e which is a variance of white noise and h of z inverse, what is h of z inverse? All you do is where ever you have q inverse you replace by z inverse right that is your h of z inverse times h of z; what is h of z? Where ever you have q inverse you should replace it by z be careful right do not thing that h of z is actually rewriting h of z inverse in terms of z, h of z is where ever you see q inverse in your h of q inverse that is what it is. So, let me just illustrate this with simple example on an MA 1 process.

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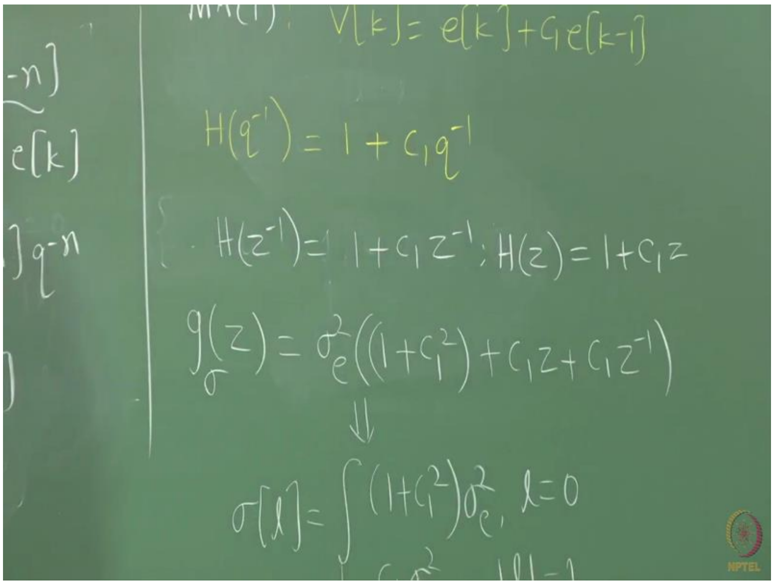

We already know the result for an MA 1 process right what you see generating equation that we have? V k is e plus e k plus C 1 e k minus 1 so that H of q inverse is, what is H of q inverse for MA 1? So, I am illustrating now how to use auto covariance generating function to arrive at the ACVF for a MA 1 process that we derive theoretically yesterday. So, what is H of q inverse 1 plus C 1 q inverse right good; now this result says if you want to compute the theoretical auto covariance function, step 1 write your h of z inverse and f of z and step 2, compute your auto covariance generating function as given in the result and from that how do I figure out the auto covariance?

So, once I give you g of z, can you figure out the auto covariance function? Yes or no go back to the definition right if I give you the left hand side as a polynomial, can you figure out sigma of l?

Student: (Refer Time: 08:45).

That is a simple that it is a simple reading exercise, you just have to read of the coefficients of the z to the minus l or z to the l, it does not matter why because sigma is symmetric. So, the bottom line is if I somehow have a mechanism where I can get g of z without going through those expectation businesses then I can actually read of the coefficients from the polynomial and write down that as a auto covariance function. So, for this example that we have H of z inverse would be 1 plus C 1 z inverse, what would be h of z?

Student: (Refer Time: 09:27).

Very good; so H of z would be 1 plus C 1 z and now you can see the answer for yourself all you have to do is multiply sigma square e with H of z inverse and also H of z. So, take the product of all the 3, what do you get for the auto covariance generating function? You get sigma square e times, what you obtain here? One plus c one square plus.

Student: C 1 z.

Plus C 1 z plus C 1 z inverse right from this can I now straight away write the auto covariance function yes and what would be that? So, this implies that sigma of l for this process is remember the coefficients of this polynomial will give me the auto covariance at those respective lags right coefficients of z to the minus l will give me; before you proceed you should make sure that the coefficients of z to the minus l and z to the l are the same because by definition sigma is symmetric and if you have done your calculations correctly then you should get we do have that situation here. So, straight away one could write this as the auto covariance as, the answer that we wrote down yesterday, do not forget this sigma square e that is a very important part of your answer and zero otherwise.

So is a very simple way of arriving at the auto covariance for an MA process.

