Applied Time-Series Analysis
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Lecture – 112
Lecture 49 - Periodogram as PSD Estimator

Look at the spectral density estimation or even you would say power spectral estimation, until now we have looked at estimation of time domain properties such as mean, covariance, auto covariance function, auto correlation and cross correlation, but as we have learnt, the frequency domain properties are equally useful in time series analysis.

In this lecture, particularly we are going to focus on periodogram which we have learnt earlier, we have introduced periodogram when we discussed Fourier analysis spectral analysis.

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The periodogram arises from the DFT and go on to discuss what are the issues with periodogram, when we use it to estimate the spectrum or the spectral density. In fact, because periodogram is an empirical spectral density if you must recall, for deterministic signals what are the issues that we have when we use periodogram and then we look at some remedies, there is an the major issue with the periodogram spectral leakage and we look at the remedies for mitigating this leakage and finally, talk about periodogram as an estimator of PSD that is the power spectral density for stochastic signals.

If you remember periodogram has been devised primarily for deterministic signals, but suppose I apply this to a finite length realization of a stochastic signal, what are the issues that I run into and of course, there is a sequel to this lecture where we will talk of estimators of power spectral density for stochastic signals where we look at non parametric and parametric estimators. What we will learn towards the end of this lecture, as far as the power spectral density estimation of stochastic signals is concerned is that periodogram is not a good estimator and therefore, the estimators that we shall learn in the next lecture will tell us how to modify this periodogram or make changes in the way we calculate the periodogram so as to obtain consistent estimates of power spectral density.

Let us move on and we will begin with the recap, the first point to recall is that periodic signals have only line spectra.

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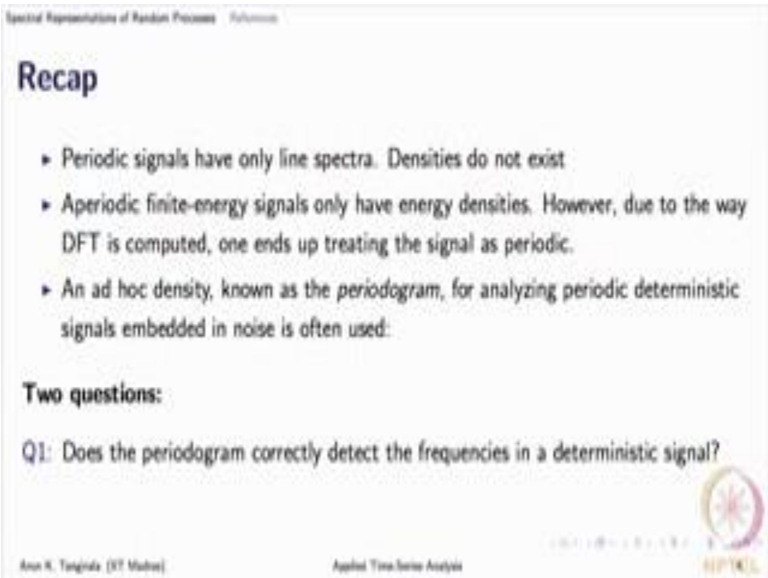

We remember that they do not have any spectral density that is because periodic signals only admit discrete at of frequencies and aperiodic finite energy signals only have energy densities, they do not have power densities. But whenever DFT is computed regardless of the nature of the underlined signal and I think I have emphasized this quite a few times when I spoke about DFT, regardless of the underlined signal DFT assumes that the infinitely long unobserved signal is periodic with the period equal to the observation length of the finite length data that you have.

In other words, DFT incepts treating the signal as periodic, as a result whether the underlined signal is an energy signal or a periodic signal, you are always going to assume whenever you compute DFT that the underlined signal is periodic. Consequently we can only speak of spectrum; power spectrum and not power spectral density; however, what if the underlined signal is actually an energy signal or maybe the data record that I have is realization of a random signal. So, in these situations, using the word power spectrum may not be correct and that is when that is why the concept of periodogram was introduced long ago by Schuster in late 1890s, this is an empherical density or you can say Adhoc density function if you recall, it is power spectrum per unit frequency and that is why it has the feel of a density function.

Now, the 2 questions that we would like to address in this lecture are number 1, does a periodogram correctly detect the frequencies in a deterministic signal? We know that the definition of power spectral density and the notion everything changes the moment you move from the world of deterministic to random signals. So, suppose I have a deterministic signal that is periodic, is periodogram suited, is it well positioned to detect the frequencies? And the second question of course, as we mentioned earlier is how good is a periodogram when I use it to estimate the power spectral density of a stochastic process?

Let us take one by one, we will first, we will address question one first.

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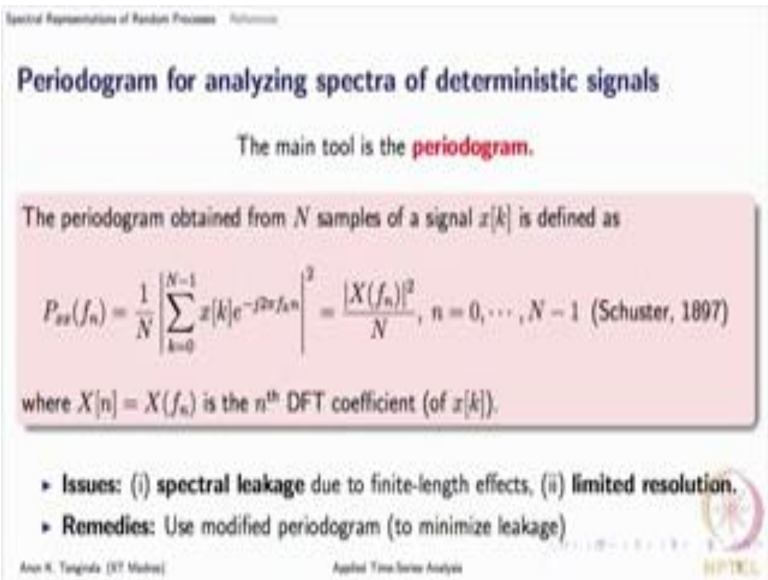

And for this, let us revisit the definition of periodogram. So, if you recall, the definition of periodogram is this expression given here X of f n is a DFT coefficient, n is the number of observations I have in the data record and I compute the periodogram at n frequencies, we know why because we need to compute at least at those n frequencies. So, as to not to lose any information and there are routines in r which compute the periodogram for you, I have mentioned this earlier, I have shown as well, there is a periodogram routine in t s a package or you can use spec dot p gram in the r basic stats package.

Of course you need to know whether, when these routines plot whether they plot on a logarithmic scale and so on, personally I feel that you can simply code this by yourself because it is a very simple expression simply compute the DFT using the FFT algorithm and write the code by yourself that is the best way to handle things note that the periodogram will be computed at n frequencies.

It turns out that when you use this periodogram for detecting frequencies in a deterministic signal and even otherwise whatever frequency content is present in the signal there is an issue that we run into which is known as a spectral leakage. In fact, I have demonstrated this in one of the earlier lectures as well, but we will take a fresh look at this and the other issue with periodogram is that it has limited resolution, what we mean by limited resolution is the frequency spacing is limited by the length of the observations that we have.

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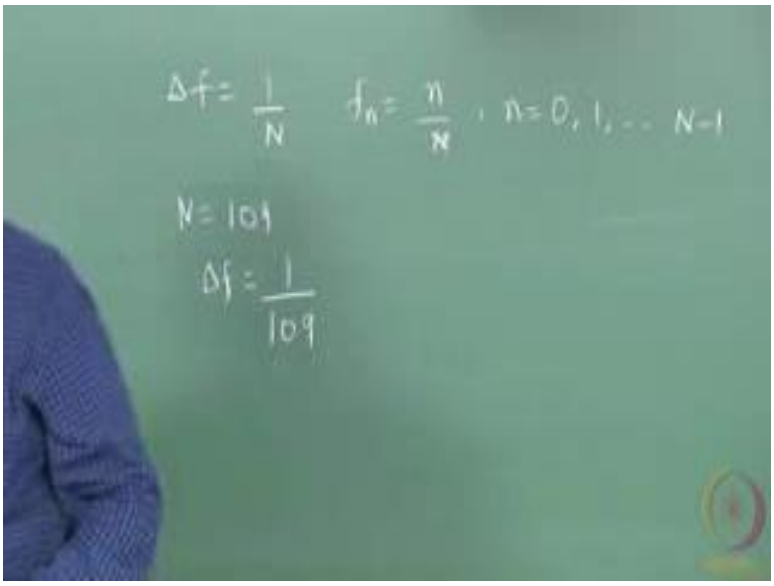

We know that in DFT the frequency spacing in n in cyclic frequency is 1 over N where N is the number of observations. Now of course, that is something that we may not be able to do much about unless I increase the length of the data record. So, in some sense it is an issue that is got to do with the number of data that is available whereas, the first issue has got to do with something else. So, it we will also look at the remedies now associated particularly with the spectral leakage.

