Applied Time-Series Analysis
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Lecture - 25
Lecture 11B - Introduction to Random Processes-5

The other thing, so we have now kind of discussed the concept of stationarity and non stationarity and so on. Let us say now the process is stationary, because we are going to confine ourselves to stationary processes. Now we address the other aspect or the other challenge in time series analysis; one challenge was to make sure that we are working with processes who statistical properties do not change with time, great. Now we come to more practical aspect where we understand there is only a single realization that we are going to work with. What allows us or what is the frame work within which we can work with a single realization. And ergodicity is that property of a process which allows us to work with a single realization and draw inferences about the ensemble.

Now, ergodicity is not just a term that is encountered in statistics you encountered this term in thermodynamics and mathematics and so on, but will restrict ourselves to statistics or random processes. So, what is ergodicity or when do we say processes. We have talked about it before; let us make a formal statement here.

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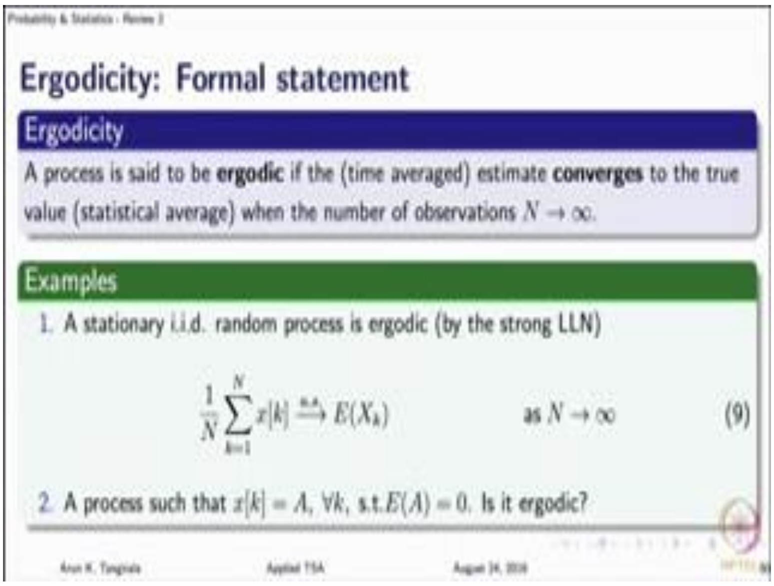

Processes said to be ergodic if the time averaged estimates, whatever you are averaging in time simple average in time that converges to the true average, what you mean by true average? The ensemble; and of course its simplicity understood the processes stationary otherwise it does not make any sense. So, ergodicity is spoken of only for stationary process.

So, this time average should co inside with a true average as the number of observations you take to infinity. For example, if you take a stationary, in fact I should not say stationary; stationary is a redundant word there for IID. An IID process, what is an IID process? Independent and Identically Distributed process; independent talks about the correlation structure of the process in time and identical tells us that each observation falls out of the same probability distribution. Such a process is called IID, it is an idealization. It is an idealization that is not realized. IIT is released, but IID is not realizable. A diluted version of IID processes is white noise and will talk about those distinctions shortly.

So, if I take an IID random process there is no need to say it stationary it is ergodic in the sense of mean for example. So, ergodicity is not a universal kind of property you have to actually look at ergodic in what property. If I were look at the mean, is this IID process ergodic in mean what this means is; if I were to compute the average in time will that average converge to the true mean. Now first of all I am given that this is an IID process which means a mean is going to be constant with time, so that is something that I have to ensure if I were to apply a test of ergodicity or conduct a test of ergodicity. The first thing that I have to make sure is the process is stationary, I am given the process is stationary.

Now, the question is whether this 1 over n sigma xk which is the time average of n observations, will that converge to mean as n goes to infinity. Now it is not so easy show this as much as a just written there.

