Computational Fluid Dynamics
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Lesson - 60
The Turbulent closure problem

In the last lecture, we have seen about Time scales and Length scales of turbulent flow.

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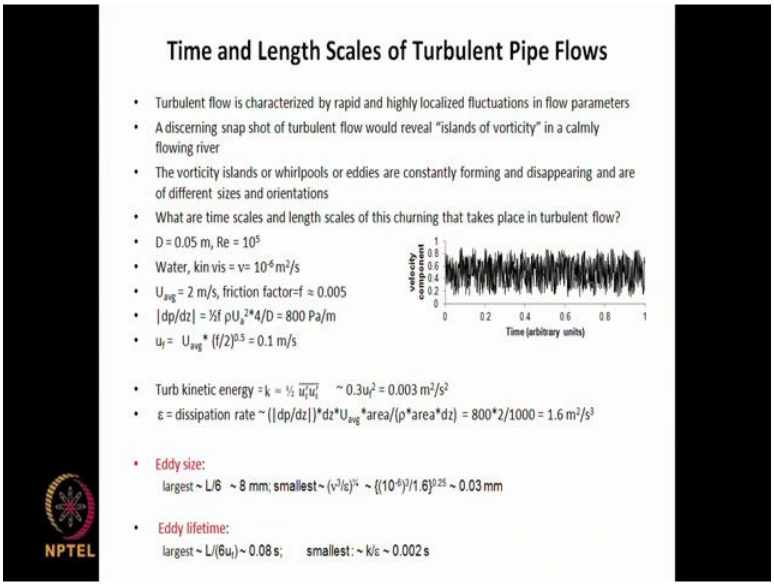

We estimated that for flow through a pipe at a Reynolds number of 10 to the power 5 that is 1 lakh. We estimated the size of eddies and the lifetime of eddies which is of the order of fractions of millimeter and a few milliseconds is the time fluctuations associated with this eddies.

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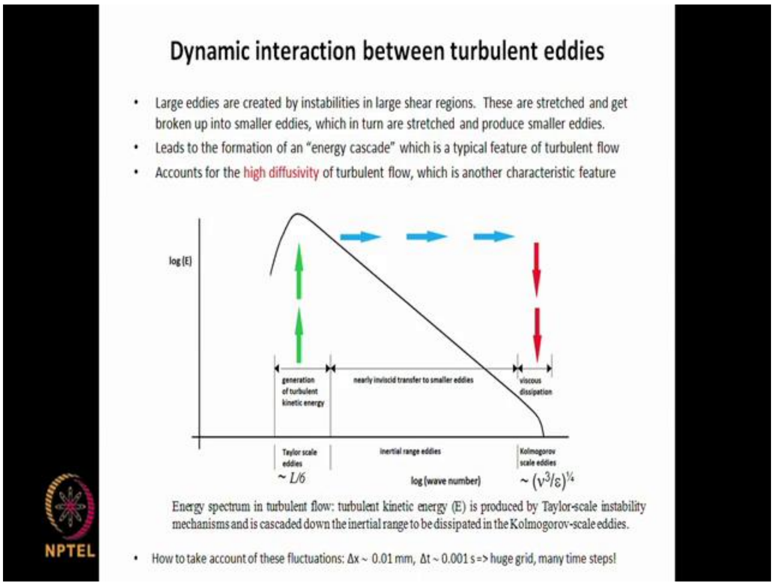

We also made the point that eddies of different sizes are there, we have very largest eddies which are created by instabilities in regions of high shear; and through the action of hydrodynamic forces, as a result of a fluid motion, these eddies get broken up into smaller and smaller eddies. And the smallest size that can be sustained is the Kolmogorov eddy size which we said was of the order of 0.1, 02 millimeters.

And the question was that what could we do when we are dealt given this nature of these turbulent fluctuations, where we have very small eddies compare to the typical length dimension of the flow or length dimension of the computational domain and even length dimension of the grid spacing that we have. Given that these fluctuations are much smaller than this length dimensions, given that the time variations of the order of milliseconds is much less than the typical time variations that we would be interested in, in terms of the system behavior or equipment behavior. Is it necessary for us to make to tackle these fluctuations?