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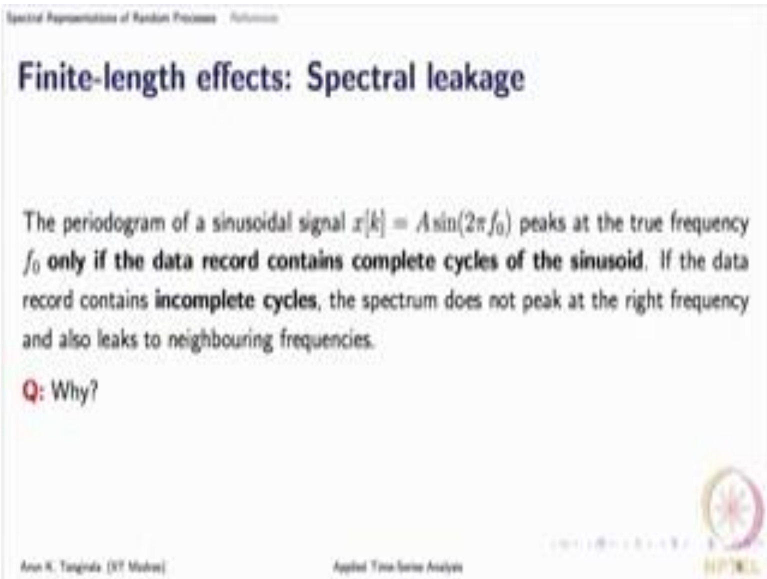

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Let us understand what this issue of spectral leakage is, a bit more closely although we have seen a demonstration earlier. So, the issue is as follows, if I have a sinusoid of frequency f naught there should be A sin 2 pi f naught k here, if there is sin wave of frequency f naught then the periodogram will correctly detect the frequency; that means, it will show a peak at frequency f naught, only if the data record contains complete cycles of the sinusoid, if there are fractional cycles then we are going to see some spurious frequencies appearing in the periodogram. This is the major issue with using periodogram for deterministic signals and the name that is given to this issue is spectral leakage and we will understand why of course, why is this happening is what is important.

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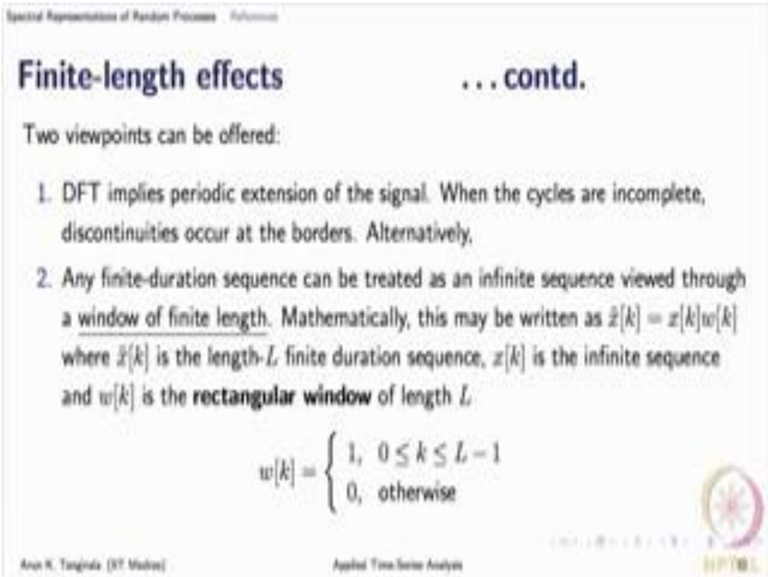

There are 2 viewpoints that are typically offered, one is that DFT implies periodic extension of the signal as we have just discussed. So, when the cycles are incomplete right when I have fraction cycles then discontinuities occur at the border and these discontinuities give rise to spurious frequencies, this is one viewpoint, these viewpoints are extremely useful because only when I have the correct viewpoint. I can find the remedy now there is nothing like 1 correct viewpoint, each viewpoint will offer me a different remedy; it turns out that regardless of the viewpoint the remedy turns out to be the same and that we will talk about shortly.

The second viewpoint is that whenever I observe an infinitely long signal over a finite duration, finite time, then you can, then the spectrum of the finite length signal that I have is a distorted version of the spectrum of the infinitely long signal and that we will prove theoretically. It is fairly easy to say that and this second view point in fact offers the remedy that we are looking for which is so called windowing we have talked about that earlier as well.

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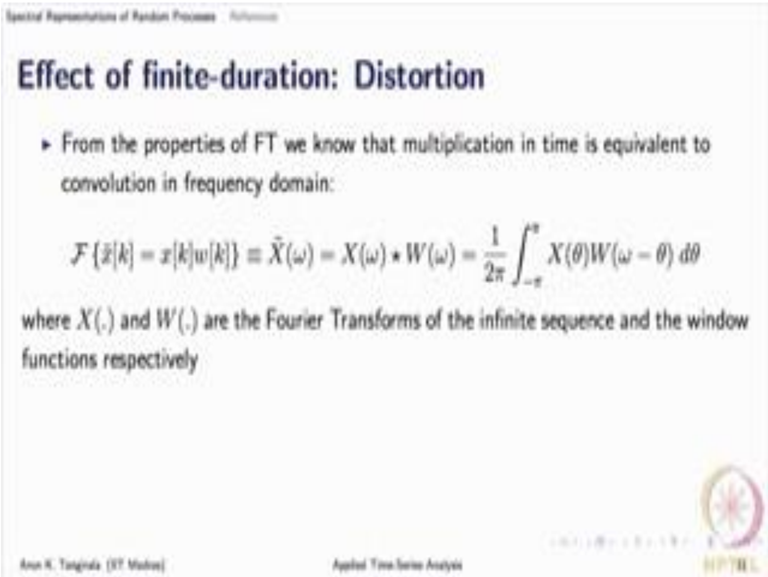

Let us look at the second viewpoint a bit more in detail or maybe I will just go past this, first let me illustrate the spectral leakage and then come back to the theoretical discussion on the second viewpoint.

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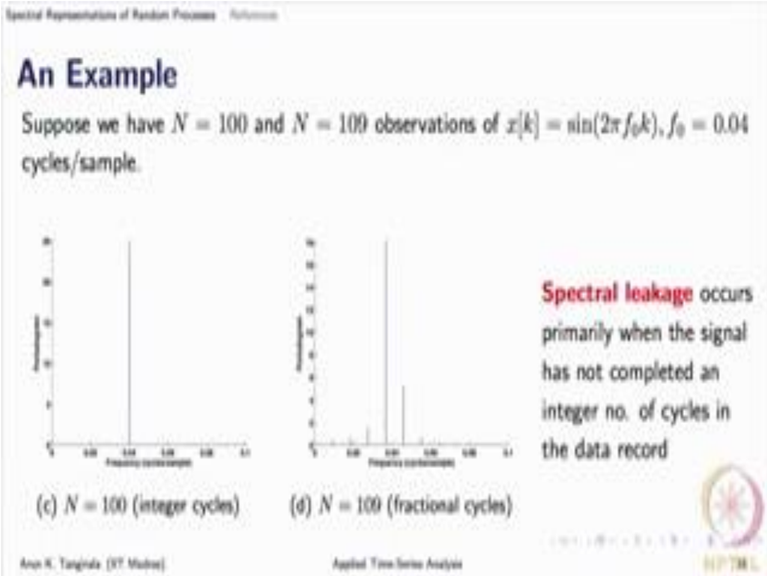

I have here on the screen for you periodogram of 2 different finite length data, but of the same signal, one record has 100 observations the other record has 109 observations and the frequency of this sin wave is 0.04 cycles per sample, on the left we have the periodogram constructed from the 100 observations records and on the right here, I have from the other one. We know that the correct answer for the frequency is 0.04 cycles per sample and you know it is a no brainer to see that the periodogram on the left shows this frequency correctly, whereas, the periodogram on the right which is being constructed from 109 observation long record, does not hit that 0.04 that is issue number 1, not only is that true, but also I have other frequencies being shown up.

