Computational Fluid Dynamics
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Lecture - 61
The Generic Formulation for Turbulence

We have seen in the last lecture the turbulence closure problem, which is that if we time average the governing equations that is the continuity equation and the momentum balance equations, then we end up with equations in which the variables are indeed the time averaged quantities denoted here by the over bar.

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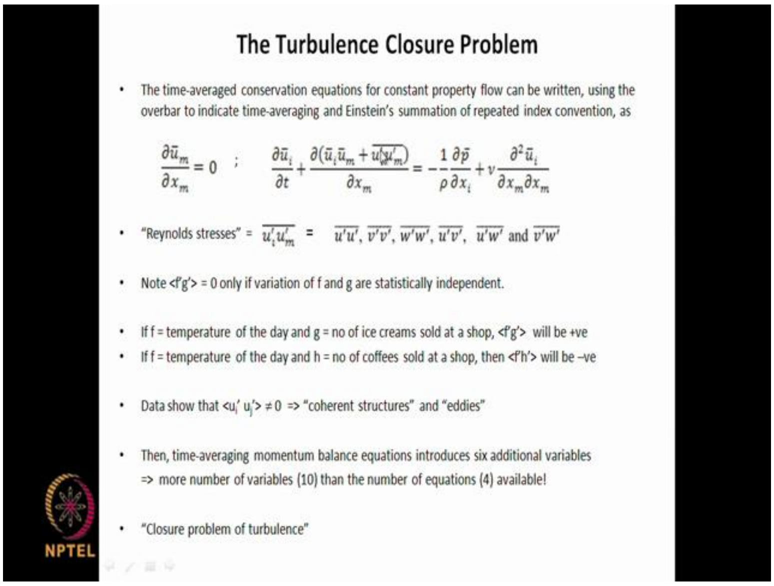

But in addition to the time averaged velocity components and time averaged pressure we also have this time averaged values of the fluctuating components u i m prime over bar.

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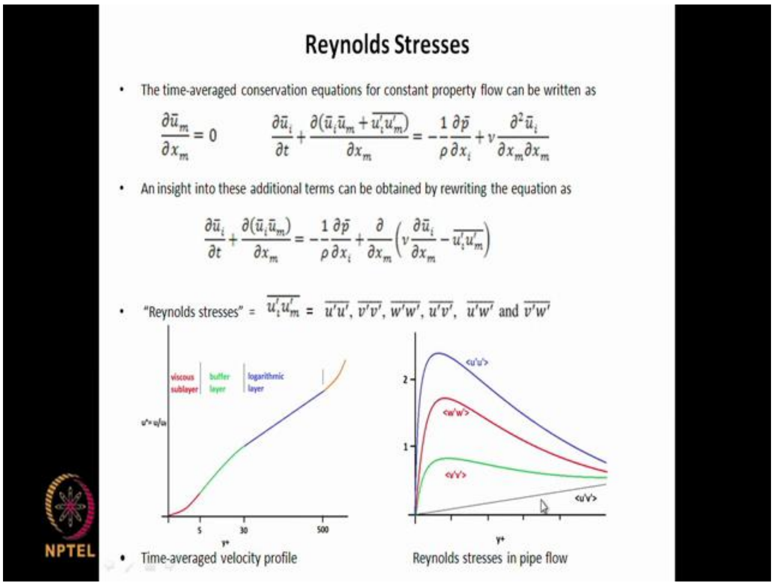

And we said that these are; in the general case these are non-0, and these also vary significantly within the computational domain and therefore, we do not know how they vary. And the variation depends on the type of flow which means that, they remain variables and unknowns this means that the time averaged equations are really like unknown things. And as a result of time averaging, we have the same 4 time averaged equations which involve together a total of 10 variables rather than 4 variables.

Now, we would like to look at how to close this problem and come up with a set of equations which has as many number of equations as the number of variables, then we can try to solve them numerically. So, in this in order to see what we have lost through the averaging process, we can rewrite the governing equation that we have here as the time averaged one, in such a way that these terms are taken on to the right hand side and we can put them in this way without making any change here when you take him to the right hand side you get minus sign here - you have dou by dou x m of this and you have dou by dou x m of this. But by clubbing this together with this term here which is the viscous system nu d u by d y for example, is a viscous stress.

