Computational Fluid Dynamics
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Lecture – 08
Tutorial 2: Converting PDE to Algebraic Equation using Finite Volume Method

We have seen the Finite Volume Method for discretizing governing equation. So, now, let us do a tutorial on this. We will go back to the triangular that problem. And, in this tutorial, we will try to derive the equations for the nine points, which we have seen earlier.

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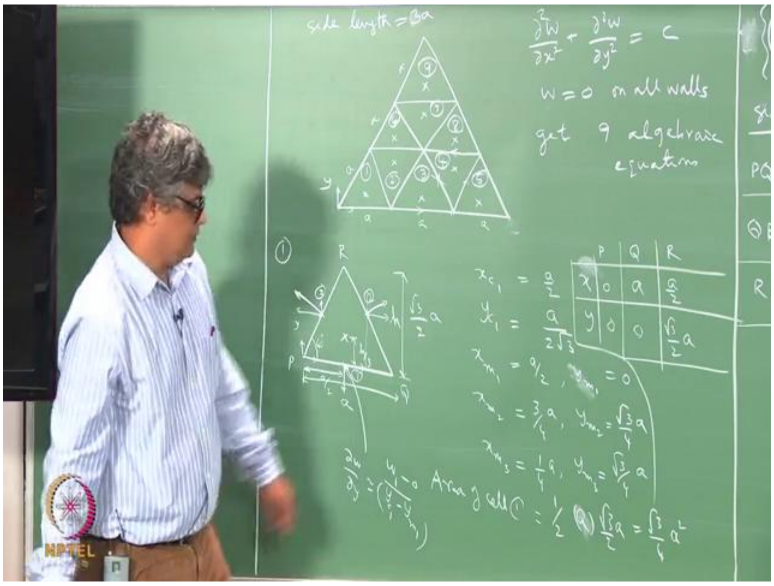

So, w e will start with our equilateral triangle, of side a here. And, we divide this into three parts. So, we have nine triangles here bit bigger, but otherwise they are all supposed to be for the same length like this. And, we can put our origin here. This is x and y. We will identify the centroids of each of these. And, let us call these by the numbers: cell number 1, cell number 2, 3, 4, 5, 6, 7, 8 and 9. We would like to discretize this equation with this boundary condition for each of the cells and get 7 algebraic equation – 9 algebraic equations. So, this is the tutorial.

And, let us get started. We have already seen how this is going to be done for – how this can be done for cell number 7. But, let us start with cell number 1. So, this cell has we can say that, this is the total length, is three times says that, therefore, the side of each of these small triangles is a. So, this is a, a and a. Similarly, this is a here like this. So, for this cell, this is cell 1; our origin is here and this is 60 degrees. So, we can say that, this length is a here. The midpoint – this is at distance of a by 2 and this whole height is square root of 3 by 2 times a. And, this height is h by 3; where, this is h. So, from this, we can say that, x c 1, that is, a x coordinate of the centroid for the cell 1 is equal to a by 2; this distance. And, y c 1 is h by 3; where, h is this. So, square root of 3 by 2 times a divided by 3 is a by 2 square root 3.

And, we can define – there are three sides to this. So, the midpoints of each of these we would like to define. So, we can say that, we can always go in the counterclockwise direction for any given triangle. And, we can always start with the horizontal side; each of this has a horizontal side. So, for example, this cell – this is a horizontal side. So, when we come to description of the sides here; so, we have side 1, side 2, side 3, because we are going in the counterclockwise direction. And, when you look at cell 3 here; then, this is cell 1 and this is side 1; this is side 2 and this is side 3. So, for each triangle, for the sake of convenience, we go in the counterclockwise direction starting with the horizontal phase cell as side one.

And then, we can go to the next one and then next one like this. And, that allows us to fix the coordinate directions of the midpoints for each of the cells. So, this is a heuristic way of doing this. And so, we can say that, x m 1, that is, this is said side 1, side 2 and side 3. So, that is why we call it as x m 1, is a by 2 and x m 2 – y m 1 is 0.