See, at least I would have see expected to see a peak close to 0.04, yes, there is 1 peak very close to 0.04, it does not hit 0.04, there are other frequencies, its cousins and brothers and so on which are showing up and these are spurious, they are not present in the sign. So, what is happening here is the power that is localized at 0.04, located at 0.04 has now leaked to neighboring frequencies and that is why this is called as spectral leakage phenomenon.

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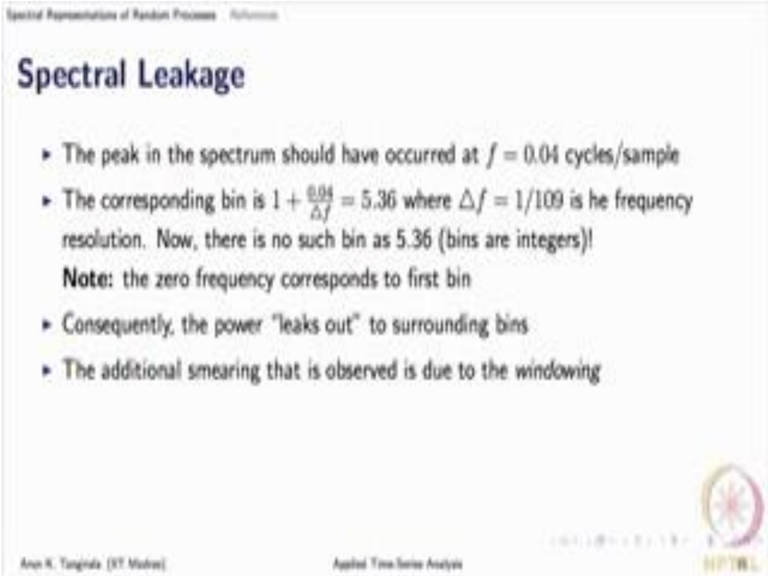

Alright, now it is very easy to understand why this occurs because with the 109 sample record delta, f is 1 over 109, as we have written on the board. So, when N is 109, I have delta f as 1 over 109 and the bin at which the periodogram should ideally peak is 5.36, but there is no 5.36 bin that one, we are adding to account for the 0 frequency, but there is nothing like 5.36. So, what happens is then it gets distributed, when I say 5 point this, this is the bin, remember we are what we mean by bin is that we have f n as n over N, small n over big N where n runs from 0 to N minus 1. So, this n is the bin number what we mean by here 5.36 is ideally, I should have seen a peak at small n equals 5.36, but small n is an integer. So, I will not be able to locate that and therefore, it just gets distributed at 5.64 and so on.

And that is what is actually happening, alright, do not draw the bin number, here for you it is I would leave it to you, go back and reproduce this and draw the bin number on the x axis rather than the frequency itself and then you will understand. So, that is the main issue, I mean this is another viewpoint that many authors gave that look, when I have a signal such that or data record such that the signal has completed only a fractional cycle then the bin sizing is such that the true frequency does not hit any particular bin. So, this is the third viewpoint, you can say whatever maybe the viewpoint, the issue is spectral leakage that is the main problem.

Let us now go back to the theoretical discussion where we said that basically the periodogram of this finite constructed from this finite length signal is a distorted version in general of the spectrum of the full length signal. So, let us look at this now first of all think of x tilde as being your finite length record.

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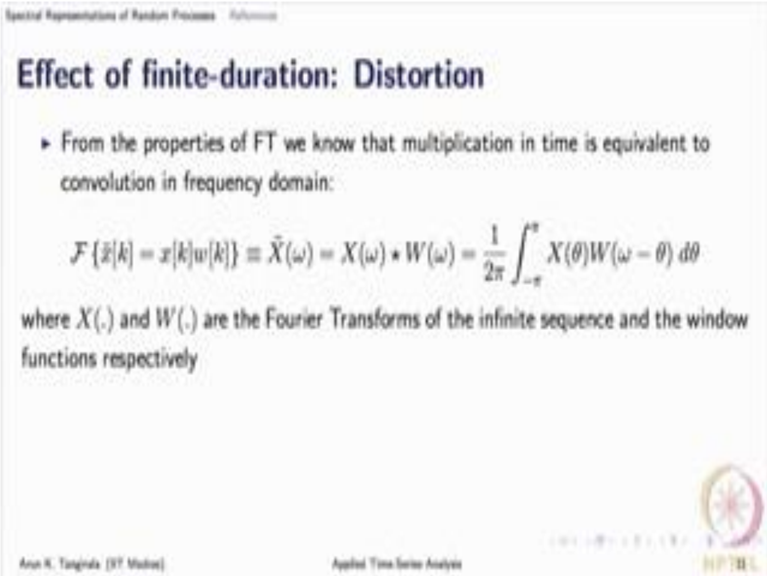

We are going to take a slight change of notation here, assume sorry s k to be the infinitely long signal and x tilde to be your finite length data, then you can express x tilde as s k times w k, what is this w k? It is a window function, how long is this window function, how wide is it? What is the width of this window function? At least over what period is it active? X is the infinitely long signal, x tilde is the finite length signal, correct. So, w is the same length as x tilde, correct, it is of length and outside this n, it is 0.

What I am doing is as I say there is an infinitely long procession going outside my house in our country, it is very common, you know they come with this band, dup dupper is all these sounds and then as a child you are watching through the window although the procession is infinitely long in your view whatever you see is through the window and typically windows are rectangular. So, here also actually what you are doing is you are essentially by looking by analyzing the finite length signal you are looking at the infinitely long signal through a rectangular window of length n. But what is its impact on the periodogram on the spectrum that you are computing for x tilde what happens is we know that product in time domain translates to convolution in frequency domain. So, as a result the fourier transform of x tilde or your DFT it applies to DFT as well is going to be a convolution of x of omega which is a fourier transform of the infinitely long signal which you do not have and the fourier transform of the window which we know it is a rectangular window.

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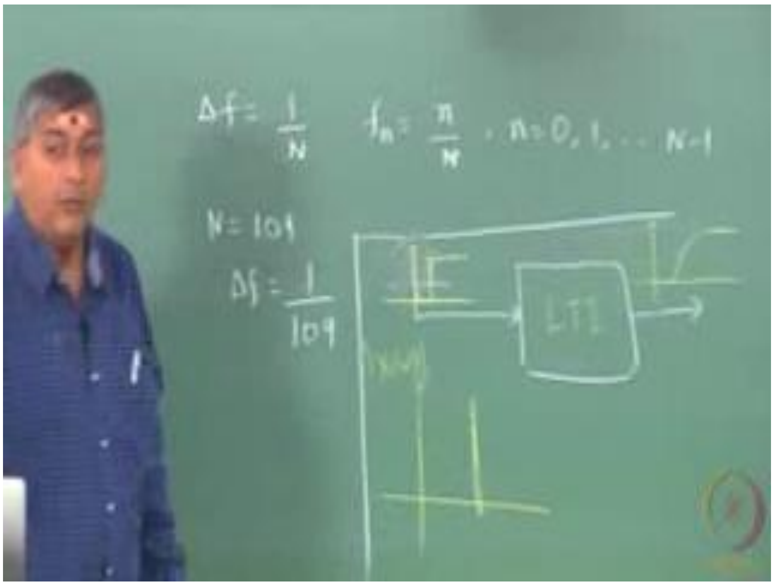

Now convolution can be viewed in 2 different ways, let us say that I have a system. So, let us do this, here I have let us say a signal which is a step being given here. So, I am going to give a step here, let us say at time t naught, this is the input to a system that performs convolution with its own impulse response, we have studied convolution earlier in time series modeling. So, let us say, this is an LTI system we know that LTI systems are governed by convolution operators.

A roughly qualitatively, if I am looking at an over dam system or let us think of a first order system, we know that the output here, if there is assume no delay, it is going to look like this that is the response that I am going to get. So, when I feed in a step or when I excite an LTI system with the step, I am going to get this response and we know LTI systems are governed by convolution operators and we also know by now that this response is nothing but the convolution of this step with the impulse response of this system, correct. So, you can think of the window function being the impulse response right in frequency domain. Now it does not matter where looking at time or frequency I have not labeled the x axis there what is important is convolution is happening there.

