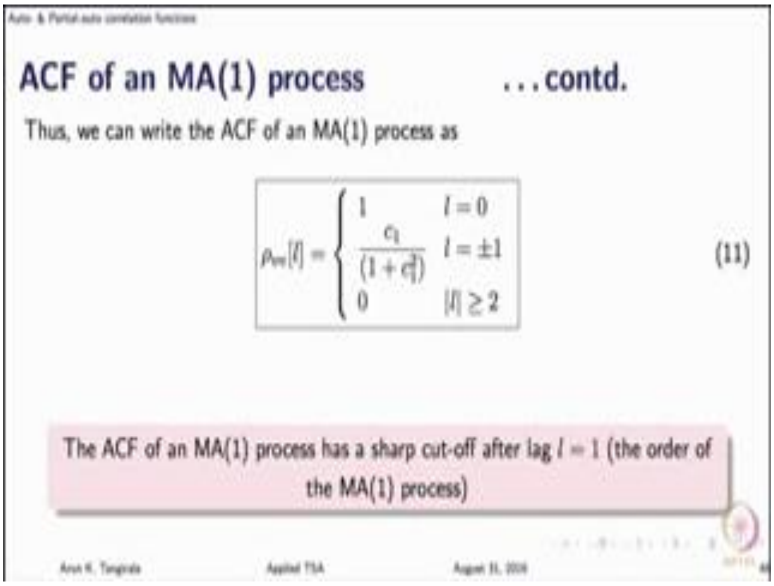
Applied Time-Series Analysis
Prof. Arun K. Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 32
Lecture 14A - Autocovariance & Autocorrelation Functions-7

Discussing the theoretical autocorrelation function of an MA one process and also in general MA processes. We will talk about the MA process in general later on, when we talk of linear random processes, but let us proceed with the further analysis of the autocorrelation function.

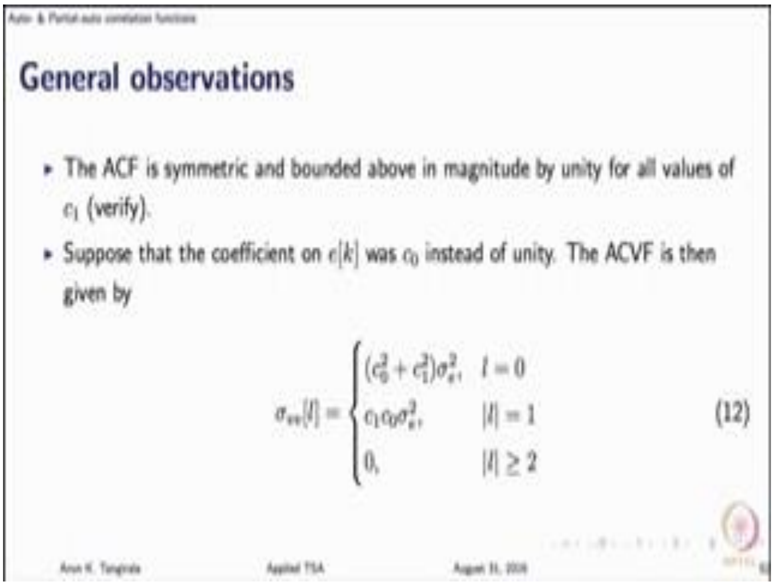
(Refer Slide Time: 00:40)



The first point to keep in mind is that the ACF is symmetric and bounded in magnitude that is when you derive the theoretical ACF, it is important to check first of all that you get symmetricity which is ok, we have assumed and we have not calculated, but sometimes it is good to check and most importantly that your ACF is bounded above by 1.

So, if you look at the theoretical ACF; at lag 1 you see that it is c 1 over 1 plus c 1 square and you should ask if this value is always less than 1 regardless of the value of c 1 and it should be, right. So, that is a quick check that you have at least a qualitative check that you have the right answer; it is not a full check.

(Refer Slide Time: 01:38)



(Refer Slide Time: 01:42)

So, let us get now going further and the other question that we want to ask is remember we assumed when we wrote the equation for MA 1, we assumed that the coefficient on e k is 1, but we could have assumed nonzero coefficient. I said that we will explain later on, we will understand later on as to why we have assumed the coefficient on e k to be unity, but suppose this was c naught it was not unity; some non unity value then the auto covariance would have been what you see on the screen and if you look at this result and really ask what happens in practice.