So, we can see that these things that we have are somewhat like stresses and these are known as Reynolds stresses, in honour of a Reynolds who started this decomposition business and then came across this way of writing it. So, this Reynolds stresses are what make the turbulent flow, are those things that make turbulent flow different laminar flow and one manifestation of that is the time average velocity profile that you have in typically in turbulent flow.

And it is expressed in some special coordinates which also shows that we need some special treatment for turbulent flows and here for example, it is plotted as in terms of y plus where y is the non-dimensional normal distance from the wall, height from the wall and here is non-dimensionalized by y plus is defined as y u friction divided by nu we have seen that already, friction velocity. And the velocity itself is on the y axis is non-dimensionalised by dividing the velocity, parallel to the wall by the friction velocity.

If you do that you can see that this is on a linear scale and this is on a log scale. So, you have 530, 500 and so on like that, and up to y plus of 5 you have a linear variation of velocity with distance from the wall. And after that, you have a strongly non-linear region which is followed by quite an extended region where velocity varies logarithmically with y. So, you have u versus log y is actually a straight line like this. So, based on this you define a viscous sub layer for y plus less than 5 you have no turbulence and for y plus greater than about 30 and stretching up to 500 and even beyond, you have a linear variation between u and lawn y, it varies, velocity varies logarithmically with the distance in the wall and after that you have wake region in which you have strong effects comment picture.

Now, when you look at what these distances are in actual quantities this y u star by nu and in our example that we had u star as 0.1 meter per second and nu is 10 to the power minus 6 meter square per second. So, u star by nu is 10 to the power minus 5, is 10 to the power 5 so that means, that when you say y plus of 5; it actually means 5 times 10 to the power of minus 5; meters is the actual height and 5 times 10 to the power of minus 5 is 0.05 millimetres. So, this distance is 0.05 millimetres and y plus of 30 is 0.3 millimetres in a pipe having a radius of 25 millimetres.

So, these things are very, very close to the wall and then y plus of 500 is about 5 millimetres. So, all this is within 5 millimetres of the wall and you have this kind of variation which is very different from the variation that you would have for laminar flow and this kind of difference and the variation of all these things is the characteristic, difference between turbulent flow and laminar flow and that is caused by this term here dou by dou x m of u i prime u m prime bar.

If this is neglected, and if you solve these things you get exactly the same velocity profile as in laminar flow. So, the true inclusion of turbulent flow effects is determined by how we deal with these u i prime u i prime and we have no idea of what these things are because they vary in a non-monotonic, non-simple way something is going up rapidly within y plus of around 20. The peak value is reached here at y plus of some 10 to 15 y plus so that means, very, very close to the wall. And then it starts decreasing, again in a non-monotonic way; in a monotonic way, but in the non-linear way and the response of different stresses is different; different components are varying in a different way. There is no simple way algebraic way of representing the variation of these in the general case.

This is the case of turbulent flow fully developed turbulent flow in a pipe. So, this is what makes turbulent flow very complicated and very complex and turbulence even today is considered as one of the unsolved problems of physics not engineering, but physics. And many generations of physicists have worked on it and fluid dynamicists have worked on it to make significant advances of turbulent flow we understand much more we can predict much more today about turbulent flow than what we could have done over the past 100, 150 years.

We would not be able to deal all that, but we will be looking at some simple things that that are easy for us to understand, especially from the closure point of view. If we have 10 variables and 10 equations then how do we solve?

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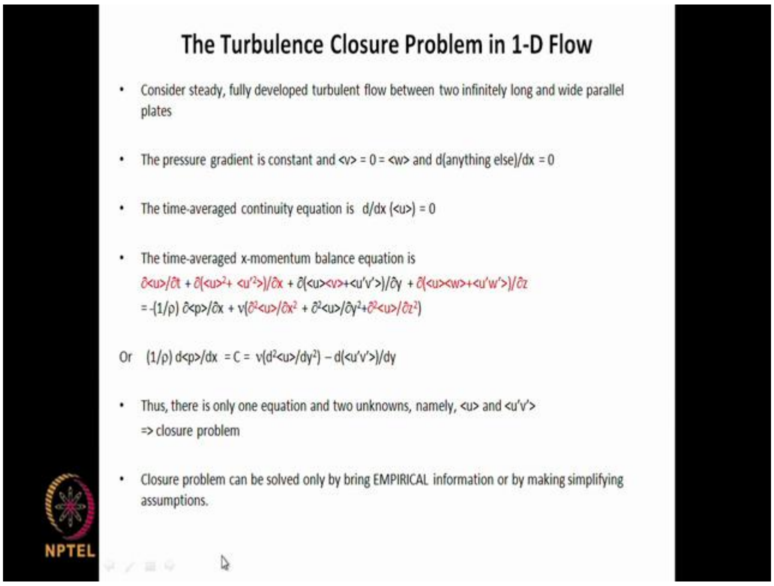