What has convolution done to the input here? It has actually distorted this corner and given me somewhat you know the sharp corners have been smoothened out you can say. So, you can say convolution operation is a smoothing operation, but for us convolution operation is a distortion operation, what is happening? Why is it called distortion? Suppose your original signal is a sign wave then how is the spectrum going to look like ideal spectrum? It is going to have a peak, this is how mode of x omega is going to look like, for now x of we will ignore the mode, assume that x is a complex valued signal. So, that x of omega is real. So, it is going to have a single peak if s k is a complex exponential of a single frequency, but the x tilde of omega which is what I am going to compute is not this, it is going to be convolved.

This ideal impulse like thing here is going to be convolved with the window function which with the Fourier transforms the window function, now what is the window that we are using here? Rectangular window; this rectangular window has a Fourier transform that looks like this let me show you then come back to the discussions, sorry.

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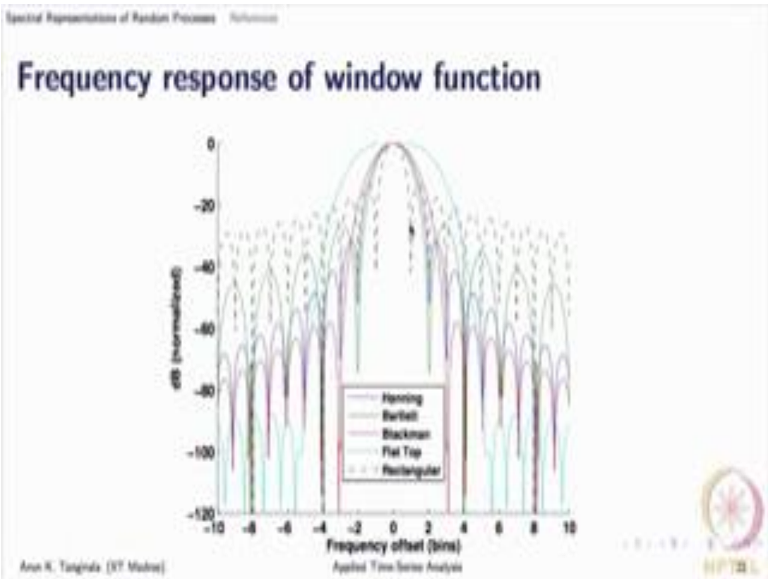

Ignore the rest of the curves, just focus on this black one here, black dash line, alright, there are many curves here, forget the rest, just only focus on the black dashed 1 and you this is how w of omega looks like.

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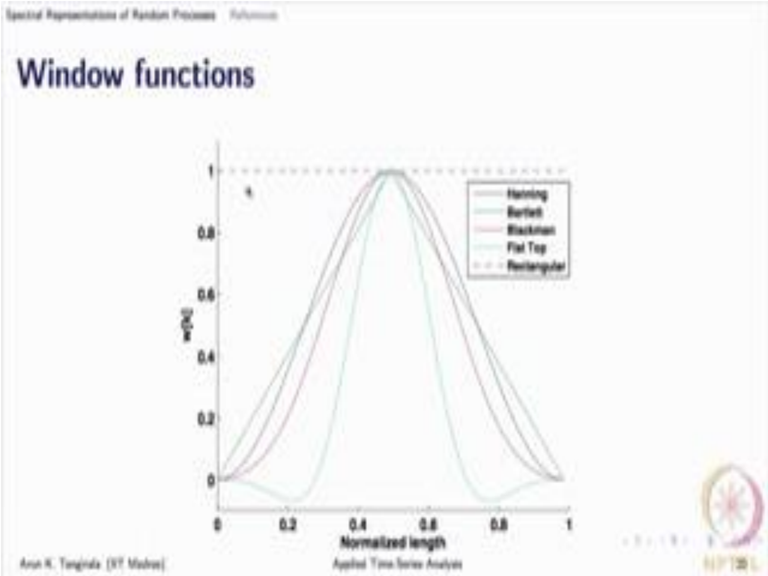

W k looks like this, again the black dash line that is the rectangular window.

What you are doing essentially when you are computing the Fourier transform of x tilde? You are actually convolving this single peak or this impulse like shape x of omega with a function like this. So, what is going to happen when you convolve these 2? This single 1 is going to spread out to neighboring frequencies, now it turns out that it will not spread out if the number of observations is equal to the, is exactly integer multiple of the period of the signal otherwise by in general you will find the leakage. So, that is the main issue here.

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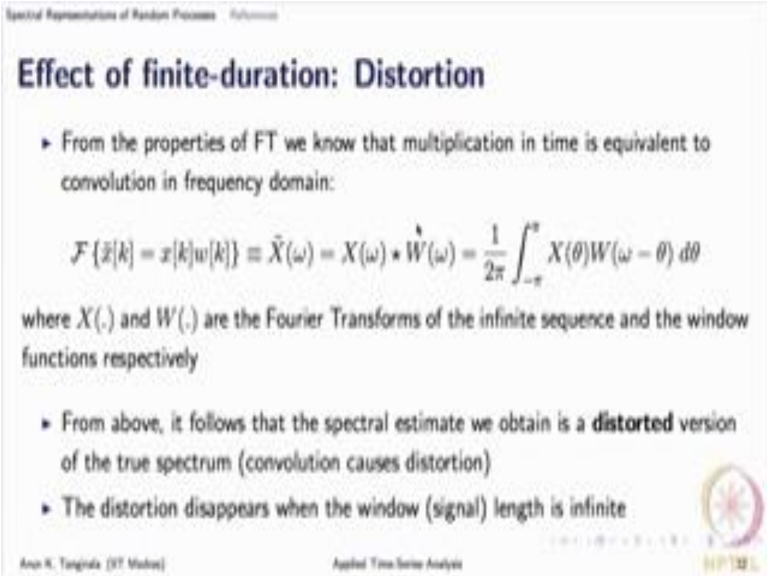

When we compute the periodogram of from that is when we look at the spectrum of finite length signals now; obviously, here we see that the window function is the culprit you can say. So, or the fact that I am observing only for a finite time is the reason suppose I had observed for longer and longer periods, what would happen? I should accept ideally to recover x of omega, the distortion to decrease and that is what will happen, alright. So, in the earlier figure that I showed you the w of omega that I showed you here is for a certain length of w k as I increase the width of w k this main lobe as we call here becomes narrower and narrower and this side lobes as we call here actually fall down.

As a result, you will not see much of a distortion. So, that is first message that we get from this analysis, in other words, spectral leakage is mitigated as I observe more and more points, as I collect more and more data over a longer period of time, it is not simply by changing the sampling rate, you are not going to do a justice. What you have to do is you have to observe the signal for a longer time which means increasing n, but that may not be in my hands, data maybe limited; somebody has observed and given it to me. So, the question is now for a fixed 10, is there a way to mitigate the spectral distortion? So obviously, now all I have to do is change this w, I will still observe for a finite time, but I am not going to observe through a rectangular window, I will change the window in such a way that the distortion is minimized that is the idea behind choosing different window functions.

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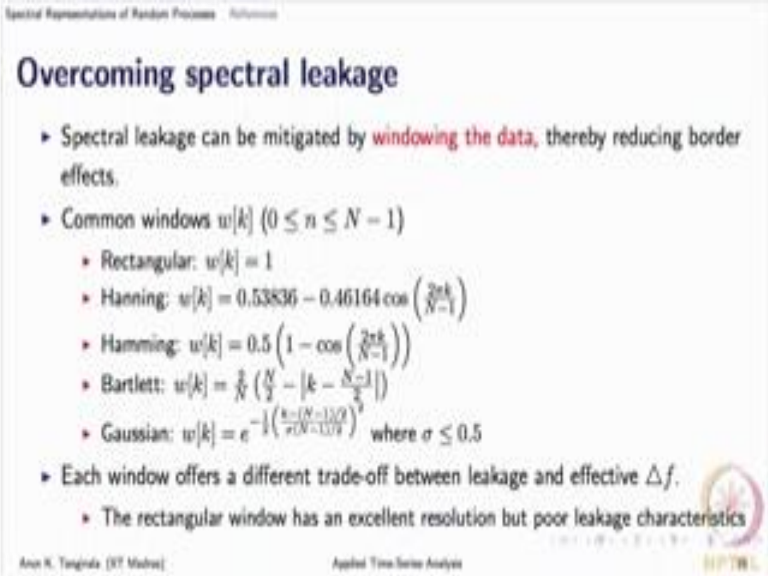

Of course, you know this idea has been toyed around for several decades now and there are several window functions that are available by default you have rectangular window that the data that you have you can apply Hanning window to the signal after you obtain the finite length data you can multiply it with a Hanning window or a Hamming window or a Bartlett or a Gaussian and so on does not matter. So, if you look at the expressions or if you look at their time plots, so, these are the different window functions, if you look at them then clearly you can see that they differ from the rectangular window in a very important respect which is that they are soft at the borders.