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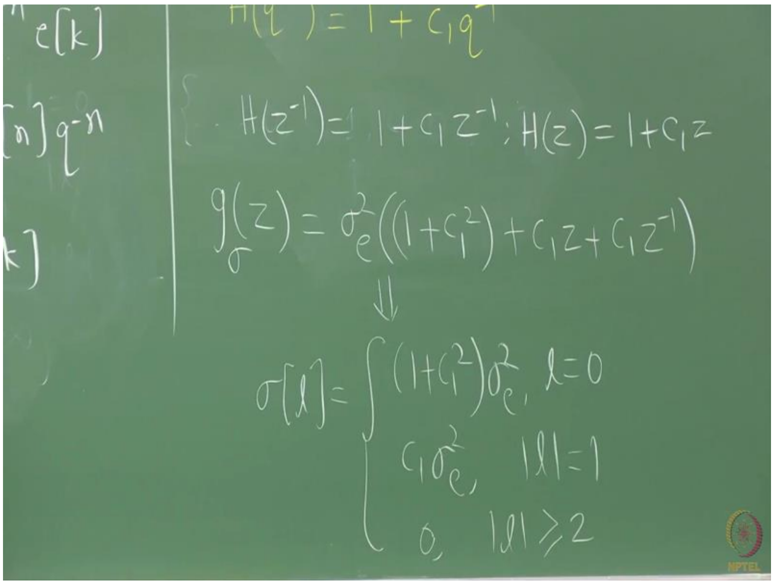


Can we use this for auto regressive process? Yes will see that shortly. So any question on this is clear? Now you can use this auto covariance generating function to see straight away that an MA M process, would give you non zero coefficient up to lag m and then 0 thereafter right, when you multiply H of z inverse with H of z for an MA M process, you would have powers up to z to the minus m is that clear or not. So, fairly easy proof to show that, the ACVF of an MA M process abruptly falls to 0 after lag m.

Hopefully now it is kind of ingredient I knew that the moving average process of order m has a certain signature. So, can we go further? Very good now of course, there is this example of ACVF of MA 2 process just go over it and just skipping this slide it is a same procedure, but let us move forward to I illustrated for MA 1 on the board on this slides you have a MA 2, let us move on to now a totally different class of processes known as a auto regressive process. This auto regressive process is something for an equation for which we wrote on the board yesterday of order 2; as a name suggest this auto regressive class of processes evolve as a linear function of the past, what is a difference between this and the moving average process? In the moving average process the process is evolving as a linear function of the shock waves, this was (Refer Time: 13:20) idea right that was his school of thinking or his view points, where as linear approach was more based on this, but in the end we will see that they are duals of each other, the auto regressive process and the moving average process is actually there is an equivalence between them there is a very strong equivalence between them and you will find also very strong parallels of these models in the linear system theory for the deterministic world.

We wrote down the moving average process as a special case of the general linear random process right, what is a general linear random process? It is this, the auto regressive process is also special case of this you may not be able to see this now, but when we talk of the linear random processes in general that is moving average, AR models and ARIMA models and so on, at that time I will show you 2 things: one that the auto regressive process is also a special case of the general linear random process and two the equivalence between AR and MA process. For now think of it this way that their exists another class of processes known as the auto regressive processes, which are basically evolving as a linear function of their past at least because you are living at the linear world, so simple AR 1 would have this generating equation that is shown on the screen v k as minus d 1, v k minus 1 plus e k and so on.

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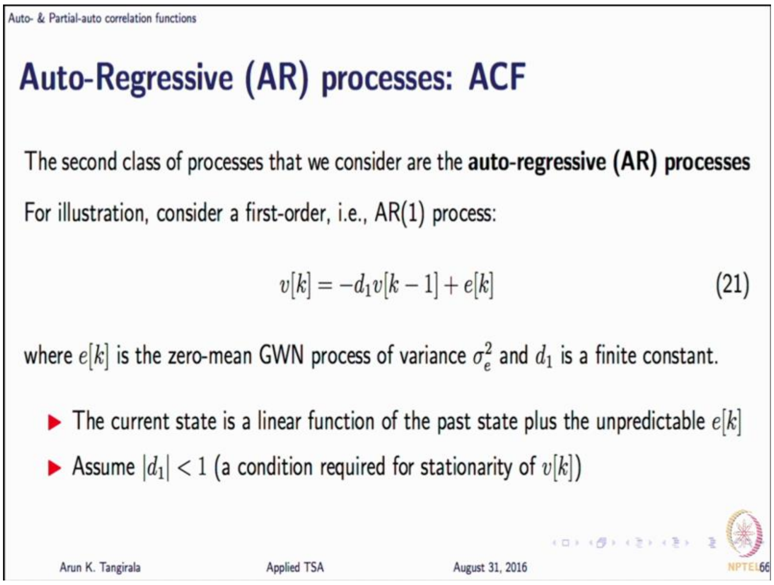


So, essentially the current state is a linear function of the past, but now one as to be careful somebody ask a question whether we can actually have any value of C 1 and guarantee stationarity, likewise here when we look at AR 1 processes or any AR processes we have to be careful in the value of the coefficients that we choose so that ultimately I get a stationary process.

