Computational Fluid Dynamics

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Lecture – 07
Flow in a triangular duct: Discretization of flow domain

We have seen the basic idea of how to do discretization using the finite volume approach for the same equation as what we had seen earlier. Now, in this lecture, we will actually go through the fine details of this and see how we can use the finite volume method to write an equivalent expression for the partial differential equation in F-shell. We go back to the example of the triangular duct.

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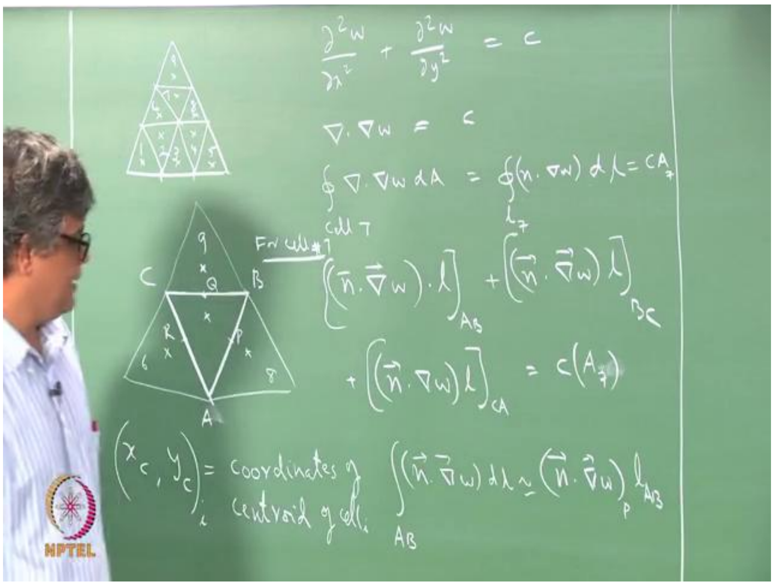

So, we go to the triangular duct here. For the sake of convenience, we are taking an equilateral triangle, and we split this up into equilateral triangles like this. We can identify the centroids, we mentioned that these are two-thirds of the height from the top and one-third of the height from here, so there be like this. We can also for our sake number the cells as cell number 1, 2, 3, 4, 5, 6, 7, 8, 9. We are doing this approximately for this when we do a tutorial on this then we can do it in with much more clarity. In this lecture, we are going to look at for example cell 7 only, and then try to convert this partial differential equation into an algebraic equation. And the way that we said that we would do this is to write this as del dot del w equal to c and integrate it over the area of cell 7, for example, we do this. And this is equal to c times and because this is over a closed area of this cell. We can write this as integral of n dot gradient of w d l along the sides of cell 7 and this is equal to c times area of the cell 7.

So, now, we would like to discretize this further; and for that we need to identify the cell 7. This is our cell 7, it has a centroid here, and it has as a neighbouring cell, we have cell 9 here with the centroid. And then on this side, we have cell 6 with a centroid here; and on this side, we have cell 8 here with the centroid here. So, this is our cell and we have identified the three immediate neighbours with which it is sharing a face, for example, this face is being shared by 7 and 9; and this face is being shared by 7 and 6; this is being shared by 7 and 8. Let us also put some names for the vertices, so that we have it is easy for us to follow and we have these A, B, C here. These centroids can be determined.

So, we know we denote by x c and y c are the x and y-coordinates of the centroid - centroid of cell i. For the ith cell we have x c i and y c i as the coordinates. And therefore, we know the centroids of cell 6, cell 7, cell 8, cell 9, so these are known from the geometry and layout of this. So, now with this thing, we can apply this equation as for cell 7, we can write that equation as n dot gradient of w over times l of A B plus n dot gradient of w times l over side B C plus n dot gradient of w times l over side C A is equal to C times area of area of cell 7. So, from the known x, y, z of coordinates of the three vertices we can evaluate the area. So, in a way that is a straightforward also given that is an equilateral triangle we can readily find that out.

So, now what do we do with this. Now if you this is a n dot so ideally this should be integral along the length, but given we are looking at CFD, and where the cells as a supposed to be small here, and we are associating with each cell are node which at only at this node we are evaluating the variable w. So, we say that integral of this along a particular side is equal approximately equal to the evaluation of this at the centre of this. So, for example, if you now so we are saying that alongside A B is approximately equal to n dot gradient at the midpoint, which we can call as for example, p here and the midpoint of this as Q and this as R. So, evaluated at p times l A B, where l is the length of the side A B here. So, we are making this approximation.