In practice what would happen is from a modeling view point, you have essentially two equations; what we mean by modeling view point is in practice I am given the series and I am given the series I construct the ACF or estimate the ACF and using these equations let us say, I sit down to estimate the model coefficients. Now in any time series modeling process, it is not just about estimating the model parameters, but it is also about estimating sigma square e; remember that sigma square e is unknown.

So, whenever you say that you are estimating the time series model; it just does not mean that you are estimating the model parameters; it is incomplete you have to estimate sigma square e as well. Remember e k is fictitious right; all we know is that e k has right now it is characteristics; we do not know anything about it is variance. So, from a modeling view point the unknowns are c naught, c 1 and sigma square e. So, these are the three unknowns that we have to estimate from the given series.

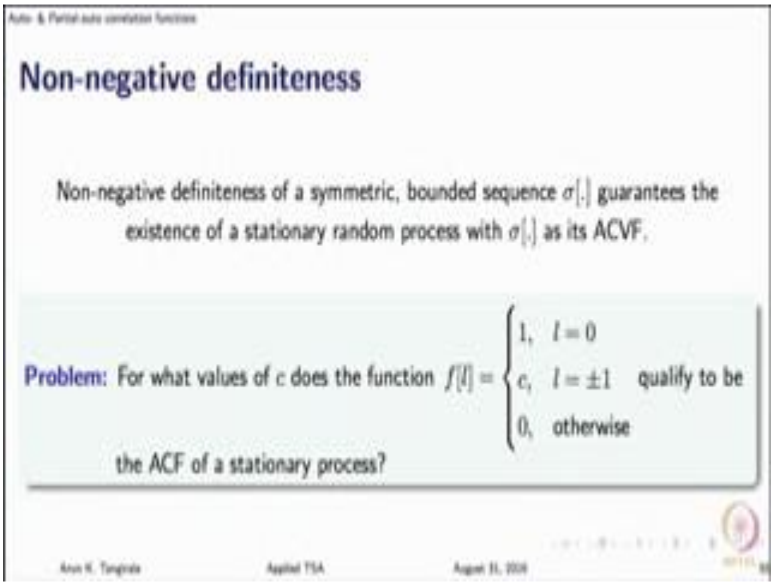
So, I estimate the ACVF and ask how many equations I have in order to estimate these three unknowns. How many equations do you think we have; what we mean by equations is; the equations that relate the ACVF to the modeling parameters - two equations and we have three unknowns. So, that should give you a hint as to why we have fixed one of the coefficients to some preset value, the pre-set value is unity of course, you may ask why c naught why not c 1 and so on, but at least it explains to you partly by we have fixed one of the coefficients to unity.

There is a stronger reason, but this is to give you preview of why we have fixed c naught to unity. By fixing c naught to unity then we have at least two equations two unknowns we do not run into an under determined problem and we have more unknowns then knowns then you have multiple solutions and among this multiple solutions, we pick the 1 where c naught is 1, you could fix c naught as two it does not matter, but remember if I fix c naught as 2; I can still fall back to the option of c naught being 1 because e k is fictitious, I can always adjust its amplitude or its variance, so that c naught is always 1 right. So that should hopefully give you some idea; we will come back to this discussion later on again when we talk about time series models in general. So, the bottom line is we fix one of the coefficients in the model; particularly the one on e k, so as to guarantee at least uniqueness or identifiability.

Now the other thing that we should remember when we estimate models from ACFs; what we are doing is we are trying to understand what signature each class of models leads behind in terms of ACF. So, that in practice from the ACF we can guess the model order and also possibly estimate the model parameters.

Now when doing so; obviously, one of the basis that you estimate the ACF and then use this results to estimate the model parameters and in that process you have to guarantee that the estimate of the ACF that you have constructed satisfies or has all the theoretical properties that a theoretical ACF has for a stationary processes and what are those properties? One is symmetricity and other is non-negative definiteness.

(Refer Slide Time: 06:31)



So here is a problem, I am given a symmetric sequence right and I claim that it is ACF of a stationary process. Now the question is whether for any value of c, does it qualify to be the ACF of a stationary process. This is the general situation that we will encounter in reality, I am given some estimates of ACF and I want to build a model from these estimates, but I have to be guaranteed first that those estimates do have the required properties; which is that of symmetricity and non-negative definiteness.

Symmetricity is not such a major issue, you can easily guarantee that by a construction, but non-negative definiteness requires special attention. So, the question is now given this function which is obviously, symmetric you can just figure that out by visual inspection. Does this qualify to be the ACF of a stationery process for all values of c?