The other aspect of this is that these turbulent fluctuations render an essentially flow through a pipe to be fully three-dimensional. If we were to take cylindrical coordinate system for a circular pipe, fully develop flow through a pipe is 1-D, but this will make it these fluctuations will make it 3-D. Similarly, steady flow through a pipe is time independent, but these fluctuations will make it time independent, so that means that no matter whether the flow is 1-D, or steady or 2-D, we have to solve all the time the three-dimensional unsteady form our equations with very, very fine grid and so that makes it wonder, we should wonder whether it is necessary to take in to account all these fluctuations what we call as a resolve these fluctuations that is do we expect, do we want our solution to show these kind of variations that are associated to turbulent flow.

Or is it possible to just smoothen this fluctuation which is also what we do in many c f d computations when we have oscillations when we have oscillative solution we try to damp and those oscillative solutions with smoothing terms. So, is it possible to just smooth out these fluctuations and get smooth solution?

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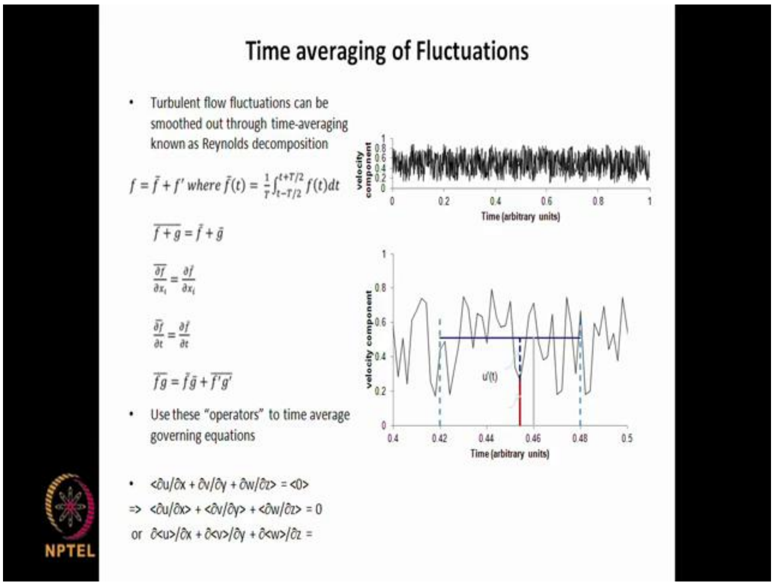

So, this is where the concept of time averaging of fluctuations has come about. Here, we have a typical time variation that is shown in the top figure top, right hand figure, where you see that over a time unit from 0 to 1, there are quite rapid fluctuations. A part of that is shown in the bottom figure in which the same time unit is varying between 0.4 and 0.5. And when you expand the time scale like this, then you can see a variation which again looks quite a bit chaotic, but now we can see the individual points how it is changing from time to time at every time step. So, if you take an interval, for example, if you take the time variation between 0.42 and 0.48, which is what we have taken here.

So, we can introduce the concept of time averaging, where if f is a parameter, it is a flow parameter like the velocity component in one direction in any direction, then we can say that its fluctuating about a mean quantity, and the mean quantity is given here as indicated by the over bar that is f over bar. And at the velocity value the parameter value f at any time is decomposed into a mean component and a fluctuating component, which is indicated with a prime so that is f is equal to f bar plus f time, where f bar that is the time average quantity is time average of f t d t over a time interval which is capital T, so that is what is shown here.

And with reference to the figure at the bottom, where you see the time variation between 0.4and 0.5, if you take the time interval capital T to be 0.06 as shown in the figure then the average value at 0.45 is integral of f b T between 0.42 and 0.48 divided by the time duration 0.06 and that will give you the horizontal line that which indicates the average value o at over that interval. So, this is the average value over this time interval now if you take this particular time, time instant then we can say that the red one is the actual value u of t at that particular time instant. And this is decomposed into the mean value which is this dotted line all web here and the fluctuating component is this part.