So, if you look at it, the rectangular window is pretty harsh at the border, it just cuts off abruptly whereas, these window functions are gentle at the ends, thereby reducing the discontinuity that you have due to fractional cycle it does not completely eliminate it. But it alleviates to a large extent and the extent to which it alleviates depends on the shape, but almost all of them have this feature that they are gentle at the borders that is the basic idea that is why you have a cosine or a bell like shape for all these window functions. But obviously, when you are gaining on spectral leakage, you must be losing out on something else life is not so easy and what you lose out on is the frequency resolution.

The frequency resolution by default is 1 over N; that means, if you do not perform any windowing 1 over N is the resolution, but once you window, what you have done, see when you multiply a signal with any of this window functions here you are in some sense effectively reducing the number of observations. Because the end points are now being treated in a different way they are not being given the same importance as a mid points correct whereas, the rectangular window gives equal importance to all observations and that is why the resolution is 1 over N whereas, here these window functions are actually giving more importance to points in the centre and less importance to the points at the end effectively therefore, you are reducing the number of observations.

You can look at it that way as a result the effective resolution is going to come down.

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And therefore, your resolution is going to change now from 1 over N to 1 over N tilde or you can say N effective number of observations and it is going to be less than or equal to N, when is the equality achieved for what kind of window? When is the effective number of observation same as the original number of observation? When I use a rectangular window, correct good, this point can also be seen in w of omega. So, if you look at this w of omega carefully what do you notice for the rectangular window the width of the so called main lobe is very small that main lobe, width of the main lobe determines the frequency resolution because that is going to sit at the centre here and determine how much of this has been captured and the side lobes will be responsible for spectral leakage.

What we observe of rectangular window is the width of the main lobe is n a small which means it is going to get most of this amplitude correctly by compared to other windows, but it is going to result in larger leakage whereas, the other windows. So, for example, if you look at the Hanning right or Bartlett and so on, you take any of this. So, let us take this green one here if you see sorry Sion color or the red one let us look at the Blackman.

If you look at the Blackman what is the difference between the Blackman window and the rectangular window the Blackman window has a wider main lobe, but very fastly decaying side lobes. So, this side lobes are like this wings the shorter the wings are better it is for spectral leakage, but that comes at the cost of wider main lobe what does it mean I have lost out on resolution; that means, this amplitude is not going to be captured as correctly as by the rectangular window. So, every window offers some kind of a tradeoff between resolution and spectral leakage like it is like Biason variability tradeoff that you seen estimation clear.

Normally what kind of windows are used generally the Hanning window is very popular, but then nothing prevents you from other kinds of windows as well there is a lot of literature available on the analysis of each of this window functions in detail and expressions are available for effective resolution I am not giving you those. But as long as you understand that the moment you window to mitigate spectral leakage you are going to actually run into resolution problems you are going to lower the resolution.

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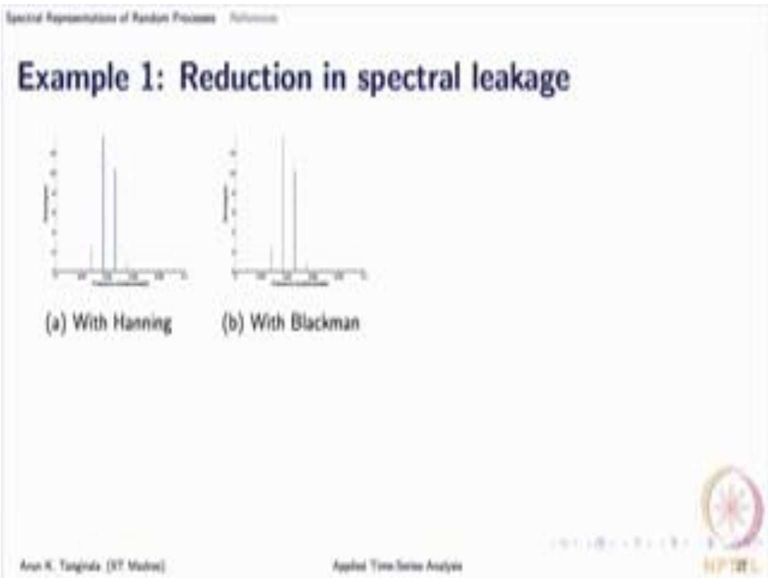

So, let us look at this example now of the previous sign way where I had 109 observations, now I have applied a wind, 2 windows different windows, Hanning and Blackman and this is the periodogram that you see still you do not hit 0.04 that it is not going to happen, but what you have managed to do is reduce the spectral leakage you can compare this periodogram that you have here. So, let me zoom in here so that you can see and compare this with what we had without the windowing. So, without the windowing I had this scenario. So, the leakage was wider, it is spread out to more frequencies now it has been controlled by the use of this window functions.

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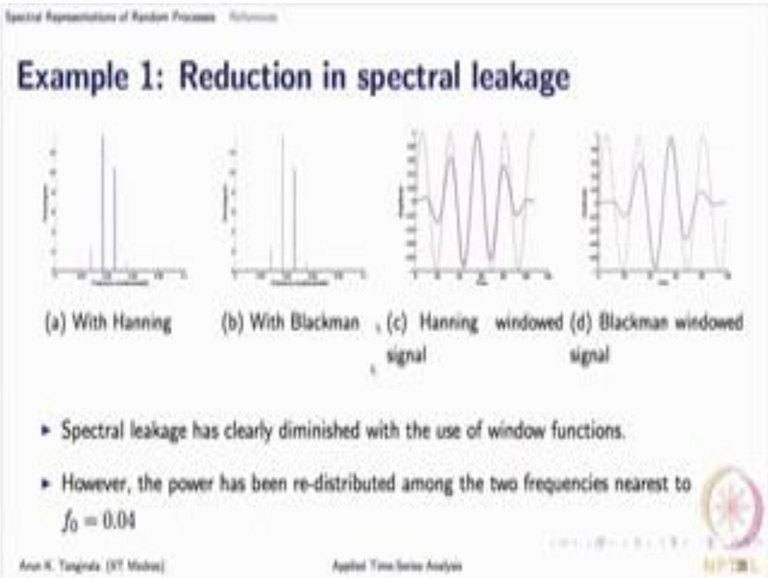

Here now of course, you know Hanning offers a different kind of mitigation versus Blackman what I am showing you on the right is the sign waves that are obtained after windowing this is the original red one is the one that you would be analyzing without any windowing whereas, the blue one is the one that you obtain after windowing. So, what have you done here? You have changed the behavior at the borders that is what you have done all right and of course, now let us get back to this question before we move on to the stochastic cases.

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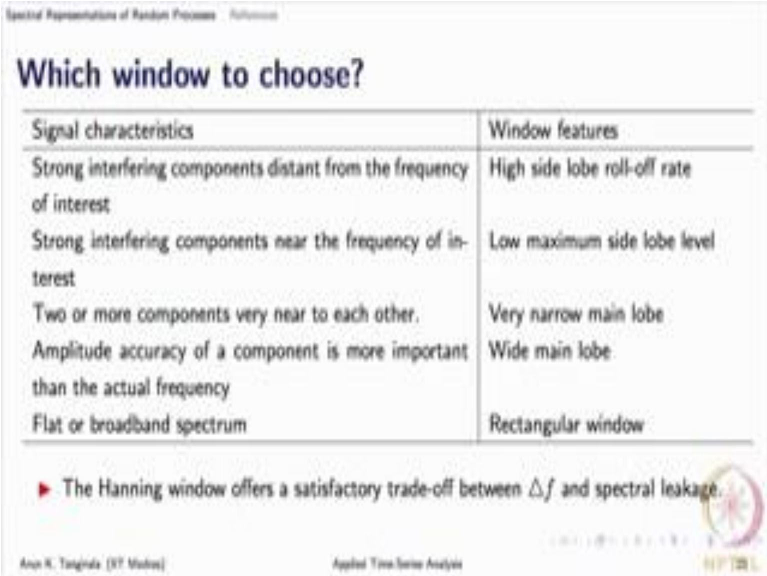

Which window to choose as I have said each window offers a different tradeoff between resolution and spectral leakage, but there are some guidelines available although there is no definitive formula depending on a signal characteristics you can choose a particular window for example, if you have strong interfering components near the frequency of interest then you want to choose a window that has low maximum side lobe level or if you have a signal where 2 or more components are very near to each other. That means, the resolution is very important for you that is what is called resolution then you should use very narrow main lobe which means maybe something of rectangular type or maybe bet slightly near that.