Now for this particular problem, there is a special way of figuring that out; when you look at this function, suppose you accept the hypothesis or the postulate that yes it does qualified to be the ACF then and MA 1 process is the suitable model; it is does not mean that is the only process, but the MA 1 process definitely has this signature.

So, we already know from our theoretical analysis that for an MA 1 process given here; we know that rho as these theoretical values; we already know this. So, what we want to figure out is given any value that is for any value of c in your function, if you do not like c you can replace it with alpha or whatever, can we come up with the model which has real value coefficients remember a c 1 is real value we are only considering real valued stochastic processes.

Student: (Refer Time: 08:54).

C less than half let us see if somebody else as say answer, so one of the answers is no that; that means, no that not all values of c render this function to be the ACF of a stationery process only c less than half and magnetite or just less than half; how do you figure the doubt? Just by equating, right.

(Refer Slide Time: 09:25)

So, what would happen is c 1 plus over 1 plus c 1 square is c as per the given function, if this function were to be the ACF and then you have to guarantee that c 1 is real value. So, simple you know solution to a quadratic equation, you require that the discriminant e greater than 0 right. So, what is the answer any other answer or is that the only answer? Requirement is c 1 should belong to r; the set of real valued numbers, what happened, is it, what is the answer?

Student: (Refer Time: 10:39).

C less than half in magnitude or?

Student: (Refer Time: 10:42).

It cannot be minus half, what happens if it is minus half what is the value of c 1? You get a complex value now; c 1 is that the case. You are not giving me with the tone of assurance; so what is the answer is c less than half in magnitude or just c less than half any answer from the other hall.

Student: Magnitude.

Magnitude; sure.

Student: Except 0.

Except 0 good; why cannot it be 0? You cannot have a white noise process, we just said c 1 should belong to it should be finite and it should belong to the real set; is 0 not include in the real number set when did we remove that. So, why should not c be 0? Can c be 0?

Student: (Refer Time: 11:58).

That is ok; is it is an every you know white noise process is an MA m process with all the coefficients being 0, perfectly there is nothing wrong with that; you can think of it that way in principle that does not violet any of the assumptions am I right or wrong; it does not matter. All the question asked is can it be the ACF of a stationery process right white noise is a stationery process, so c can be 0; so you can have minus half.

You are still doubt full anybody from the other hall, how do we arrive at the answer by the way.

Student: (Refer Time: 12:54).

You write the quadratic equation. So, what is the equation that you have? 1 plus 1 over c times c 1 right; sorry minus plus c 1 square equal 0. Let us now keep the; so is the answer confirm? What is the answer final answer?

Student: (Refer Time: 13:36).

Mod c 1 less than half, so that is the answer.

Student: c less than half.

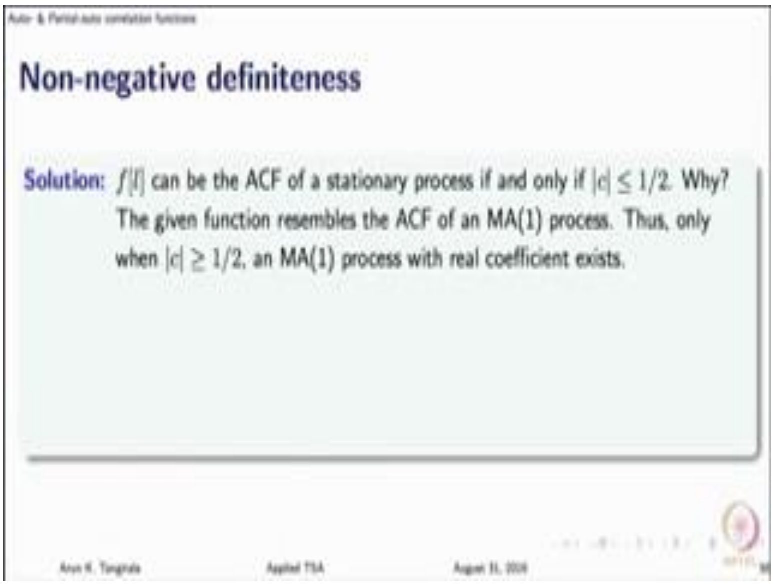
Sorry, c less than half I am sorry correct.

Student: (Refer Time: 13:45).