So, let us reduce complex to the problem from 3-D flow to 1-D flow and we are considering steady fully developed turbulent flow between two infinitely long and wide parallel plates. So, here you have pressure gradient is constant, the velocities v bar and w bar in the other two directions are 0 and only u bar is non-0. It is like 1-D flow that we know for the corresponding laminar flow between two infinitely long and wide parallel plates which is fully developed.

And since it is fully developed d by d x of anything else is 0. So, if you now look at what these things mean to the time averaged continuity equation, you get d by d x of u bar equal to 0 that is anyway understood because it is fully developed flow. So, there is nothing much coming from the continuity equation. The time averaged x momentum balance equation, where x is in the flow direction has this thing and all the red quantities here are 0. So that means, that this term drop out its steady state and variation with respect to x is 0. So, these two terms will go to 0 v bar is 0, so this one is 0 and this is non-zero and this is dou by dou y which is non-zero and here w bar is 0 and variation with respect to z of anything is 0. So, this term is 0.

And pressure gradient is constant, variation with respect to x is 0 and variation with respect to z is 0. So, you have a resulting equation which is like this, a pressure gradient which is a given constant is equal to nu dou square u by dou x square or in this case d square u bar by d y square because it varies only with y, minus d by d y of u prime u prime bar. Now you have this single equation, continuity equation does not give us any information. Single equation involving two variables u bar which is a function of y and this u prime v prime bar it is again a function of y.

So, even in this particular case of the simplest possible condition of fully developed steady one dimensional turbulent flow, we still have the problem closure that we have more number of variables and the number of equations available. So, this is the turbulence flow of closure problem and it can be solved only by bringing empirical information or by making simplifying assumptions. And both have been done over the past century.

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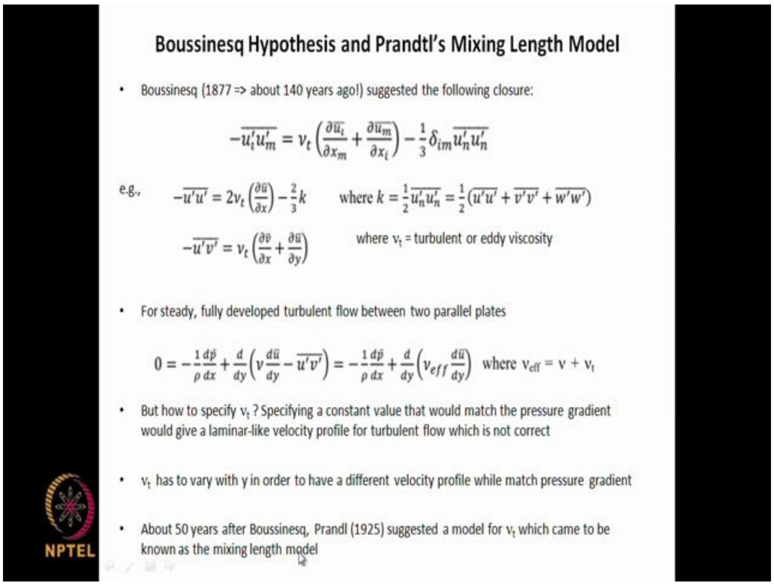

Going back to 1877, so which is about 140 years ago where Boussinesq made the hypothesis that this u i prime u m prime which are like stresses can be represented as a turbulent stress times dou u i bar by dou x m plus dou u m by dou x i. So, this is like exactly the way that we are treating tau i j according to the navier stokes equations. It is written as nu dou u i by dou x j plus dou u j by dou x i plus two-thirds of a plus lambda times dou u m by dou x m. So, this similar kind of formulation is brought in here except that instead of the kinematic viscosity now we have eddy viscosity. This nu t, subscript t is turbulent or eddy viscosity, kinematic viscosity and this particular term is made sure that when you sum all these normal components we do not get 0. So, it is just something like this.