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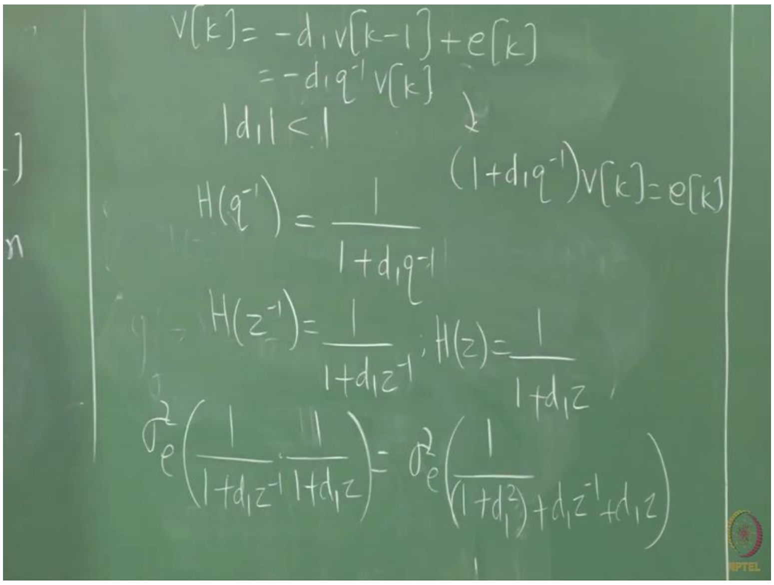


For AR 1 it is fairly straight forward to see that for this generating equation to generate a stationary process one has to have d one less than one in magnitude; we will go we will we will go or a more formal result which will tells us what is a general condition for an auto regressive process of order p so as to guarantee stationarity. General conditions on the coefficients; for now it is fairly straight forward to see if d 1 is greater than one in magnitude, what would happen? V k will just blow up right and that will spoil the stationarity, for now we will assume therefore, that d 1 is less than one in magnitude.

Now, I want you to compute the ACF of this AR process using the auto covariance generating function; idea you can apply the auto covariance generating function even to this process first what is a first step?

Student: (Refer Time: 16:28).

Right H of q inverse, what is H of q inverse for this processes any idea?

Student :( Refer Time: 16:35).

So, to see that your answer is of course, 1 over 1 plus d 1 q inverse, but how does one get that again apply the shift operator here, rewrite this as minus d 1 q inverse v k that the first baby step or the intermediate step before you arrive at this answer. So, that you see, but this can be written this entire equation can be written as 1 plus d 1 q inverse operating on v k, to produce here e k; it is a you can see a duality there we had H of q inverse purely as a numerator polynomial, now you have as a radical function is only the denominator polynomial right it is a polynomial in the operator it is not polynomial in the variable, that is your H of q inverse therefore, H of z inverse would be 1 over one plus d 1 z inverse and H of z would be 1 over 1 plus d 1 z, any questions until this point?

So, now what is auto covariance generating function? Make use of the fact that remember that d 1 is less than one in magnitude you should make use of that and then may be some standard you know series inversion method that you have to use to compute the auto covariance generating, do you have the answer? There are there is another way of deriving the auto covariance for an auto regressive process, I will talk about a tomorrow in the tomorrow class, but now let us quickly use auto covariance generating function and arrive at the answer.

So, how do you calculate here sigma square e times 1 over 1 plus d 1 z inverse times 1 over 1 plus d 1 z, what you get? Very simple polynomial at least at this stage, sigma square e times 1 over, what you get.

Student: (Refer Time: 19:22).

1 plus good; 1 plus d 1 square that is it.

Student: Plus d 1.