Less than or equal to half very good, so let us see the fortunately there is an alternatively way of checking this answer which we will learn very soon and which will also be useful in solving the assignment question that I have given . So, what we know for sure is all MA 1 processes, any MA 1 process can never produce an ACF at lag 1 greater than 0.5 in magnitude right and of course, it is going to be 0 afterwards good. So, these are called admissible values of the you can say ACF sometimes given the ACF, you have to figure out what are the admissible values of the constants and so on. But typically one constructs the allowable values of ACF regions; so, you say that is the admissible values.

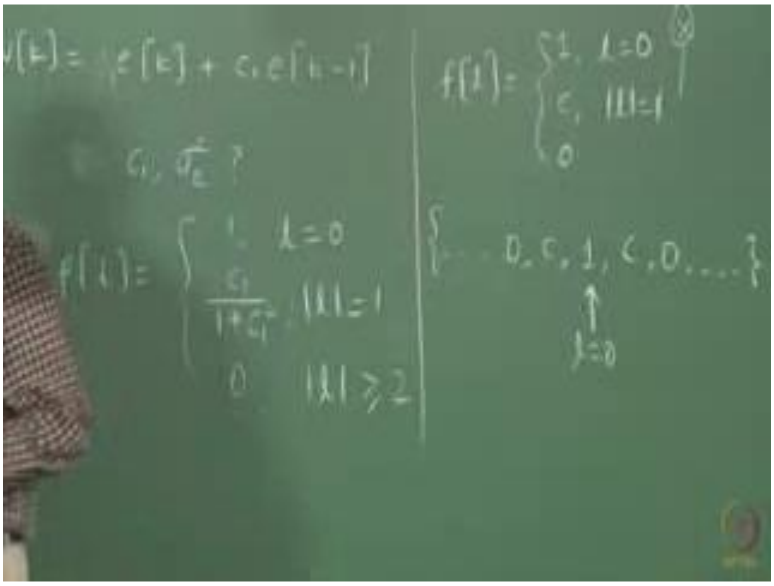
So, let us move on and ask are they alternative ways of checking this answer. Remember that this condition that we are actually deriving is nothing, but implicitly the condition of non-negative definiteness.

(Refer Slide Time: 14:59)



And you can show that; if c is greater than half; it should not be greater than or equal to there is a small error I will correct that on the slides; that when c is greater than half, it should be less than or equal to half sorry that is the mistake there.

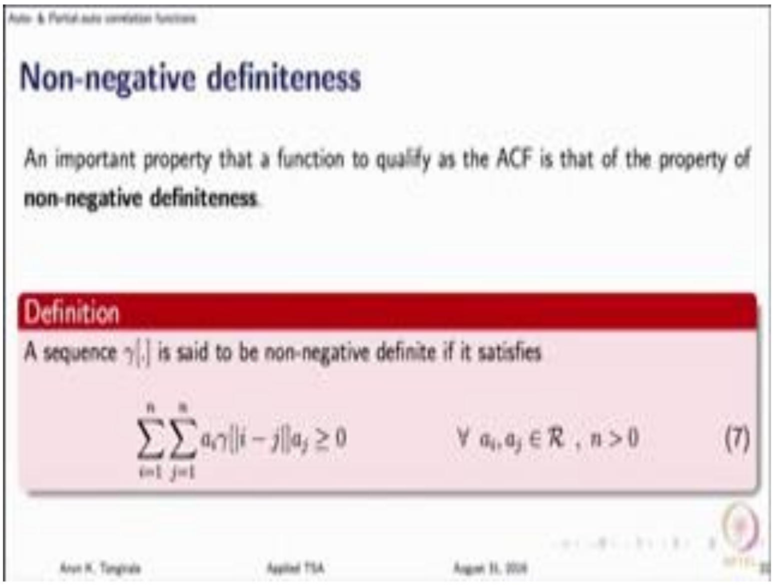
(Refer Slide Time: 15:25)



Now, what happens is if c is greater than half, it turns out that the ACF sequence that comes out that is what you mean by ACF is remember now, we are given f of l as 1, c and 0 this is set lags 1, set lag 0. So, if I consider this to be the ACF then I have this infinitely long sequence sorry. So, I have c and then 1 and then c here is a lag 0 and 0 thereafter. So, this infinitely long sequence it turns out is not non-negative definite if c is greater than half.

How do we prove that? In this special case you can show by hand and this is the problem that there is solved in the book by Brockwell and Davis and I am just giving you the solutions straight away from that book, but you can show that they there exists a bunch of coefficients. Remember the definition of non-negative definiteness will have to always guarantee that for any set of coefficients the summation that we had; sigma a transpose the sigma l again a for all values of n should be greater than or equal to 0, for all values of those constants let me actually pull up that definition.