So, if you put for example, u prime u prime bar. So, that is i equal to 1 and m equal to 1 in this you get 2 nu dou u bar by dou x minus two-thirds of k, where k is half of u prime square plus v prime square plus w prime square bar quantities. This is the turbulent kinetic energy which we have seen earlier, and similarly if you want u prime v prime bar which is what we want here and that is given as nu t times. So, in this case i equal to 1 and m equal to 2. So, you put i equal to 1 and m equal to 2 you get dou u bar by dou y and then you put m equal to 2 here and i equal to 1 you get dou v bar by dou x.

So, this is the way of writing this. So, these Reynolds stresses are expressed in terms of eddy viscosity and the conventional treatment of viscous stresses. So, if you were to substitute this thing here, this is the equation that we have of fully developed turbulent flow between parallel plates and you substitute this expression here and you can write this as d by d y of nu effective time d u bar by d y. So, this is almost like the laminar flow except that you have new effective which is nu plus nu t.

Now you can say that, now let me put nu t to be some constant and if you put nu t to be a constant in such a way that the pressure gradient in turbulent flow would match with what is predicted here. For the same flow rate then you would have the difficulty that the solution of this with a constant nu t since nu is also constant, that means, that nu effective is constant that means, this should be like a parabolic velocity profile, but we know that in turbulent flow we do not have a parabolic velocity profile.

We have just now shown a variation which is quite characteristic of turbulent flow which is different from laminar flow and you cannot get that by making nu t as a constant. So, nu t has to vary with y in order to have a different velocity profile, if you make nu t as constant then this whole thing is exactly like a fluid with a higher viscosity. But that is not all what turbulence is turbulent flow is.

So, it must have a different velocity profile by matching pressure gradient; small mistakes here. So, the quest for a proper formulation of nu t given that this possibility is the possible way of modelling after 50 years has led to success through Prandtl's Mixing Length Model, where prandtl in 1925. This prandtl is the same prandtl of the prandtl number for heat transfer and all that. So, this model has come to be known as the mixing length model. So, we will just see what it is.

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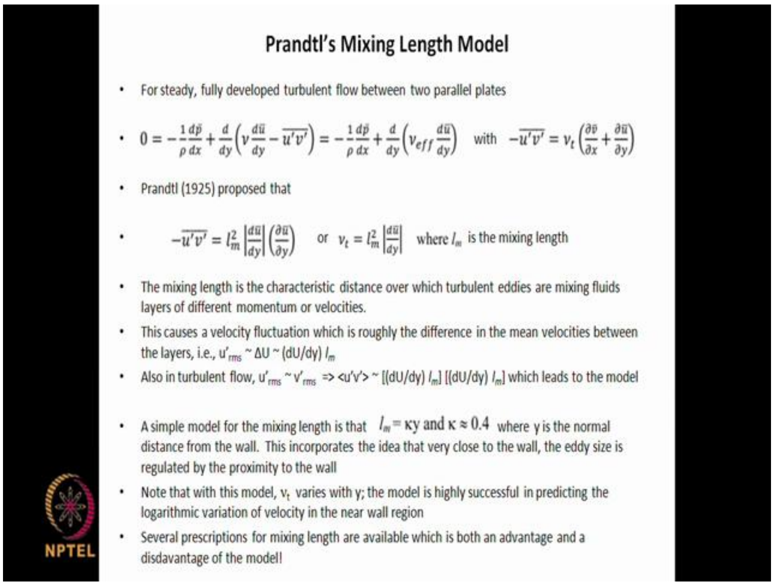

For steady fully developed turbulent flow between parallel plates we have this expression, once you substitute the boussinesq eddy viscosity hypothesis you have this and prandtl suggested that this u prime v prime bar it should be written as l m square times d u bar by d y times d u bar by d y. Where l m is the mixing length, and if you now were to write this because it is 1-D flow dou v bar dou x is 0. So, u prime v prime bar is nothing, but nu t times d u bar by d y. So, if you would take this out that will give you a nu t of l m square times d u bar by d y where l m is a mixing length. Why this is made in this particular way? There can be many picturizations of this the mixing length is can be seen as a characteristic distance over which turbulent eddies are mixing fluid layers having different momentum and velocity.