And on the other hand if you have a flat or broadband spectrum which means you have many many frequencies closely spaced it is better to use a rectangular window. So, the Hanning window is in general used by default, but you should not necessarily you need not necessarily stick to default look at your signal understand its features and then make a decision.

So, summarize now when it comes to using periodogram for deterministic signals it is a good estimator, but it runs into this issue of spectral leakage and this spectral leakage becomes particularly important when you have small sample sizes when you have 1000 or 2000. You can repeat this example that I have shown for 2000 observations, you may not see much of a spectral leakage or 2005, keep the frequency the same. So, 1 over 2005 is going to be the frequency resolution, still it theoretically will not hit the 0.04, but you will not be too far away from it, but when the number of observations is fixed and we do not have any control over it, it is better to use this window function and then compute the periodogram.

The periodogram that you compute after windowing the signal is called modified periodogram so that is the story of spectral leakage and the mitigation now let us move on and ask what happens. Now when I used the spectral density for estimating PSD sorry periodogram for estimating the power spectral density for stochastic signals now first let us recap.

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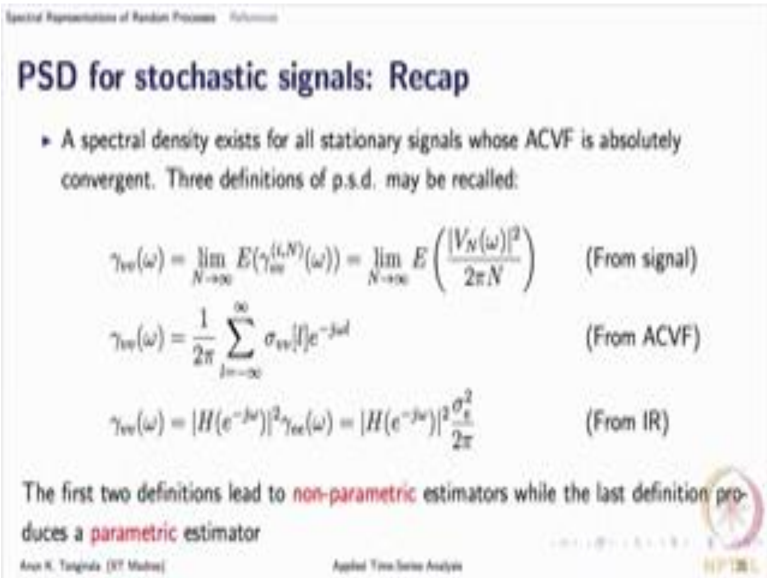

Here the definition of power spectral density for a stochastic signal, it is extremely important, we have 3 different ways of looking at it and I am showing you those 3. In fact, there is forth one which from Venus generalized harmonic analysis, but will not get into that one is the semi formal route that the spectral density is defined as expectation that is average of the periodogram in the inner bracket you have periodogram of the finite length realization and then you take an ensemble average and then let limit n go to infinity. So, this is the standard semiformal approach presented in many texts which we have also gone through.

This is directly from the signal on the other hand you have the Wiener-Khinchin relation which allows you to compute spectral density from ACVF and the third one here is from the parametric model let us focus on these 2 because what we are going to discuss in the next few minutes is based on these 2 definitions here in the next sequel we will focus on the third definition because the third definition assumes that you have built a time series model.

Now we are not going to talk about that we are saying I have a data record I have a finite length realization if I apply a periodogram as an estimator if you use it as an estimator how good is it as an estimator. Now one of the key things that you should observe which we have emphasized earlier also is this spectral density of stochastic signal is an average quantity unlike for the deterministic signal it is an ensemble property it is a population property whereas, I am going to have only one realization. So, naturally you should expect some issues because ideally if I want a good estimate I should do some kind of averaging. If you recall the auto covariance estimation what is the auto covariance estimator that we have used, you have said the auto covariance estimator is one over n sigma v k assume it to be 0 mean or if it is not then use a sample mean this is the estimator that we have used for auto covariance estimation.

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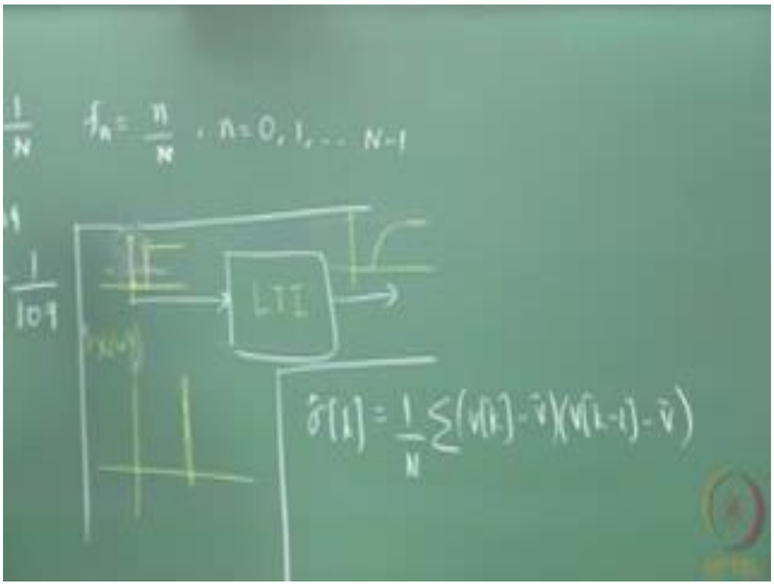

We are doing some kind of averaging here, but we are averaging in time that is all we can do, but some averaging is happening whereas, periodogram which is the inner most one. It is not going to perform any averaging in the frequencies or in any sense therefore, I should expect the periodogram to be a poor estimator of the power spectral density when it comes to stochastic signal that is the key now of course, as with the spectral leakage one can offer different viewpoints and we will briefly talk about that and will come back to that.

By the way what therefore, we realize is that the spectral density is an average property and consequently the periodogram is a poor going to be a poor estimator of the power spectral density that is just an intuitive argument, but the main technical drawback of using periodogram is that it is not a consistent estimator what this means is as I increase the number of observations the periodogram does not converge to the truth which is a key requirement sorry.

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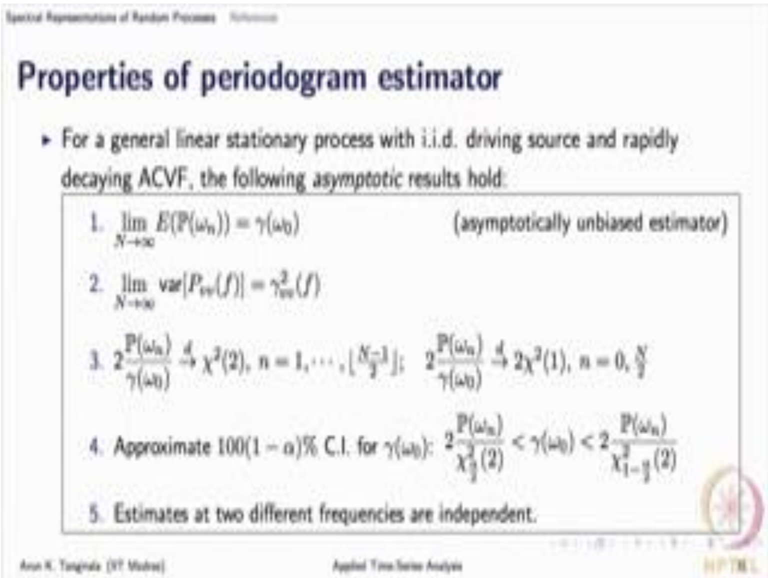

Here is a list of the properties of the periodogram one that it is an asymptotically 1. So, the first property says that the periodogram is an asymptotically unbiased estimator which is good news, but the bad news is the second property that the variance does not shrink to 0 as n goes to infinity. In fact, the variance as n goes to infinity is a square of the power spectral density at that frequency which is not what we desire we want this variance to go to 0 right this variability here is not in frequency you have to understand at a specific frequency if I were to repeat my periodogram estimation on different realizations the variability is not going to shrink as n goes to infinity.