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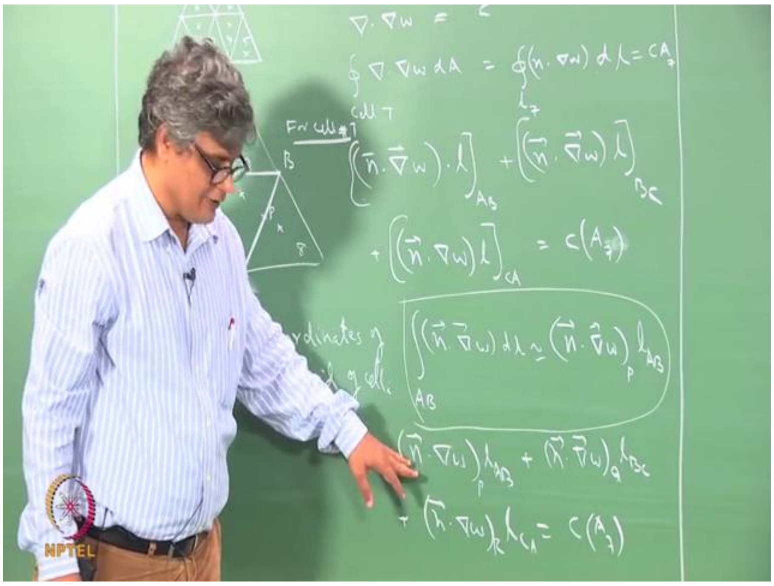

Now it is a question of so using this approximation here, we can write this thing as gradient at P times l A B n dot Q times l B C R times l C A equal to C times area of cell 7. So, the reason why we explicitly wrote up like this is that in order to evaluate these gradients and the surface normal vector at p, we need to know the x, y coordinates of P, Q and R. So, we also need to know x and y coordinates of P, Q and R for cell 7.

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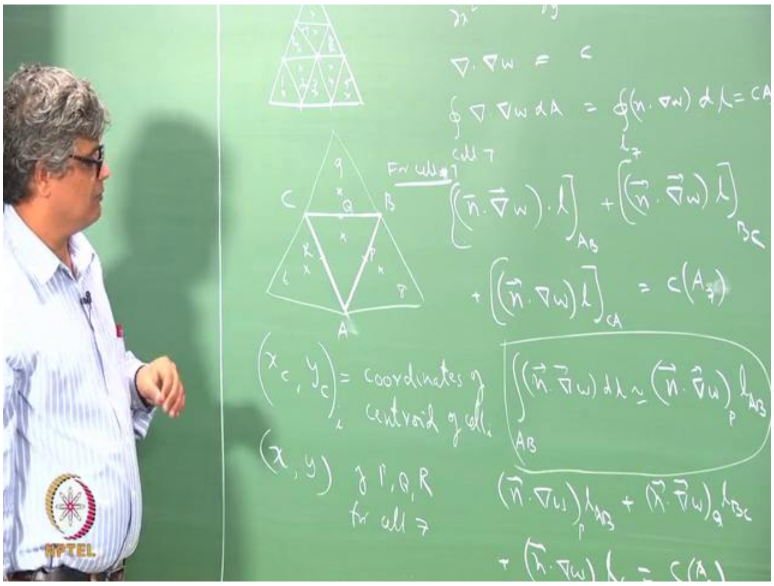

We will see why that is needed and that can be worked out from the geometry. So, all the geometric information related to this, what is a length, what is a area, where is a centroid, where as the centre points of each side all that thing is to be is essentially known from geometry.

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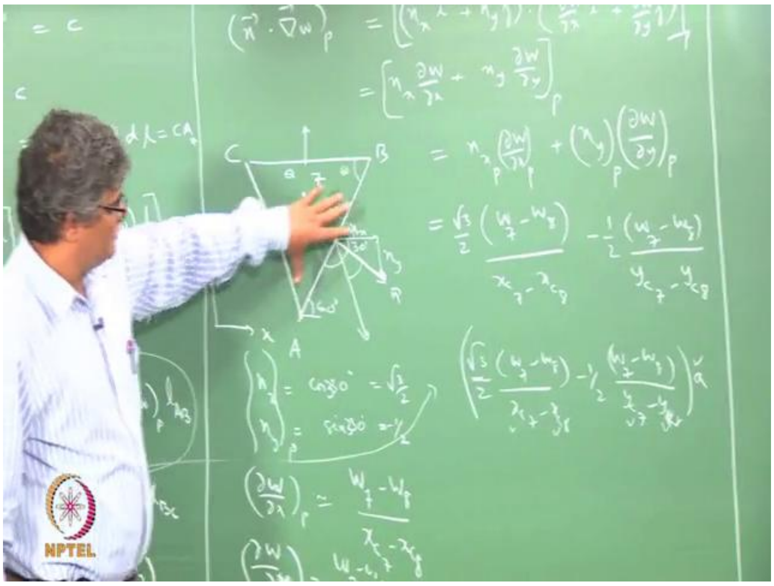