So, the mean is this whole thing plus the fluctuating component which happens to be minus at this point, so that the actual value is less than the mean value here. If you go to some other time, for example, here then you have a mean value plus a fluctuating value, which is positive. Again here a fluctuating value is what is above this and the mean value is what is below this. So, over a particular time period capital T, the value the fluctuating component is both positive and negative within the time period where we are averaging the time average values is constant. So, the time average value such that if you integrate the fluctuating component over the time interval, then it becomes zero. So, this is how we can we define the time averaging of any particular parameter f.

In fluid flow, we have several parameters we have u component v component w component we have pressure all these things are simultaneously varying, temperatures varying concentration all these things. So, this time averaging operator is such that if you take the time average of the addition of two parameters f plus g, so time average of f plus g is just the sum of the time average quantities, so that is what is shown in the first relation here. Time average of f plus g over bar is equal to f bar plus g bar.

Similarly, time average of a derivative that is dou f by dou x i time average is the derivative of the time average quantity, so that is dou by dou x i of f bar which is the next relation. And the relation after that says that even if you are looking at time derivative of f that is dou f by dou t and then you take time average of that then that is dou by dou t of f bar.

So, even time average of a time derivative is the time derivative of the average quantity, but there is a difference between the left hand side and right hand side of this particular thing here. On the left hand side, when we talk about dou f by dou t that is variations over very small times of the order of milliseconds, and then microseconds. On the right hand side, when we are talk about time average of the that derive a time derivative of the average quantity, we are talking about time steps time intervals delta t s which have much, much larger than the capital T that you have used for time averaging.

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Given that the turbulent fluctuations in turbulent flow are very rapid and highly localized; to such an extent that these are much smaller than the typical length scales and time scales of actual flow variations. For example, length of the computational domain or the time scale of simulation that we may want to do like 1 hour or 10 minutes or like that, and that these can even be smaller than the grid spacing that we typically employ. And the delta t is that we usually use for simulations given all these things is it possible to smooth over this fluctuations by doing some kind of averaging.

This is the idea behind time averaging of fluctuations, where if you take a parameter f, which can be the velocity - the instantaneous velocity component in a particular direction pressure, or temperature, concentration of a species any parameter can be decomposed into a sum of a time average value f bar plus a fluctuating value f prime. Where the time average value is evaluated as integral of f dt over a time period which is capital T by 2 before and capital T by 2 after.

You take this example here this is actual time variation. Within that we are focusing over a smaller window between 0.4 and 0.5 here. And since we are expanded the window, we can see a more the variation more clearly here which like this. And if you choose capital T the time period over which we are averaging to be 0.06; so that at 0.45, we want to know what is the time average value. We integrated between 0.42 and 0.48 and then we say that the average value over this time interval is this value here.

Now within this interval, the value of f is changing; and at specifically at this time, the true value is what is shown in the red one. And this value is decomposed into a time average value which is this whole thing plus the fluctuating value which is given by the dotted line here. And in this particular case the fluctuating value is negative, so that the time average value minus this value has given as this one here.

If you take some other instant may be at this particular instant here, now you have the instantaneous value is greater than the mean value so that means, here u plus f plus is positive. So, over this time interval, f plus f prime is both positive and negative, in such a way that the average value of f prime over this time interval is 0 and average value of f over this time interval is f bar. So, at any time instant, f is decomposed into f bar plus f prime. In a case, where f bar is constant with time, for example, you take this thing if you take if you take the average here then it is almost looks like it is constant. Then if f bar is constant for every parameter then you can say that it is a steady turbulent flow in which a time average quantities are steady means that you have a steady turbulent flow.

So, even though within this time period over which you are integrating f is constantly changing, you can still claim it to be a steady turbulent. So, this f is therefore, some integrating function, so the f bar - the time averaging is an integrating operator here. Now, when we write our equations we are always writing in terms of this plus this term plus this term and like that, and in a particular term, you can have different parameters. Here, we are taking two parameters here. And if you take the time average of sum of f and g, where each of f and g is a different parameter, then you can plug this into this you can show that the time average of f plus g is just the time sum of the time average quantities f bar and g bar.