So, this mixing of different fluid layers causes a velocity fluctuation. For example, you have a low velocity moving layer and a high velocity moving layer above it. And if because of turbulent motion this fluid from here having this low velocity comes here then that immediately produces a velocity fluctuation which is characteristic of the velocity difference between the two layers, and so that characteristic velocity is delta u where u is the time averaged velocity which is function of y. So, therefore, you can write this delta u as d u bar by d y times l m the mixing length. So, this gives us an idea of the mean velocity fluctuation u prime and in turbulent flow all the velocity fluctuations are roughly of the same order of magnitude.

So, you can say that u prime rms is roughly equal to v prime rms. So, that if you are trying to evaluate this u prime v prime bar, you can now say that this is proportional to d u by d y times l m and this also proportional to this and that gives you l m square times d u by d y, whole square that is this one. Since we want to have a viscosity which is always positive, we write this as mixing length square times modulus of d u by d y. So, that is the way that we can explain prandtl mixing length model.

And what this actually is not over because we still need to specify what the mixing length is. So, that many possibility proposals have been made for mixing length - one simple thing is that mixing length is now being looked upon as an eddy which is causing this mixing and if you have a wall here then you cannot have a huge eddy, the size of the eddy close to the wall is governed by the distance from the wall.

That means, that you can say that mixing length is proportional to distance from the wall and based on measurements of the velocity profile and all that we suggested this kappa proportional to constant to be 0.4. So, if you were to put this kappa y 0.4 y in this then you know nu t and if you know nu t then you can evaluate this and then you can put this here and you can predict for a given pressure gradient what the velocity profile is and you can see how well it matches. And thereby you can try to find some constant kappa which matches with the measured velocity profile and that is how this value of 0.4 was arrived at.

So, this particular model is such that with l m as being proportional to kappa times y and mixing length itself, viscosity itself being proportional to l m square, that is y square and then you have d u by d y which itself varies with y. That means, that in this model you have nu t viscous turbulent viscosity which is a function of y and we said that this functional dependence of y is necessary for us to get a velocity profile which is different from that of laminar flow velocity profile. But it is this particular form which actually gives us a velocity profile which matches with what is measured experimentally.

So, in that sense that is why it is been so difficult to get this and that is why this particular model has survived the test of times and it is still used in for very simplistic things. It has been very successful, highly successful in predicting the logarithmic variation of velocity in the near wall region, and several modifications have been proposed for this. This model is with those kind of modifications is supposed to work very well in the near wall region. So, that is y plus of maybe up to 200 up to 500 like that.

As we go deeper and deeper into the flow and farther and farther away from the wall then other effects come in, so at that point this model will be failing a bit. So, what we have considered is essentially flow along the wall, parallel to the wall. But the same idea can also be extended to other types of flows for example, flow between two mixing layers - we have high velocity jet and low velocity jet, they mix and then they form a mixing layer which is growing bigger and bigger.

And if we want to look at what is the expansion of this kind of jet, rate of expansion with distance then you can again come up with a prescription for mixing length you can also have wake flow, you have a bluff body behind that you have low velocity region which again ultimately becomes fully developed flow and the velocity in the wake region can also be predicted by having it is own mixing length model.

So, there have been several kinds of mixing length models proposed for different kinds of flows. These several models can be considered both an advantage and also disadvantage, when you look at the model. It is an advantage because the model can be applied not just for flows which are parallel to the wall in pipe flow, but they can also be applied to boundary layer flow, they can be applied for as I mentioned flow over a sphere type of things and flow defined bluff bodies, wedge flow all these kind of flows can be treated with an appropriate model for the mixing length. That enables us to do turbulent flow calculations over these kinds of aerofoils and those types of practical applications.

But it is also a disadvantage because, for each case you have to come up with a mixing length model and how do you know which one is correct. You have to do experiment and if you have to do experiments and fix this value then what are we actually getting from the prediction. So, there is that kind of problem that is there.

In the next lecture, we will look for something more generic where you do not have to specify this mixing length and then, if we can have a model which calculates turbulent viscosity without having to have this (Refer Time: 24:08) prescription for mixing length then that would be more beneficial. And that is the kind of model that has been developed much later in the 1940s 50s 60s, there have been lots of developments on turbulent flows and we look at once a generic model as part of this course.