Computational Fluid Dynamics

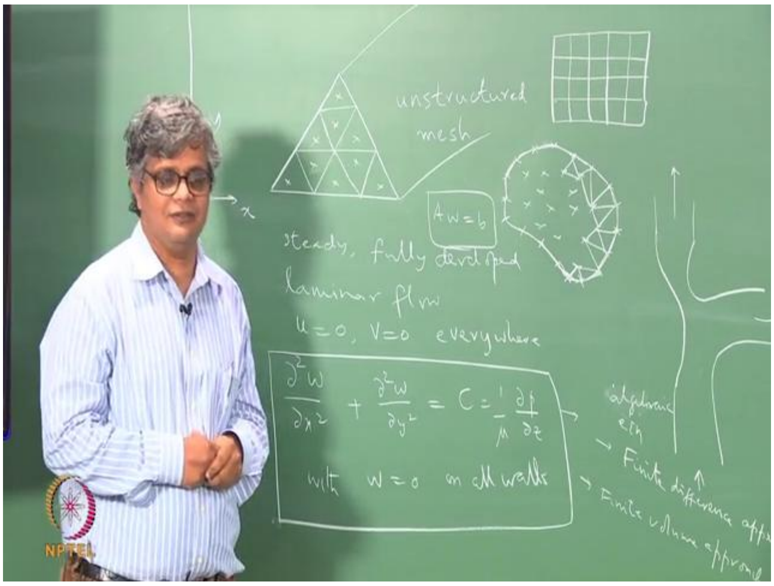
Prof. Sreenivas Jayanti
Department of Computer Science and Engineering
Indian Institute of Technology, Madras

Lecture – 06
Flow in a triangular duct: Problem formulation

In the last lecture, we have seen the essence of computational fluid dynamics approach to the solution of a partial differential equation. We discretize the flow domain into small cells, we distribute a number of grid points within the flow domain and then, we write finite difference approximations for each of the derivatives that occur in that equation and then, convert at that particular point, the partial differential equation into an approximate difference equation featuring the variable values at the grid points as the unknowns in that equation and usually this equation requires knowledge of the neighbouring values at the neighbouring grid points also. So, we have to do this for every grid point, where we want to have a solution and then, in the process we also make use of the boundary conditions and we may convert the partial differential equation into a set of algebraic equation and then we go through some specialised techniques for the solution of this algebraic equation.

Now, this particular approach is very good. It is flexible and before we go much more deeply into the underlying concepts which are actually given us an idea how to go about treat systematically for complicated cases, we would like to introduce different kind of problem which again looks very simple from the outside, but when you actually want to deal with it, it becomes a bit more complicated and this is the case where we would like to look flow through a triangular duct a fully develop flow through a triangular duct which is different from what we had earlier.

(Refer Slide Time: 02:13).



This is a duct of triangular cross-section and again we have x going like this, y vertical and z going in in the outward normal direction as per the right hand coordinate system here. So, you have flow going through this and we would like to see how the velocity changes here. Again here we have the case of steady fully developed laminar flow and we will see that again we have the case of u equal to 0, v equal to 0 everywhere and w is not equal to 0 and its governed by the same equation that we had in the previous class which is dou square w by dou y square this dou square w by dou x square plus dou square w by dou x square equal to c, where c is a pressure gradient is a constant involving the pressure gradient in the z direction. So, we have this and we need to solve this for this particular shape with the same boundary condition as before that is with w equal to 0 on all walls as the boundary conditions and this is the problem that we like to solve.

So, if we want to go about this problem in the same way as we had done earlier, we would have to put points at intersection of coordinate lines. Remember that for the rectangular that we drew constant lines of constant y like this and constant x like this and then, we put the grid points and then, we express the derivatives in terms of their neighbouring points here and here if you want to do the same thing, we will have difficulty because this line here is inclined and this line is inclined and it is not possible to have a similar kind of grid in this triangular cross-section and this problem persist not just for triangular cross section, even for a cross section like this, even for a shape like this and in such cases simplistically speaking, it is very difficult to use this way of doing the CFD solution

These kinds of shapes are not uncommon. In fact, as we saw in the example of the t junction where we are looking at shaping the junction in a certain way like this, we have a shape here which cannot be put in a mesh like this in Cartesian and mesh like this. So, we still need to compute the flow going through this and then, coming out partly and coming out partly here. So, when we have these kinds of irregular geometries, then the way that we have used here that is using lines along lines of constant values of x and y to identify the grid points is not going to work easily. So, it is very cumbersome and it is going to give us a lot of problems and it brings in approximations. So, for problems like this which are really quite common, we have a different approach and that approach is what we are going to study in its simplest form. We go through the same arguments as what we had done earlier which is replace a partial differential equation with an approximate form at a grid point and then, we do that for every grid point and that results in a set of algebraic equations in place of one partial differential equation and then, we go about solving it, but the way we convert the partial differential equation into algebraic set of algebraic equations is different.