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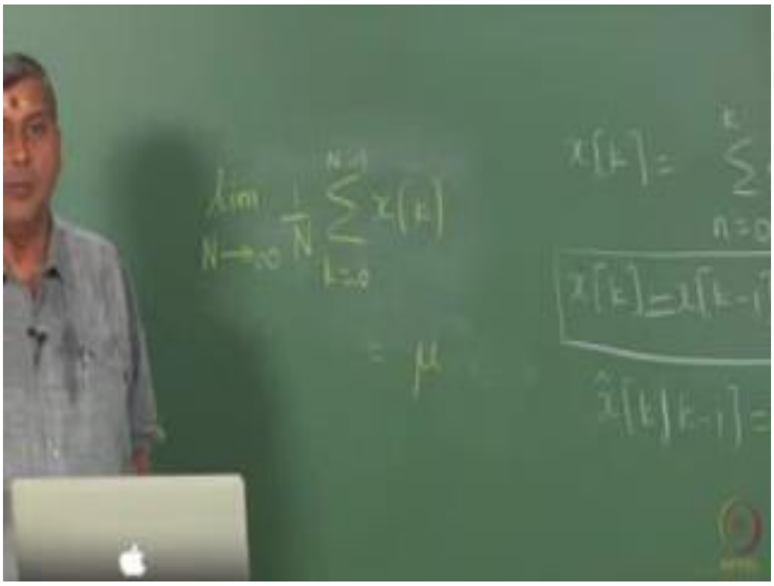

One has to essentially show that in the limit as n goes to infinity. This 1 over n sigma xk or 1 to n does not matter is mu, where mu is the mean of the process. It is not straight forward to show that this is the true. I mean this is of course known by the law of the large numbers now. There are true versions of the law of the large numbers now; there are two versions of the law of the large numbers weak law and strong law, based on the assumptions that you make of the process. But there is something else that I have written there is an abbreviation a dot s dot right, what does that stands for almost sure. Why are we talking about this almost sure and so on? The reason is we are now looking at convergence of random numbers or random variables, xk is not a deterministic variable it is a random variable.

So, I am adding up random variable and I am talking of limits of such random variables. And we are asking if the limit of the summation convergence to mu is mu at deterministic quantity or random one. What do you think?

Student: (Refer Time: 05:22).

Sure, everyone is sure. Yes, so mu is a statistical property and for us it is a fixed one. We are talking if sum of random numbers, the sequence of random numbers when I add up. This should converge to mu. What do you mean by sequence here? I evaluate the summation for n equals 1 then I evaluate the summation for n equals 2. So, call each of the summations as S n. So, I have here S 1, S 2, S 3. And so what is S i? S i is or S n is the summation is average 1 over n sigma xk.

So, as I keep evaluating it says ultimately I should hit mu. We are used to thinking of sequences of deterministic numbers, but not random numbers. Therefore, a separate theory of convergence is required which will not talk about now will talk about the very briefly the theory of convergence of random numbers in estimation theory. In estimation theory will ask the same question; as I increase the sample size will estimates converges to the truth. But, underneath that statements is a (Refer Time: 06:45) box we have to open, because this convergence at we are talking about is not the straight forward convergence that we encounter, is like what you see here the convergence of random number to a fixed one.

How can a random number converged to a fixed one, its strain right it is a contradiction, because S 1 is it a random number or not. It is a random variable right, because it is some of random variables. And so is every element of the sequence how can a sequence of random variables converge to deterministic one since to be somewhat contradictory. But it is possible in some sense, so we will have to define in what sense to be mean convergence. We cannot do it in the plane vanilla law sense that we applied to deterministic numbers.

Almost sure says that there exist some n after which it has reached the truth, but when it reaches we do not know. In finite n it will reached, that is somewhere at there exist a (Refer Time: 07:50), but what is that finite we do not know. Will this converges to 0 in 100 observations 1000 10000? We do not know, but it will converge; that is what is almost sure convergence. There are two other forms of convergence known as the convergence in probability and what is known as a mean square convergence. What you see here in the statement is a strongest form of convergence. Almost sure, is still want you know mathematicians try to be as correct as possible without getting into any legality any legal issues.

So, almost sure is there is a probability, but I am almost sure. In fact, sometimes you will see a replacement for this almost sure as a statement with probability one, still there is a probability under what is this probability one business. We will talk about that later on, right now what you should understand is we are looking at convergence of random numbers to a deterministic quantity and we cannot use a plane definitions of convergence. And getting back to the discussion and ergodicity you can show that this process is ergodic in mean.