Similarly, this is x m 2. And, x m 2 is half way between these. So, it is exactly at this. So, this is 3 by 4 a. And, y m 2 is half height of this. So, that is h by 2. So, that is square root of 3 by 4 a. And, this is the midpoint of the third side x m 3. And, this is at an x distance of one quarter of a and y distance, is at the same height as this. So, that is square root 3 by 4 a. So, this is the geometric information.

In addition to that, we can – let us call this as P Q R – the three vertices. And, we can also put here x p – x and y of the three vertices. For P, it is 0, 0; it is an origin. And, for point Q, the distance is a here and y is 0. And, for R, this is x of R is a by 2 and y of R is square root of 3 by 2 a. And finally, we want to have the area – area of cell 1. This is half into base into height. So, that is half 3 a; no, this is a here and height is 3 by 2 a. This is 3 by 2 a – square root of 3 by 2 a. So, this is square root of 3 by 4 a square. So, we have –

for this particular cell, we have all the geometric information directly from the location of this cell with respect to everything else.

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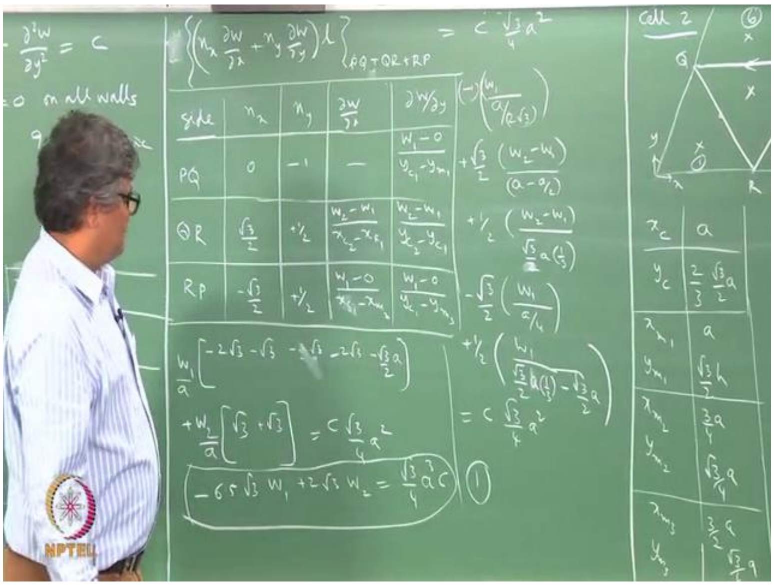

And, we can now look at the discretization of this, which we can write as n x dou w by dou x plus n y dou w by dou y times l for cell i – for side i over all the three phases. So, we can write this as over P Q plus Q R plus R P equal to the constant c times the area, which is square root of 3 by 4 a square. So, now, we need to evaluate the n x, n y and all these things. So, let us start with for side – so, let us do this systematically. So, we can say that, in the form of a table, we have side PQ, QR and RP. So, for each of these, we can write n x, n y, dou w by dou x and dou w by dou y, so that it will be easy for us to evaluate these terms and then multiply by the length and all that.

So, we come to this PQ; n x is 0 and n y is minus 1 for the midpoint here. And, here n x is square root of 3 by 2 and n y is minus half – no, no, n y is plus half. And, here it is like this. So, for RP, this is minus square root of 3 by 2 and plus half. So, the three normals are like this. So, here this is n x n y, n x n y, so, n x is negative and equal to this. And dou w by dou x – since this is zero here, we do not need to evaluate this. And, dou w by dou y, at this point here. So, we need to estimate this in terms of known quantities. And, when you consider this triangle here, what are the known quantities or to be known quantities? This w at this point is being evaluated. And, it is also given that, w at this point is 0. But, unfortunately, in the equation, we do not need the w at this point; we need to have the gradient dou w by dou y at this point. So, we can estimate the gradient dou w by dou y at this point roughly as w 1 minus this velocity, which is 0 – divided by this distance. And that distance is y c 1 minus y m 1; and, y m 1 is 0. You can see that, y m 1 is 0. So, we can write the gradient here as w 1 minus 0 divided by y c 1 minus y m 1.