The third property here talks about the distribution property of the periodogram as an estimator it says at the end points for the frequencies, it is a chi square distributed variable with single degree of freedom of course, normalized periodogram. Whereas, at the all other points it is a chi square distributed variable with 2 degrees of freedom which is that is not an issue, as I said the issue is the second point here that the periodogram is not a consistent estimator of the power spectral density gamma of f then you can construct 100 times 1 minus alpha person confidence intervals and so on. And the periodogram estimates at 2 different frequencies are independent now this is another problem we do not want the estimates to be independent why? Because if you look at the truth that is if I were to draw a gamma of f; the true gamma of f for any random process it is going to look smooth suppose I have a colored process like this or even white noise process these are all the different power spectral densities that you can see for different processes

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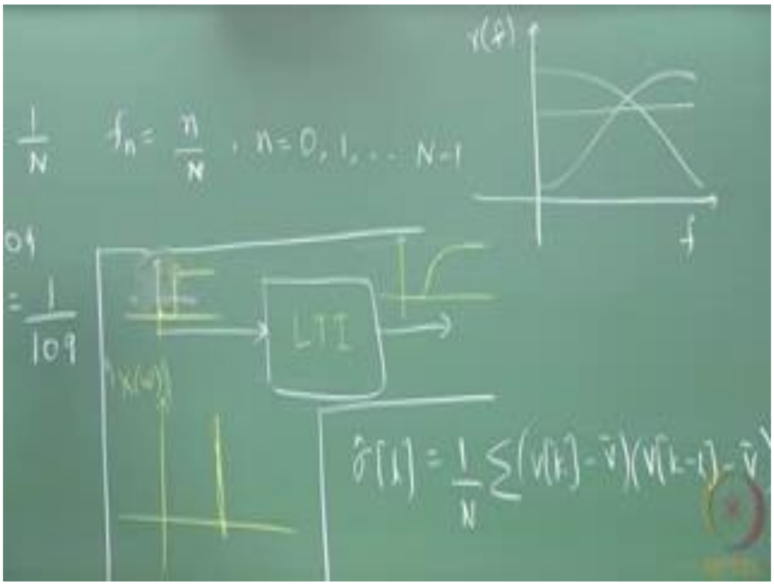

What you notice there is a certain smoothness there is a certain relation between the power spectral density at different frequencies right. U-nfortunately the fifth property says that the estimates of 2 different frequencies have no relation in them which is not a good thing, they are suppose to be closely related and that again is tied to the second property also implicitly. So, that is a main issue.

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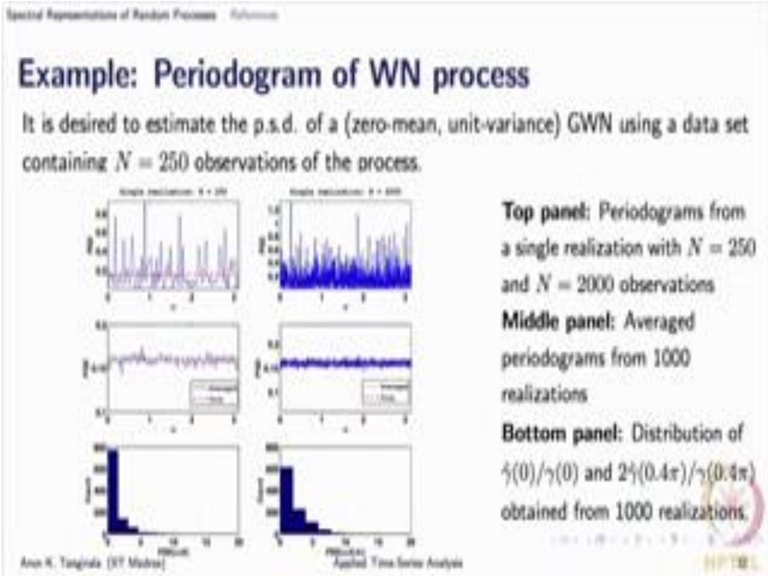

Let me show you as an example, here I have taken a Gaussian white noise process and I have taken 250 observations. So, on the left here, so, let me zoom in here on the left here I am showing you the periodogram of the 250 observation long realization and you can see remember this is a Gaussian white noise process. So, if periodogram was a good estimator how should the periodogram look like almost a flat one correct, but what do you notice instead? A very erratic looking periodogram frequency to frequency and the as I increase the data length things do not change from 250 to 2000, the periodogram is still erratic for across frequencies. Why is this erratic jumping from one frequency to another which property tells me that this is going to be erratic? The fifth property that we talked about earlier which is that? The periodogram estimates at 2 different frequencies are independent correct.

Now what I do here is I take the 250 observation long realization and generate many such realizations and compute periodogram for each realization and average that. What you notice? The fluctuations have come down right and now I get the red line that you see is actually the truth as the legend says the is not the averaged one much better looking than the one for single realization right, why has this happened? Because now we are closer to the definition, the definition of power spectral density as we discussed earlier says that it is an average property correct and that is the case with 2000 observations as well of course, it is looking more dense because frequency resolution is much better the frequencies are closely spaced.

And on the bottom here I am showing you the distribution of the periodogram at 2 different frequencies at 0 frequency and at some other frequency you can see that, remember the property told me that the distribution of the periodogram at 0 frequency is a chi square periodogram is a chi square distributed variable with one degree of freedom whereas, at other any other frequency apart from the end point, it is a 2 degree of freedom chi square distributed random variable, how do I know that this is 1 degree and this is 2 degree?

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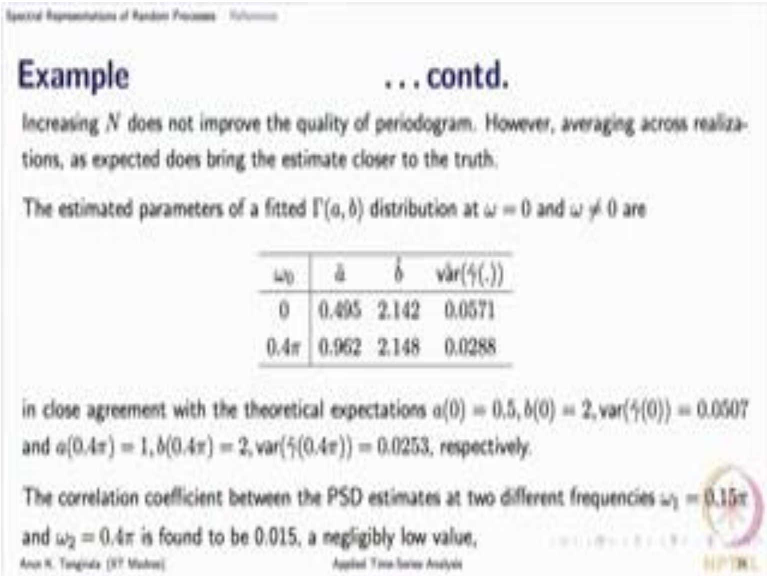

I can do a distribution fitting and that is what I have done, I fit a gamma distribution, remember that chi square distribution is a special case of gamma distribution and how do I relate, well I look at the value of a of gamma distribution and twice the a is the number of degrees of freedom.

Gamma distribution is a big one; it is a super boss of which chi square is a member. So, if I when I do the fitting for the periodogram at omega, this should not be omega naught it should be omega at 0 frequency a is roughly 0.5 twice that is the degrees of freedom. So, which means one degree of freedom at any other frequency a is roughly one. So, twice that is 2 degrees of freedom. So, which confirms that the results theoretical results that we had earlier that is this property number 3, alright. So, through an example we have shown the issues associated with periodogram.