In this particular case we cannot fit a triangle into a rectangle. Here it is possible through a coordinate transformation, but it is not so easy to do that and it requires advance level of mathematics. So, we would like to use another intuitive approach for the solution of this particular thing and the power of this method is what will show shortly. So, the basic idea is not to be restricted to have these kinds of rectangular cells making of this, but we can have cells like this.

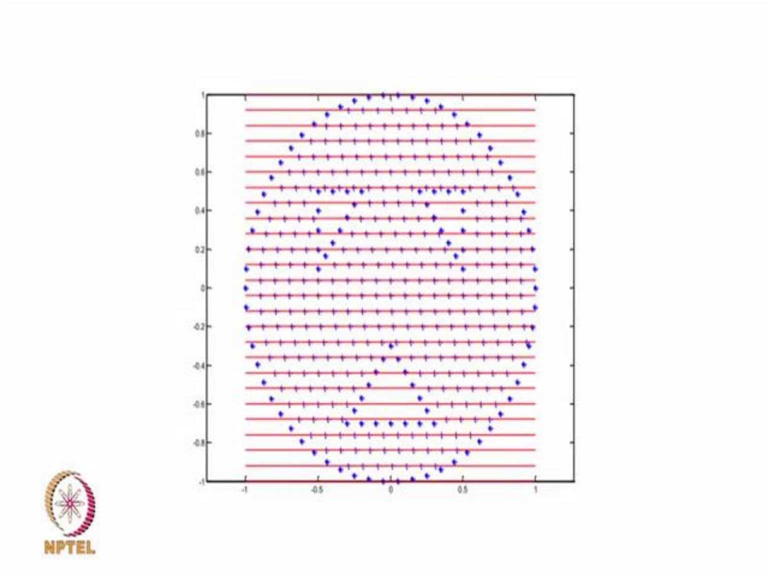
We can have triangular cells and these triangular cells we would like to then solve this equation around this and this kind of triangular cells kind of thing can easily enable us to deal with even geometry like this. So, if you have curved domain like this, we segment it by putting number of points, and we make it into linear segments and then we can join them by straight lines. So, this is what we mean by linear segmentations where the curvature is too much. We can put smaller segments for example here we can put too small segments and make it like this or then, we put lots of points in between and then, what we call as join them by triangles, so that the whole shape inside the flow domain is now made up of tiles of triangular shape in this particular case. So, if we have a way of dealing with non-rectangular mesh, this is known as unstructured mesh. So, if you can describe of flow domain in terms of an unstructured mesh and if we have a way of converting our partial differential equation into an algebraic equation at the centre of this, then you can get a value of this here at the centre of this like this and if you make them small, then you can get many more points.

So, this approach which is used to convert the partial differential equation into a difference equation into an algebraic equation, we have done this using finite difference approximations where we have substituted a formula for dou square w by dou x square which is valid only when the grid points are along lines of constant values of the coordinate lines and here we do not have the grid points here along lines of constant values of coordinate lines. These grid points are not arrived at the intersection are not located at intersections of constant values of coordinate lines and for this case, you cannot write down the previous finite difference approximations. So, what we do is that we use a different approach which is known as finite volume approach. So, using finite volume approach we take the governing equation like this and then, we integrate the governing equation over a cell.

For example, in this particular case we have a cell here, we integrate this and using that in the process of integration we will convert this into an integration across the faces and then, we can evaluate the fluxes associated with this and we make use of estimate the fluxes and then, convert this continuous partial differential equation into an approximate form valid at the sentence of each of these cells. So, that approach is known as the finite volume approach and as a result of this discretization, we end up again just as we have done in the other case equations which describe which have variable as variables the values of w at all these nodes.