And the time average of a space derivative that is time average of dou f by dou x i is equal to dou by dou x i of the time average quantity. So, there is dou by dou x i of f bar. This applies even to the time derivative you can say the time average of dou f by dou t is equal to dou by dou t of f bar. The difference between the left hand side and the right hand side here is that in the left hand side this dou f by dou t can be where this delta t here can be a lot of milliseconds. And here this delta t that we are talking about to describe the time variation of the time average quantity, this delta t must be much, much larger than the slowest of turbulent fluctuations.

In the case of pipe flow, the slowest fluctuation was associated with the largest eddy and that was at the order of 80 milliseconds. So, this delta t that appears on the right hand side here must be much, much bigger than 0.08 seconds. So, may be the order of 10 seconds; 10 seconds is still quite a lot small in terms of the typical time variation of a system or an equipment that we are interested in.

If you are looking at start up of a chemical reactor or a plant, then that may take hours and to get a value every 10 seconds is still pretty good resolution time resolution of how the system parameters are changing over this hours, or over this days during which the start up takes place. So, this particular approximation that as a result of time averaging, we are no longer allow to talk about changes over milliseconds, but may be changes over seconds and tens of seconds is not to severe and approximation, and we can make this kind of thing.

Now interesting thing happens, if you take the product of two quantities and time average it, the time average of a product f g like product of u v that appears dou by dou by of u v is a term which appears in the momentum equations and f is u and g is v. So, if you write down the time average of this product then you can show that it is a product of the time averages that is equal to f bar plus g bar plus the time average of the fluctuating components f prime and g prime.

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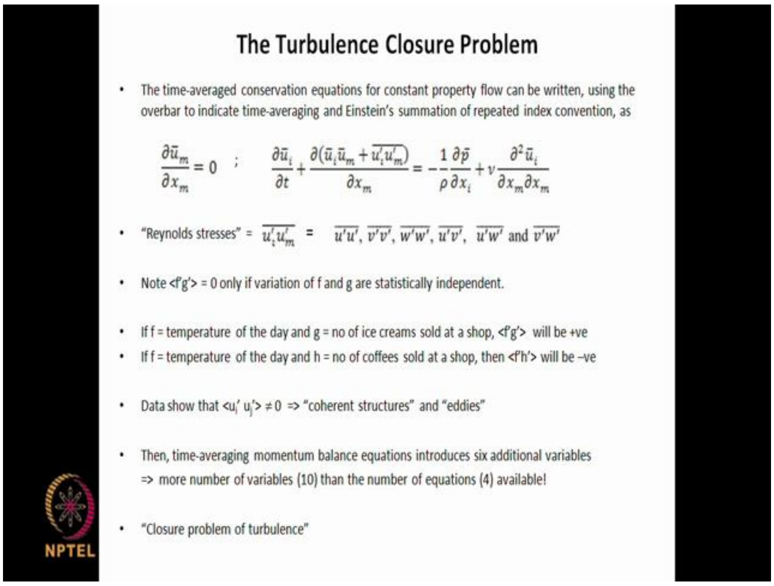

So, this f prime and g prime are that time average quantities not necessarily zero, and they are zero only if f and g are statistically independent. For example, if you say that f is the temperature of the day at a particular shop, and g is the number of ice creams sold at that particular shop. Then every day there is going to be variation in terms of temperature of the day, the peak temperature of the day. And then every day there is going to be variation in terms of the number of ice creams that have been sold in that particular shop on the particular day.

But if temperature is high on a particular day then more people prefer ice creams, so f bar is the variation from the mean value of the temperature on a particular day. On a particular day, it may be hot day so that means, f prime is positive. And g prime is the number of ice creams sold the variation on that particular day compare to the mean value. Since, on a hot day more people prefer ice creams on that particular day you will see more ice creams sold, so g prime is going to be positive. So that means, that on the average you can expect the time average of f prime g prime, the time average is indicated by these greater than and less than symbols here, this will be positive.

Now if you look at the same shop and then it is also selling coffees and since on a hot day fewer people prefer hot coffees over ice creams; then you can say that on a hot day the coffees sold that number will dip so that means, that h prime which is the variation of number of coffees or the mean on a hot day f prime is positive and h prime is negative. So, on a time average basis you can say that the f prime h prime will be negative. So, these f prime g prime bars are not necessarily zero, and they depend on what kind of process we are looking at.