Now, we come to side RQ and we can evaluate the dou w by dou x at this point as w 2 minus w 1 divided by x c 2 minus x c 1. So, w 2 minus w 1 divided by x c 2 minus x c 1 based on the two nearest values that we have here. And, we also need dou w by dou y and that will be this value minus this value divided by the vertical distance. So, that is w 2 minus w 1 divided by y c 2 minus y c 1. Now, you come to this one here – this side here. And here, dou w by dou x is to be estimated from this and the immediate neighbor. There is no immediate neighbor, because it is a wall. And we know that, the w velocity at this wall is 0. So, we are bringing in the information of the boundary condition here. So, we can say that, this is equal to w 1 minus 0 divided by x c 1 minus x m 3. That is the third point here, because that is the third phase. And similarly, this is w 1 minus 0 divided by y c 1 minus y m 3. So, in order to now discretize this over this cell, we need to multiply this by this and this by this and then this by this and this by this; and then, this by this and this by this. And then, add up all of them and put that equal to this.

So, let us see. We have – we can substitute the values here. So, that gives us minus w 1. We have minus 1 times w 1 divided by – y m 1 is 0; y m 1 is 0. And, y c 1 is a by – So, this is what we are getting from this – plus square root of 3 by 2 times w 2 minus w 1 and x c 2 minus x c 1. So, this is x c 2 is this and x c 1 is x c 2 – this is at distance of a here and this is a by 2. So, the distance between the two is a by 2 – a minus a by 2. And, here we have plus half into w 2 minus w 1 divided by the vertical distance. This is at h by 3 from the top and this is h by 3 from the bottom. So, this is 2 h by 3. So, this must be h by 3. This is 2 h by 3 from the top of this. And, the top is the same as this. So, this is h by 3 from the bottom. So, this distance here is h by 3. And so, this is – and h itself is square root of 3 by 2. So, this is what we have from this.

And, here we have minus square root of 3 by 2 times w 1 divided by – x c 1 is a by 2 and x m 3 is a by 4. So, the difference is a by 4 plus half times w 1 by – y c 1 is h by 3 and y m 3 is square root of 3 by 4 a, is what we have here. So, this is h is 3 by – square root of 3 by 2 by h times 1 by 3 minus. And, this whole thing here is equal to constant times the area, which is 3 by 4 a square – So, this is the equation here. And, in this equation, a is given. So, everything else is all the w's are coming with coefficients. So, let us just add up all the things and get an equation for cell 1. So, from the – we can say w 1 times – in the first term, we have – this is in the denominator; so, this goes as minus – so, we can divide by a here, because a is occurring in all of these, is what we have. And, here we have a by 2 and that goes into the numerator and – so, that and this cancels out – minus square root of 3. And, here again we have – there is a mistake here. So, this is 2 square root 3 will be going in the denom – in the numerator. So, this is minus 2 square root 3 divided by 2, is from here.

And, here we have 4 going here. So, that becomes minus 2 square root 3. And, here we have – so, this is a by 3 and this is a. So, this is minus 2 a by 3. So, this will be one-third of – 1 by square root of 3 – 2 a by 3. So, 2 2 cancel out. So, you get a by square root 3 – minus a by square root 3. And so, this is minus – So, this is all for w 1. And, for w 2, we have – So, we have 1 by 2 here. So, that goes as square root of 3. And, here we have 2 root 3. So, this is plus root 3 here. And, that is all there is for w 2. And, there is nothing else. So, this is equal to – So, we can take the a to this side and we can evaluate all this. So, this is 2.5, 3.5, 4.5, 6.5. So, I am getting minus 6.5 square root of 3 w 1 plus 2 square root of 3 w 2 equal to square root of 3 by 4 times a cube times c. So, this is an equation that we have for cell 1. So, let us also fill this table for cell 2 and then we can do this at leisure. So, for cell 2 is this.