So, now if you are looking at of at point P here, we can write this as the x component of the normal vector plus n y is the y component dotted with dou w by dou x i plus dou w by dou y j. So, this is what it is and because it is a dot product, this becomes n x dou w by dou x plus n y dou w by dou y and this whole thing is evaluated at point P. So, this is also at point P. So, now, we have to look at point P and then say what is a so for this point P, we can find the surface normal vector to be like this. So, we have n x. So, let me just for the cell 7, this is B here, and A, this is point P; and so point p has this as the outward normal vector to the cell to the side A, B. And this has a component of this much as n x and this much as n y. Now the gradient here is of some direction, which is known from the gradients we can see; and so the cross product the dot product of these two will give us this particular value here.

So, in this particular case, we have an equilateral triangle. So, this angle is 60 degrees and this angle is 60 degrees here. And this is our x-direction, and this is our y-direction and therefore, this is 30 degrees and this is 60 degrees. So, you can say that n x is cosine 30 degrees and n y is sin 30 degrees. So, this is square root of 3 by 2, and this is equal to half this is for point P. And n y is actually it is not actually this is minus 30 degrees, because we should be counting all the way from here. And so from that point of view this becomes minus half, so it is not 30, it is actually 330. So, we always count the angle in this direction. So, we can actually put this as 330 here. So, in this way we have no difficulty special difficulty in working out what n x and n y are for the point P.

Now we need to get dou w by dou x at point P, because we need to evaluate this. So, we can we are writing this as n x at point P times dou w by dou x at point P plus n y at point p dou w by dou y at point P. So, each of this is being estimated from the geometry of this triangle and the side here, we have evaluate n x and n y. And the gradients here are evaluated from the difference between the neighbouring points. Now, if you come to this point here this is the value of this is 7th cell and this is 8 cell, and we can say that the gradient here dou w by dou x at P can be written as w 7 minus w 8 divided by x c 7 minus x c 8, where w 7 is a value of the velocity for the cell 7, and this is the value for the velocities for cell 8 that is dou delta w. And this change is happening over a x distance of as you move from here to here, the x displacement is the difference between the x coordinates of the two points, so x c 7 minus x c 8. We said right in the beginning that the centroid of each cell needs to be worked out it is assume to be known and therefore, we can we know these values here.

So, similarly, we can write down dou w by dou p dou x dou y at point P as w 7 minus w 8 divide by y c 7 minus y c 8. We need to keep in mind that the same order will also appear here w 7 minus w 8 and x c is 7 minus x c 8, we cannot say that x c 8 minus x c 7. If you put here 8 and 7 then it is again 8 and 7 like this, so that is how we can we can evaluate this. And using this, we can write this down here as square root of 3 by 2 times w 7 w 8 divided by x c 7 minus x c 8 minus 1 half of w 7 minus w 8 divided by y c 7 minus y c 8. So, this is our n dot delta gradient of w at point p and this has to be multiplied by the length here l A B. So, we can take a smaller to be the length of the cell side, because we have chosen an equilateral triangle. This is side length A and A and a like that. So, we can just take that value here and therefore, the evaluation of this particular term here this particular term will now be given as square root of 3 by 2 times w 7 minus w 8 divide by x c 7 minus x c 8 minus 1 half of w 7 times a.

We similarly evaluate this quantity over cell side B, B C; and this quantity over cell side CA here and then we can work it out. We can add all the three and we have put that equal to C time area of 7. So, let us just do that and we can already see that the evaluation of this term has given us introduced the values of w 7 and w 8 and the coordinates. So, if you look at this term here side a is known here, and all these x c's, x c's are known and y c's are known here; and the only unknowns are w 7 and w 8. So, we can also bring in the whole thing and that will give us an algebraic equation. So, now let us go here to the side B C, and here we said this is Q, here we have this is n x. So, having evaluated for the side B A, let us go to side BC and we associate the midpoint as Q here, and the outward normal vector for side B C is vertical 90 degrees, because side BC is horizontal as per this diagram here.