So, we cannot straight away say that at the time of time averaging this f prime g prime bar here can be reduce to zero. So, this cannot be made zero it can be zero only if you know that f and g are statistically independent they are statistically independently varying. So, otherwise we have to retain this using this operators of time averaging, we can time average a governing equations, so that we can reduce we can eliminate the influence of this rapid fluctuations from that.

So, if you take the continuity equation you have in three d incompressible flow you have dou by dou x plus dou v by dou y plus dou w by dou z equal to 0. Now that is an equation and we can do any mathematical operation on that and we should be with that.

So, we do the integration as given here and then so effectively we are doing time averaging of both sides of the equation and so you get time average of the LHS is equal to time average of the right RHS and this would not change the equation the validity of this conservation equation.

Now if you now look at this there are three terms its sum of three terms we can apply this rule here and we can write this as time average of the individual derivatives. So, time average of dou u by dou x plus time average of dou v by dou y plus time average of dou w by dou z equal to 0. Now we look at this and then we can apply the derivative rule here and then we can write this as dou by dou x of time average u here dou by dou y of time average v and dou by dou z of time average z equal to 0.

So, by doing this time averaging, we have converted this instantaneous equation, the mass conservation equation in which the instantaneous values u, v, w are appearing from that into we have converted into mass conservation equation in which only the time average values are coming. And what is the game that we have made time average quantities will change much, much more slowly so that means that you can have much bigger time steps than what is needed if you want to solve the original equation. And that is the idea that is advantage that we wish to gain through time averaging average out this fluctuations.

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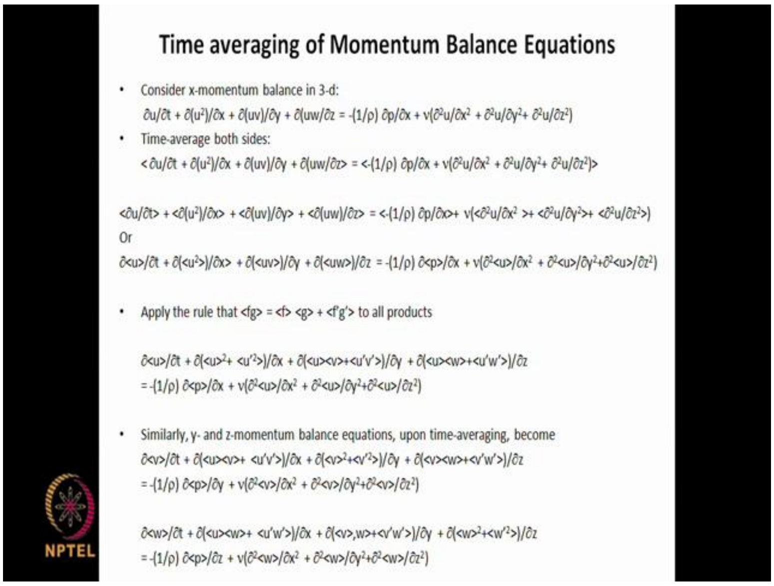

Now if you do the same thing for the momentum equation slightly more difficult here, but the same idea will be there. So, you have considered the x-momentum balance equation in 3-D. So, you have dou u by dou t for incompressible constant properties. So, you have dou u by dou t plus dou by dou x of u square plus dou by dou by of u v plus dou by dou z of u w equal to minus 1 by rho, rho is constant incompressible flow, dou p by dou x plus mu the kinematic viscosity which is constant dou square u by dou x square plus dou square u by dou y square plus dou square u by dou z square.

We can neglect the influence of gravity here. So, we time average both sides, so time average of LHS equal to time average of the RHS. And since this is some of these terms, we can write this as individual terms time average of dou u by dou t plus time average of dou by dou x of u square time average of dou by dou y of u v like that.