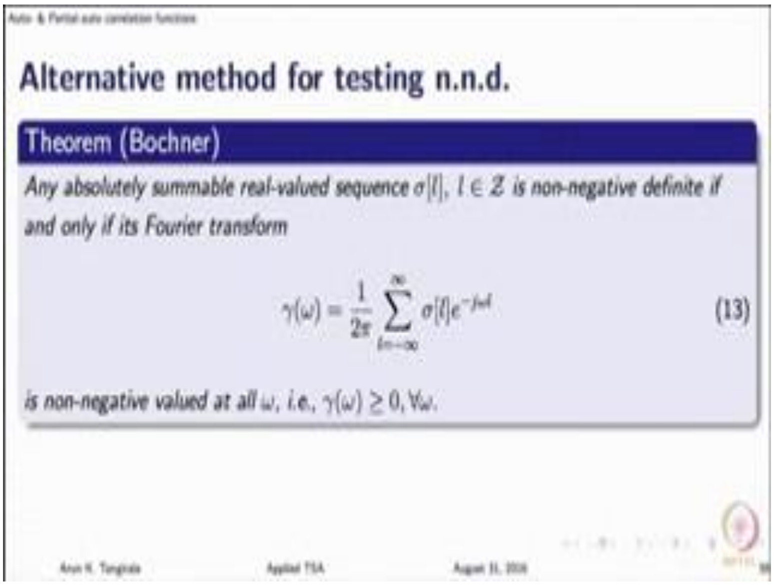
(Refer Slide Time: 16:48)



So, remember recall this definition that we had from non-negative definiteness; we said that for all values of these coefficients a I should guarantee that the summation yields a value of greater than or equal to 0 on non-negative number essentially. It turns out that when c is greater than half, you can find 1 set of coefficients that is a bunch of a's that this yields a negative value for some n, that is some of number of terms in this summation.

Now, that is a very hard way of checking for non-negative definiteness in general; here it was easy because we would now we will look at this pattern of f and say that if it were to be the ACF then MA 1 process is likely then I come up with the requirements on c for this function to qualifies as an ACF or for this function to be non-negative definite, in a general situation; it becomes tougher. So, there is fortunately a fantastic way a very simple way of figuring out the non-negative definiteness of a sequence and this is due to a theorem by Bochner in function analysis. It is a very powerful theorem because it makes its simplifications throughout in time series modeling and we will see a grander appearance of that result later on in spectral analysis. But for now, I will just give you the theorem under some conditions and the theorem says any absolutely summable that absolutely summable means that the sequence should be absolutely convergent; a real value sequence.

(Refer Slide Time: 18:23)



Now, we are not talking of ACVFs this is the result from functional analysis; there is no reference to ACF or ACVF there. This result is used in the theory of random processes, so the result says theorem says that any absolutely summable real valued sequence is non-negative definite; if and only if, so it is a necessary and sufficient condition, if its Fourier transform. Fourier transform is slowly making its way through and the Fourier transform for your convenience has been given the definition and it is denoted by gamma; gamma of omega is the discrete time Fourier transform you can say discrete indexed Fourier transform of the sequenced sigma is non-negative valued at all omega right. By some accident you must have come across Fourier transforms or accidently may be remember it, so this expression should recall such a familiarity should reinforce such familiarities.

So, you have gamma of omega is 1 over 2 pi sigma, the summation of sigma l e to the minus g omega l; the summation runs all the way from minus infinity to infinity. Why is this absolute summability requirement; any guesses? At the moment do not think of sigma as auto covariance or anything it is just a sequence.

Student: (Refer Time: 20:05).

Sorry.

Student: (Refer Time: 20:09).

I do not know. So, I do not understand what you are saying.

Student: (Refer Time: 20:15) at omega equal to 0.

At omega equal to 0 gamma of omega should be finite; what is that got to do with the absolute summability.

Student: (Refer Time: 20:24).

No that is sum of this absolute sum. So, what we have been by absolute summability is that the sequence mode of sigma l should be convergent, this is what is absolute summability. You have learnt convergence of series long ago right, there are absolutely convergent series. So, this sigma should actually constitute belong to the absolute convergence series class, why is this requirement necessary any idea?

Student: (Refer Time: 21:00).