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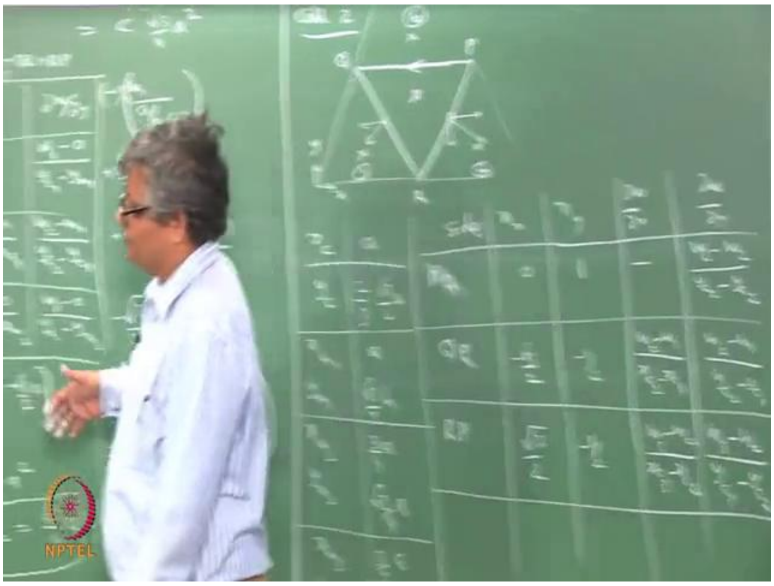

So, as per this figure, cell 2 has immediate top neighbor of 6, left neighbor of 1 and right neighbor of 3. And, these are the centroids for each of these. And, this is our coordinate origin. So, just as before, we can start with this as P, Q and R. We start with the horizontal thing; go in the counterclockwise direction; then, come back apply this. So, we can say that, for this phase, x c and y c. x c at a – is at a distance of a here and y c is at a distance of two-thirds of h. So, two-thirds of h is square root of 3 by 2 times a. So, this is equal to square root of 3 here. So, this is – now, this is our x m 1. So, that is this point here. And, that is at a here. And, y m 1 – this is this side. So, that is h. So, it is square root of 3 by 2 h. This is our m 2. So, that is the second point – midpoint for the side QR. So, that is x m 2 is at this distance, which is a by 4 or 3 – 3 a by 4 and height is half of this. So, y m 2 is – half height is square root of 3 by 4 a. And, this is our x m 3, is this distance plus half of a. So, this is 3 by 2 a. And, y m 3 is at the same height as this. So, that is square root of 3 by 4 a. So, we have important coordinates of important points here. And, we can similarly, write down the coordinates or the vertices; and, the area is already known.

So, let us go inside; and then, n x, n y, dou w by dou x, dou w by dou y. So, now, for the side starting with PQ, n x is 0 and n y is 1. And, here and – So, dou w by dou x does not matter. And, dou w by dou y is w 6 minus w 2 divided by y c 6 minus y c 2. For the side QR – please note that, this QR is for this cell. So, for this QR, we have this is n. So, n x is minus square root of 3 by 2 and n y is minus half. And, dou w by dou x is this minus this divided by this vertical – horizontal distance. So that is w 2 minus w 1 divided by x c 2 minus x c 1. And, dou w by dou y is w 2 minus w 1 divided by y c 2 minus y c 1.

And finally, for the side RP, we have n going out like this. So, this is n x and n y. So, n x is square root of 3 by 2; and, n y is minus half. And, dou w by dou x is based on these two things. So, this is cell 3. So, w 3 minus w 2 divided by x c 3 minus x c 2. And similarly, dou w by dou y is w 3 minus w 2 divided by y c 3 minus y c 2. So, we have all the information – just as we had for cell 1, we also have the information for cell 2. So, we would multiply this by this plus this by this – this by this times this by this – plus this times this plus this times this – equal to the right-hand side, which is nothing but c times the area of the cell. So, in this way, we can simplify and get a similar equation for cell 2. And, we have to do that for each of these things. And, we will be able to get the nine equations. So, in this tutorial, we have seen how we can write an approximate form or the governing equation for a given cell making use of the finite volume formulation of this. And, this again gives us one set of equations.

In the next lecture, we will put down all the equations that we get from all the 9 cells and then we can see how we can solve.