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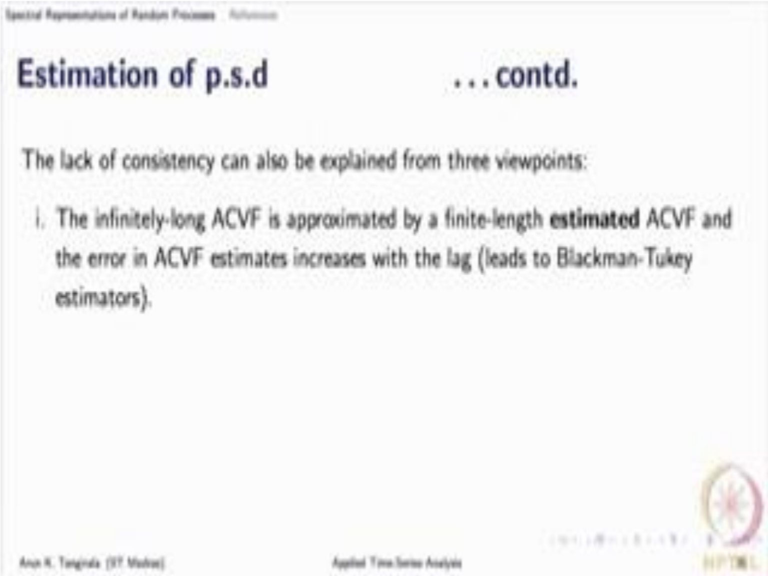

Let us now complete our discussion with a quick discussion of the 3 different viewpoints which we will again take up in the next sequel. The first issue has got to do now people came up with different viewpoints at different times and therefore, suggested different remedies one of the early viewpoint that was offered is go back to this definition of periodogram here, the definition number 2 here and you will notice that ideally you would require the infinitely long ACVF, but you only have finite length ACVF that is one issue. And the second issue is that the errors associated with the ACVF at large lags is much more than the errors associated with the ACVF estimates that smaller lags. So, somehow you have to do some kind of windowing here this windowing is different I mean the reason for the windowing is different from the spectral leakage thing, but what by mean by windowing is you have to give more importance to ACVF at lags closer to 0 then the ones that are farther away because of the error distribution.

The second view point is this comes from this; the true PSD is a smooth one whereas, the periodogram as you saw in the example is an erratic one. So, there is no smoothness in the periodogram at all. If I bring about some smoothness by somehow how do I bring about smoothness? I can take estimates over a band and then take average that is called the smooth periodogram, so that was offered by Daniel.

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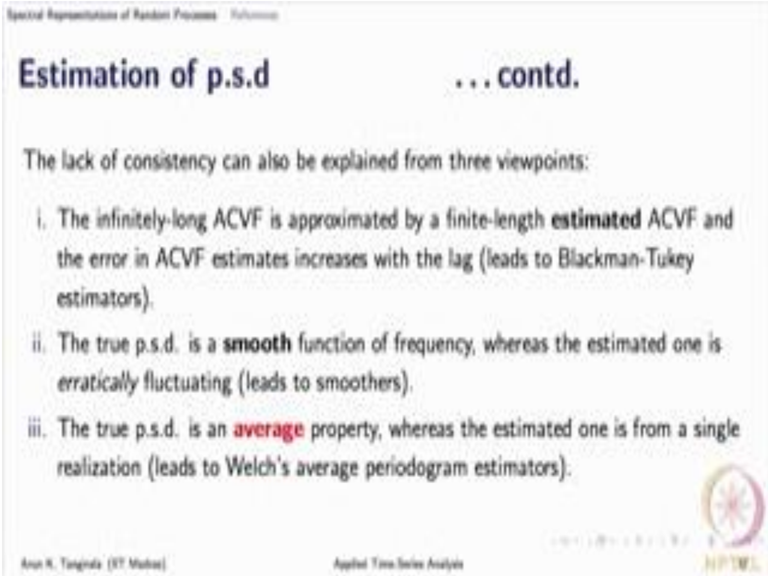

And the third viewpoint is that the true PSD is an average property we have talked about that earlier whereas; the periodogram is not an averaged one. So, if from this viewpoint you can actually generate artificial realizations how do we do that I take the real single realization chop it chop them up right and then take compute periodogram over each slice and then average. So, I am generating multiple realizations by chopping up a single realization slicing up that is the idea of behind wells and Bartlett's averaged periodogram method.

Each of this viewpoint gives rise to different remedy, the first one was proposed by Blackman Tukey where they apply a waiting function to the ACVF, the second one was perceived by Daniel, it results in Daniel's motor, the third one was perceived viewpoint was perceived by Bartlett and Welch which gives rise to average periodogram method.

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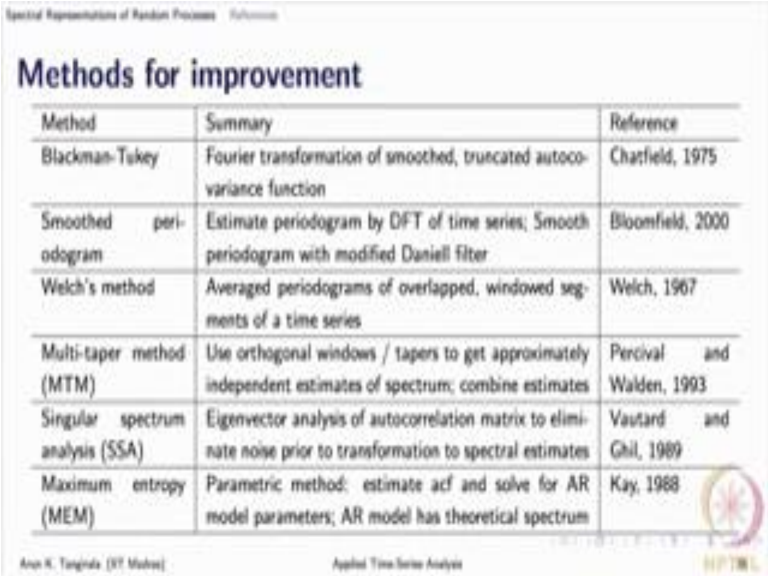

Method	Summary	Reference
Blackman-Tukey	Fourier transformation of smoothed, truncated autoco-variance function	Chatfield, 1975
Smoothed peri-odogram	Estimate periodogram by DFT of time series; Smooth periodogram with modified Daniell filter	Bloomfield, 2000
Welch's method	Averaged periodograms of overlapped, windowed seg-ments of a time series	Welch, 1967
Multi-taper method (MTM)	Use orthogonal windows / tapers to get approximately independent estimates of spectrum; combine estimates	Percival and Walden, 1993
Singular spectrum analysis (SSA)	Eigenvector analysis of autocorrelation matrix to elimi-nate noise prior to transformation to spectral estimates	Vautard and Ghil, 1989
Maximum entropy (MEM)	Parametric method: estimate acf and solve for AR model parameters; AR model has theoretical spectrum	Kay, 1988

As a result you have a number of methods available today I have discussed the three main viewpoints whatever methods you have today are built on this viewpoint. In the next lecture we will study what is Blackman Tukey method I have already given you the logic behind it the philosophy and look at the smoothed periodogram Welch's method and then briefly talk about the MTM I will not talk about the singular spectrum analysis or maximum entropy of course, maximum entropy in some sense is burgs method. So, in the parametric estimator world we will run into burgs estimator as well.

Each of this is basically going to offer some kind of again trade of between resolution and variance. Remember all of these are trying to bring about consistency in the periodogram you can show that all of this result in consistent periodogram estimators, but at the cost of resolution like if you take Welch's average periodogram method we said we will take a single realization slice it up, the moment I slice if I have thousand observation long realization and I divide it into four then I will have only 250 observation in a single slice. So, the resolution goes down right. So, you can see straight away that I will sacrifice the resolution for the sake of bringing in consistency.

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That is the idea and that brings us to the close of this lecture where we have looked at periodogram as an estimator for both deterministic and stochastic signals. For deterministic signals, it is not much of an issue only issue is spectral leakage, so I work with the modified periodogram by applying a window function. Whereas, with the stochastic signals the issue is lot more serious which is that of consistency itself and then again we have learnt 3 different viewpoints, we understood why it is not consistent by looking at 3 different viewpoints and then realized that there are different methods that come up as a result.

The next lecture, we will dwell on these different remedies that are available and also talk about parametric ways of estimating spectral density, the parametric method would be to first fit a time series model and then use this expression that we had earlier the third one. The biggest advantage of the parametric 1 is it does not suffer from neither the does not suffer from the spectral leakage as well as resolution because the moment I have h I can compute gamma at any frequency whereas, with other methods that I have discussed the frequencies at which I can compute the periodogram is limited by the number of observations or the number of lags and so on.

That is the big advantage of the parametric estimator, but of course, it heavily depends on the model that I fit. So, I have to be quite careful when I fit the model. Well that is about it I mean you should now understand the prime difference between these 2 classes of estimators alright. So, that brings us to the close and we will meet in the next class.

Thanks.