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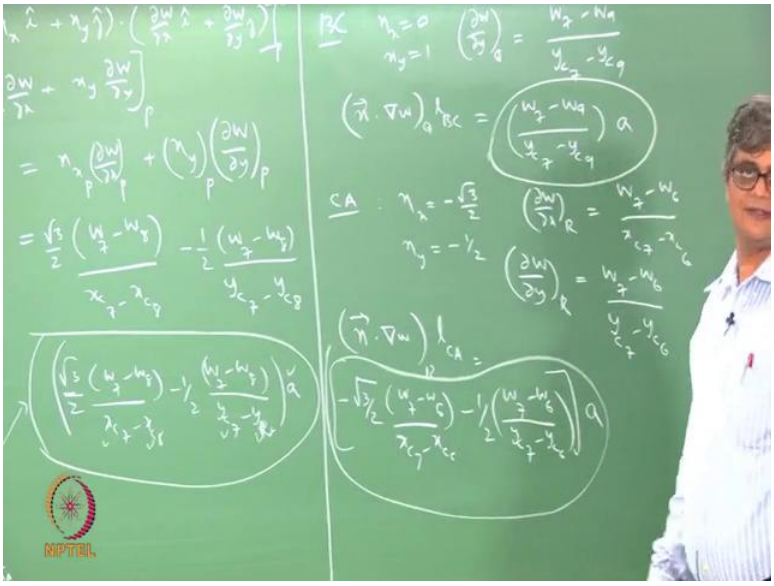

And therefore, for the side BC, we can say that n x is 0 and n y is 1. And if you look at this expression here, we are multiplying n x times dou w by dou x. If n x is 0, it does not matter what this dou w by dou x is we do not need to evaluate this. So, we only need to evaluate n y dou w by dou y. So, we can write dou w by dou y at point Q as the value of w at 7 minus the value of w at 9 which is a immediate neighbour upwards neighbour divided by the distance between the two vertical distance between the two. So, we can write this as w 7 minus w 9 divided by y c 7 minus y c 9. Why we brought in y c's and all these because it is only at the centroid, we know the value of w, either it is centroid of 7 and centroid of 9. So, we can write it like this.

And therefore, n dot gradient of w at Q times l B C will be equal to w 7 minus w 9 divided by y c 7 minus y c 9 times a, where a is a length of the side B C. Now we come to side C A here, again we put the centre of this face at R, and this has an outward normal vector like this. So, this has n x and n y, and this is again 30 degrees. So, it is actually 210 degrees. So, for side CA, we can say that n x is equal to minus square root of 3 by 2; and n y is equal to minus 1 by half. You find out what the angle is and then you substitute in the cosine theta then you should able to get this. And now we need to get dou w by dou x at R can again be written as w 7 minus w 6 divided by the horizontal distance between the two.

So, we can write this as w 7 minus w 6 divided by x c 7 minus x c 6, and dou w by dou y at R can be written as w 7 minus w 6 divided by y c 7 minus y c 6. So, n dot gradient of w for R times l c a can be written as square root of 3 by 2 times w 7 minus w 6 divide by x c 7 minus x c 6 are there is a minus here minus half times w 7 minus w 6 divide by y c 7 minus y c 6 and this whole thing times a. So, we can add this part, and this part, and this part, and set that equal to C times area of the cell, and that will give us the evaluation of this partial differential equation over the cell 7.

And we can see that when we do that, what we have is an equation of this plus this plus this equal to C times area, where c is a constant and area of the cell is known here, and the coordinates are all known and those are numbers. So, we have those become the coefficients of w 7, w 8 and all that, so that will give us a linear equation involving w 7, w 8, w 9 and w 6. So, the node itself and the three immediate nodes which are surrounding it and that becomes an algebraic equation for this particular cell.

And we do that for every cell and from that we can get 9 equations - 9 algebraic equations. And we need to solve all the 9 together obviously, because this equation cannot be solved on its own, because we need to have w 6 and w 8 and all that. When we solve all the 9 together, we will be getting the velocities at each of this and from that we can get the velocity profile or velocity contours and so on. So, the way that we discretize the governing equation using the finite volume approach is a bit more complicated than what it is for finite difference formula. In the case of finite difference, we directly substituted the finite difference expression here, and we immediately got it, whereas, here we have to evaluate the direction cosine of each of the faces and then the gradient at each of the faces and then go through all these derivation.

But the advantage of this finite volume approach this particular approach is that it can be applied to cell of any shape. Whereas, the finite difference approximation for this can be applied only on a structured grid where the grid points are at the intersection of lines of constant coordinates ah curves so that is a disadvantage with finite difference method. But if you can use finite difference method the discretization of the governing equation at any point is pretty straightforward in the case of finite volume method is much more tedious, but it can be done.

So, we will follow this up with a tutorial in which we will go back to this thing the same triangle and derive all the 9 equations using this particular approach. And at the end of that, we will have 9 equations for the 9 variables which we can solve.