And similarly, time average of the pressure gradient term, time average of the individual viscosity terms here like this. And we can apply the corresponding rules that we had earlier for the derivatives time average derivatives. So, we can write this as time average of u bar and here we have time average of u square and you have dou by dou y of time average of u v dou by dou z of time average u, w. And here you have rho is anyway constant, so dou by dou x of p p bar and here say two derivatives are there. So, they come out of the time averaging. So, you can say dou square by dou x square of u bar dou square by dou y square of u bar dou square by dou z square of u bar. Now, on the left hand side here we see dou by dou x of u square, so that is a product u times u and here you have dou by dou y of u v that is again product of u v.

And we know that time average of f g is the product of the time average quantities plus the time average of the fluctuating quantities. So, we can write this as dou by dou t of u bar here plus dou by dou x of u bar square plus u prime square bar. So, this is the time average of the fluctuating component the product of fluctuating component because g and f are the same u here you get u prime square. And this time here becomes dou by dou y of u bar v bar plus time average of u prime v prime bar; and here again you have dou by dou z of u bar w bar plus time average of u prime w prime. So, we have this equation here.

We can write similarly the y-momentum and z-momentum equations. And they become dou by dou t of v bar plus dou by dou x of u bar v bar plus u prime v prime bar, and dou by dou y of v bar square plus v prime square bar. And we have already mentioned that this is not necessarily zero and this is not necessarily zero that depends on whether u and v fluctuate independently or there is some correlation between them. So, like this and you have the z-momentum equation is dou by dou t of w bar plus dou by dou dou x of u bar w bar like this.

Now if you look at these equations what we would like to have are the equations momentum equations conservation equations expressed in terms of time average quantities, so that we can make use of large delta t and large delta x. And we do not need to have a very fine grid and very fine time step. And when you look at that we can see that that is being satisfied by these conservation equations, because every term has a derivative and that derivative is talking only about time average quantities. So, from that point of you, we have succeeded in representing the momentum and continuity equations mass conservation equations in terms of time average quantities.

But we have a problem and the problem is that we can write the overall equations for all of them using Einstein's summation of repeated index convention that is in this term you have m is a subscript which is an index which is being repeated here, because it appears twice in this term. So, then that means, that this implies sum over the three quantities m equal to 1, m equal to 2, m equal to 3 and that gives us dou u 1 by dou x 1 plus dou u 2 by dou x 2 plus dou u 3 by dou x 3, and that becomes the time average co the continuity equation. And similarly the time average momentum equation is like this.

Now what do we see here we see that you have the time average velocities here and you have time average velocities appearing here, but here is a new term u prime bar u i prime bar, and u i prime time u m prime over bar, and this is the average time average pressure and this time average velocity.

So, this term and this term, these terms and these terms are not those are the ones that you wanted. But in the process we also got these extra terms and these terms is zero only if the u and v are independently changing. And we know that u prime u prime bar is not zero, it is zero because u prime is positive and u prime is negative, but when you take u prime bar times u prime time u prime then that is always positive. So, the only way that u prime u prime bar is zero is when u prime is zero completely all the time that is when you do not have turbulent fluctuations when you have laminar flow.

But if you have turbulent flow then u prime square bar is not zero. And similarly, v prime square bar is not zero. And these terms are appearing the momentum equations so that means, that you have a problem as a result of fluctuations, as a result of time averaging, you are introducing six additional variables that is u prime square bar, v prime square bar, w prime square bar, u prime v prime bar, u prime w prime bar, and v prime w prime bar. And actually measurement shows that these are not zero.

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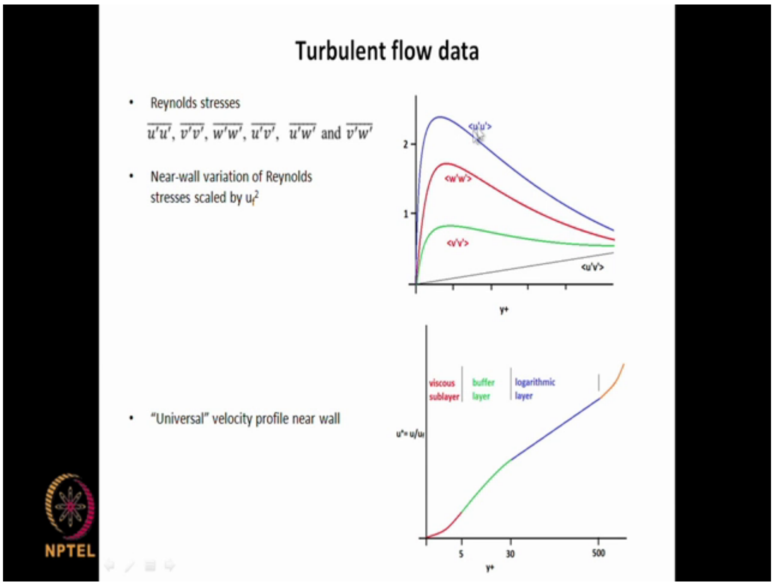