Correct very good, for the convergence of the Fourier transform, this is not a Fourier series be very careful this is a Fourier transform. It is essential that sigma should be absolutely convergent and we will revisit that requirement when we explicitly talk of a Fourier transforms. For now you should remember that this absolute summability is essential for gamma of omega to exist; to exists meaning to have a finite value nice.

And the other thing that you should remember that I have not specified here gamma of omega to be real value; you can add that it is important that the gamma should be real valued or you add the statement that sigma should be symmetric, it is 1 on 1 relation if gamma has to be real valued, the sequence has to be symmetric. I hope you see that; it is very straight forward to see that only symmetric sequences will give you a real value gamma. So, either you say that gamma should be non-negative real valued, it kind of understood because we do not talk of non-negative complex valued it does not make sense. So, it is implicitly understood that sigma is symmetric here.

Now, how do we use this result? By the way before you use this result, you should remember that it is sufficient to check for non-negative value. So, we are talking of non-negative definiteness of sigma and we are translating that to non-negative valuedness of gamma at each omega; they are two different things. On one hand, we are talking about an entire sequence and its non-negative definiteness. Remember non-negative definiteness of sequence does not necessarily mean that at each index; the sequence should be non-negative value that is not necessary.

Now, we are translating that to a requirement on gamma that it should be non-negative valued at every omega; which is the much easier thing to check. All you have say is gamma of omega should be greater than 0. So, now how do we use this result; all you have to do is we have this function here and if someone claims that this is the ACF or alternatively you want to find the constrains on c such that it qualifies to be the ACF; remember ACF has to be symmetric non-negative definite then all. That means, is I take the Fourier transform of this sequence and construct gamma of omega and impose a condition that gamma of omega should be greater than or equal to 0 at all omega. So, can you do that and now come up with the answer; do you get the same answer as we have obtained before with the other method. What is gamma of omega for this example? Yes.

Student: Sir what is the meaning of non-negative (Refer Time: 24:20).

The physical meaning of that has got to do with our ability to basically conceive a stationary process that is; it is very hard to actually pin point to the value of the ACF, the physical meaning is best understood when we move to the spectral domain at this moment it is not so clear. We will show you later on that this gamma that we are referring to in a general sense when adopted in the stationary processes assumes interpretation of a spectral density. Since you ask this question, I just giving you the answer, but we will prove that formally later on, will go through some formal results later on.

The non-negative definiteness of ACF translates to a very important requirement that the power spectral density of the stationary process should be non-negative value; that means, I should be able to construct for this process for which I am claiming some sequence to be ACF, a density function that is non-negative value. If the ACF or ACVF does not satisfies their property, we cannot construct a process that will have positive or non-negative value density function; by definition density functions are always non-negative. So, physically what this mean says that the frequency if you imagine that there are some frequencies contributing to the overall power of the process; their contributions should not be negative right, you cannot think of something contributing negatively to the power, you can only think of contributing 0 or greater than 0.

So that is the physical meaning; it is starts off with a mathematical requirement, but it is slowly translates to that requirement. So, our ability to find a stationary process with positive valued or non-negative valued power spectral densities. So, it has got to do with our ability to decompose or think of what free what is the contribution of each frequency component to the overall power. In other words our ability to find a realistic process, so what is the answer; do you get the same answer what is gamma of omega for this example.

Student: (Refer Time: 26:38).

2 c.

Student: (Refer Time: 26:44).

So is it 2 c cos omega or 1 plus.

Student: (Refer Time: 26:48).

Right, so that should give you the answer; right for gamma of omega to be greater than 0; obviously, then c has to be less than or equal to half in magnitude. Now obviously, you should recognize that this is the much easier way of arriving at non-negative definiteness rather than breaking our heads and figuring out the existence of a sequence in that summation that will violate the conditions. So, all test for non-negative definiteness of sequences can be translated to a requirement on this, but there are this restrictions that the sequence should be absolutely summable which means there can exists non-negative definite sequences that are symmetric but not necessarily absolutely summable and that can also qualified to be the ACF of a stationary process.

Remember the requirement that we have placed through this theorem is that whatever function is being tested should be absolutely summable, without performing the test you should not use this result. You may ask does there exist a general result for any function that is symmetric and I want to perform a test of non-negative definiteness yes you can, but it is bit early to present that result, but I can tell you that whatever theorem that this is called a Bochner's theorem. There is a more general result and when it was applied and kind of tailored to the random processes field, it became to be known as Wiener Khinchin theorem. We will learn later on in the spectral representations and so on and that is a fundamental result in theory of random processes. We will talk about it later on let us move on.