Now, this a very painful thing to do right, what you can do is all we need is to get let us say if I ask you for the ACF values are different lags, there is another way one way is to do a long division of this or we can do a long division of the individual polynomial, that is write this down as an infinite series and what would be that?

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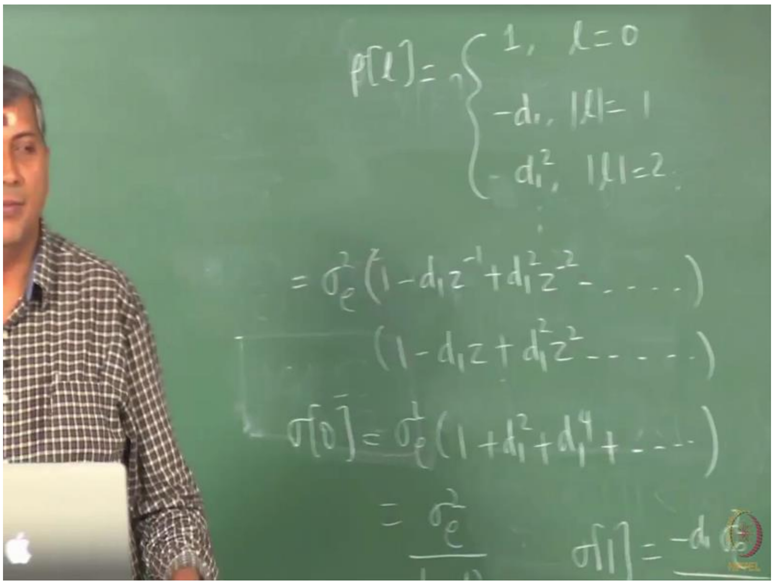

1 minus d 1 z inverse plus d 1 square z inverse square and so on minus d one cube z to t he minus 3 and so on times; what do you get here? 1 minus d 1 z, minus d 1 square z square sorry plus d 1 square, z square and so on, is that clear?

Now, what is auto covariance at lag zero?

Student: (Refer Time: 20:36).

At lag zero the auto covariance at lag zero is simple the coefficients of z to the zero and what would be that sigma square e times 1.

Student: (Refer Time: 20:54).

Plus d 1 square plus d 1 raise to 4 and so on right, now does that infinite sum up to given that d 1 is less than one in magnitude.

Student: (Refer Time: 21:08).

One over very good; 1 over 1 minus d 1 square and that is the variance of the process; we will show that this is the answer for the variance by different method tomorrow by the sigma square e or 1 minus d 1 square, what would be the ACF at lag 1? Any guesses ACVF at lag 1. At lag 1 you would have sigma 1 would be minus d 1 again using the conditions that you have here sigma square again you get same sigma square e 1 minus d 1 square, what about at lag 2? D 1 square times sigma square e or 1 minus d 1 square and so on. So, straight away one can write here as and will just take couple of minutes more and we will adjourn. So, we have here will write the auto correlation based on this of course, 1 at lag 0 that goes without saying, it is by definition; at lag 1 what do I have? Minus d 1 very good what about at lag 2?

Student: (Refer Time: 22:32).

D 1 square right and in general does is go to 0 at some point?

Student: (Refer Time: 22:42).

Only asymptotically right at no finite lag does this ACF go to 0. So, in general we can say for an AR 1 the generic solution for an AR 1 is rho l is minus d 1 raise to mod l for all lags that takes care of lag zero also.

Student: It should be auto covariance.

Which one?

Student: A correlation.

Where here no this is auto covariance right; this is autocorrelation, you in order to complete this solution you will have to give the variance so that you can always construct the auto covariance sequence. In other words unlike the moving average process the ACF of an auto regressive process although we have done this only for first order if this is a general result, the ACF dies down only exponentially, does it die down? Because d 1 is less than 1 in magnitude, ACF as to die down, how it dies down? It may go depending on the value of d 1, it may go through in a swinging mode or it may decay in a monotonical exponential decay, not monotonical necessary always, but exponential decay unlike MA, which are the shark cut off. So, what this tells us is I can find out whether the underlining process as an auto regressive characteristics, but unfortunately I cannot determine the order.

So, tomorrow what will do is, step one we will actually first thing we will do is we will go through an alternative method will where we will set up the Yule walker equations to figure out and to arrive at the theoretical ACF and also look at why we are unable to figure out the order and use the partial autocorrelation function to figure out the order.