For example, this is a typical variation of u prime square bar, and v prime square bar, w prime square bar, and u prime v prime bar in pipe flow and these are varying we will see more about this with respect to space and so these are not zero.

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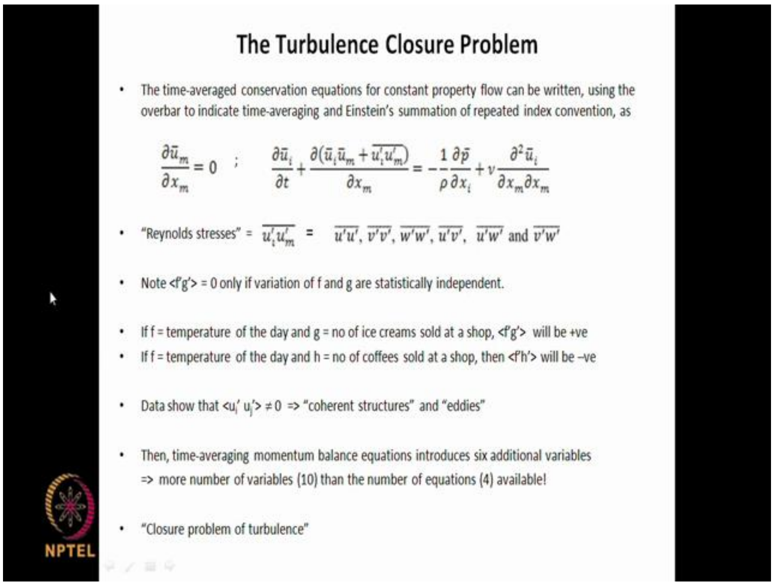

So, if these are not zero, and these are remaining in the equations so that means that this set of four equations that is one continuity equation, three momentum equations, we have four equations. And the feature 10 variables you have the three velocity components that is u bar, v bar, w bar, we have p bar the pressure the time average pressure and in addition to that you have these six components six terms which are known as Reynolds stresses.

We will see why this called Reynolds stresses and these are also non-zero, so that means that you have ten variables and you have only 4 equations and so that means, that as a result of time averaging you have introduced more number of variables you have not got more number of equations. So, you had the same four equations, but involving time average quantities and six of the time average quantities which are not zero and their variation is not known apriori.

And simple measurement show that even for simple flows they vary in a complicated non simple way. So, this is what is known as the turbulence closure problem. When we want to do the time averaging of equations in order to smooth out these oscillations and represent the conservation equations in terms of time average parameters of the flow then in the process in some of the equations, we have extra terms and these extra terms are such that these are not necessarily zero.

And we do not know how they vary in the general case, because in even in simple case of fully develop pipe flow as we move away from the wall which is given here as dimensionless distance from the wall here that is why we put it as y plus dimensionless normal distance from the wall then each of these quantities is varying in a non-monotonic way. They are going up to a peak and then they falling back like this. And each of them is reaching different values and slightly a different times and this u prime v prime bar is also changing. And so the fact that these are exhibiting a complicated special variation and which is not known apriori means that these terms will remain as variables in the equations.

And if these are variables in the equations, we have ten variables and you having the four equations. So, you cannot solve the problem. You do not have a close problem you have problem of closure of the mathematical problem involving time average governing equations, and we need to do something more about these equations, we need to bring in extra equations, so that will be the subject of the next lecture.

In the next lecture, we will see what kind of closure models we can bring which will give us reasonable solutions for the turbulent flow.