Now, there is another process that we can look at and ask if it is ergodic. This process is a peculiar one; as you can see on the screen this process actually holds on to a constant value, how can a constant signal be random? What is random about this signal that I we are looking at case two? What is random about it? Any idea, I said that A falls out of a; A is a random variables such that when I say expectation of A 0 is understand A is a random variable, it is a 0 mean random variable.

That is all needs to be known, we do not worry about the distribution. How can you justify that x is random, when x is constant actually. What is random about it?

Student: (Refer Time: 09:56).

Sorry, right. So, this starting point whatever it began there is randomness about it and then it has start to that random, so that is what impacts to this randomness to the signal. Suppose I have such a random signal is it ergodic, what you think? It still has the flavor of randomness do not think just because its constant one realization is constant, but if I look at the collections they are not the same and in that sense it is random. So, do not get confused here that xk is deterministic signal.

The only the initial there is some randomness about the initial point once that is fixed then condition on that it is you can say it is deterministic. Now the question is whether this process is ergodic? What you think? Yes or no? Any answers from the other hall? You just have to ask what is the time average, and whether that co insides with the true average. What is the true average? Very good, that is easy answer. What about the time average? A, is it ergodic? It is (Refer Time: 11:12). So, this process is not ergodic. So, there are many examples of such processes and many such processes which are not ergodic and we have also talked about ergodicity before.

Ergodicity can be spoiled, can be lost by the way you are measuring. Suppose, I have a sensor bias; for example in process one which is IID the true average is 0, assume that it is 0 average process or you can say it is mu, but I can easily have a biased sensor which introduces bias at every instant in time. What happens is the time average would be a shifted version of the true average and then the process is no longer ergodic. So, sensor bias is can actually spoil the ergodicity property. Therefore, one has to make sure that the instrumentation; the sensors of the measuring devices that you are using are free of biases.

Not only in terms of hardware sensing we have talked about an examples also determining the average maximum number of people; the part to which the maximum number of people visit or theater that the maximum number of people go to and so on. In that case also the survey mechanism, the sampling mechanism that you use should be unbiased one. You cannot pick a person who does not go to park at all and come to a conclusion. So, that will introduce a bias. You have already introduced a bias by selecting such an individual. So, all kinds of sensor biases will spoil the ergodicity property, but of course sensor biases in the hardware sensing easier to detect when you know something a priory about the process and so on.

So, why are we discussing this ergodicity? Because we want to sure that whatever inferences that we draw from single realization wholes for the ensemble; why did we discuss stationarity whatever realization I draw from a single realization holds in future also. So, stationarity allows us to walk ahead in time, march in time with a model that I have built. And ergodicty property allows us to make inferences about the entire processes, that is walk along the ensemble direction. Therefore, we have taken care of things.

Now, are there formal test for ergodicity? Well yes, some but there true complicated for us to learn we will not worry about it we will assume that the process is ergodic; both the process and the measuring mechanism preserves ergodicity property and then keep moving along. When everything else fails then we can re visit the ergodicity property that is all will do, so will keep that at the back of our mind.

So to close this discussion; ergodicity can be given loose interpretation, it is a crude interpretation. If a process is ergodic then it more or less means that given sufficient time you have unraveled all possibilities. But that is a very loose interpretation; do not go by that formal interpretation. It is like I have given sufficient time, I have observed the process for sufficient time and I have seen all possibilities, and therefore I have seen the ensemble kind of thing. And as I said ergodicity is not necessarily a characteristic of the process, but also of the measuring mechanism.

So, that kind of a brings close to the discussion on stationarity and ergodicity. And now assuming that the process is stationary, assuming that the process is ergodic will study the theory of first of all determining whether process is predictable, how do we build models for such processes stationary ergodic processes. We do not keep repeating that the process is ergodic very soon will drop that term. We will only keep talking about stationarity, and that also we may drop for a while because will confine ourselves to stationary processes.