The only difference between the finite difference method and the finite volume method is how we do the discretization of this equation at this discrete points located throughout the domain. So, at the end of the discretization, we again have a set of equations which can be written as a w equal to b and in many cases, we can make use of the same solution techniques for the solution of this as we do for the finite difference methods not in all cases, but definitely in many cases. So, in this lecture we are going to look at the finite volume approach. We would like to get an introduction to the finite volume approach of converting a partial differential equation into a set of algebraic equations. We look at specifically for the case of a triangular cross-section. Before that I would like to show some pictures which will show you what kind of how a finite triangular mesh looks different from this.

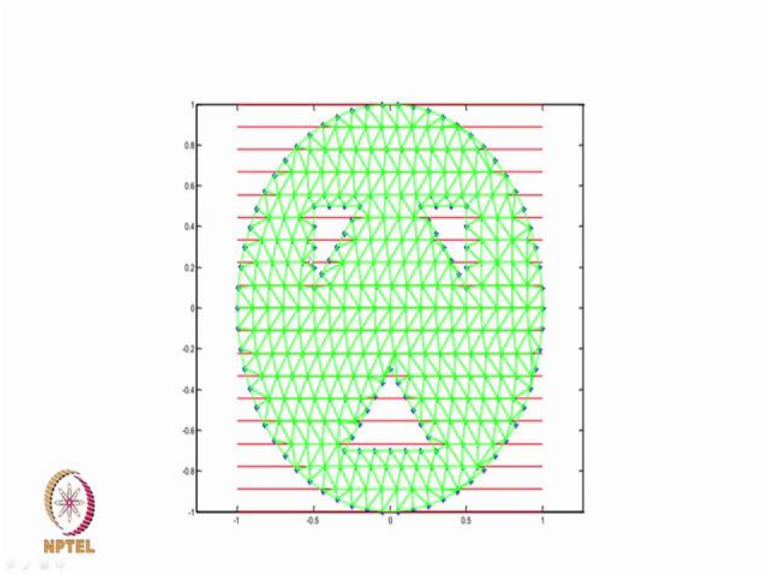
(Refer Slide Time: 14:42)



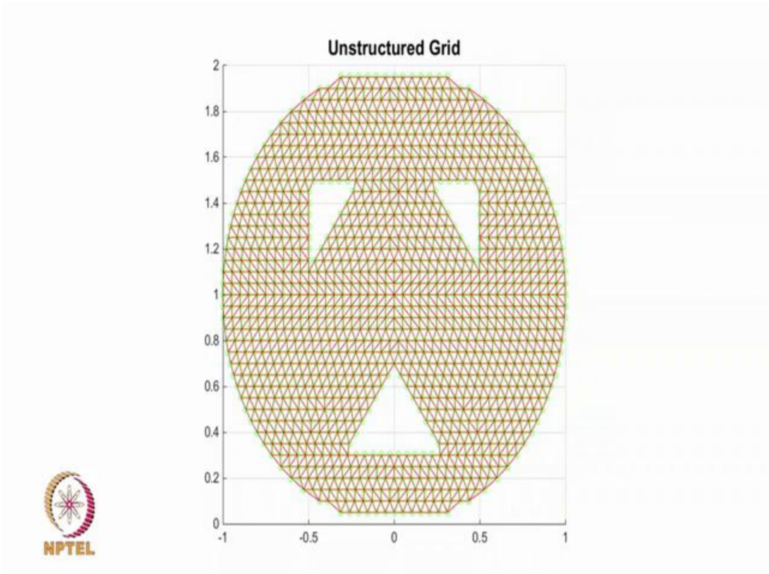
Yes imagine that we are looking at this funny face that you see on your screen here and this as is a circle with three triangles may be the mouth and the two eyes and we would like to describe this, we would like to convert, we like to put lots of grid points here in such a way that we can solve the governing equation at these things and here we used a special algorithm which we will discuss later on to put the grid points, locate the grid points throughout the domain here and only within the domain because what is there here this part is not in the flow domain. This is outside the flow domain. Again this is outside the flow domain.

You can see that there are boundary points which describe this and there are no grid nodes in the domain outside here and here and here. So, we distribute all these grid points and then, we make use of special algorithms to join up these grid points into grid nodes and then, the boundary nodes in the form of triangles here and we need specialised algorithms here because we do not want overlapping of the triangles. We would like to make sure that when you join up all the triangles, we cover the entire area nothing more and nothing less, ok. So, this requires an algorithmic approach as in a case like this may be we can join by hand, but in a more practical case, difficult case, we would like the computer to do it because it will take a very long time for us to do it manually. So, we need to have an algorithm for doing this and we look at certain algorithms later on the course.

(Refer Slide Time: 17:21)



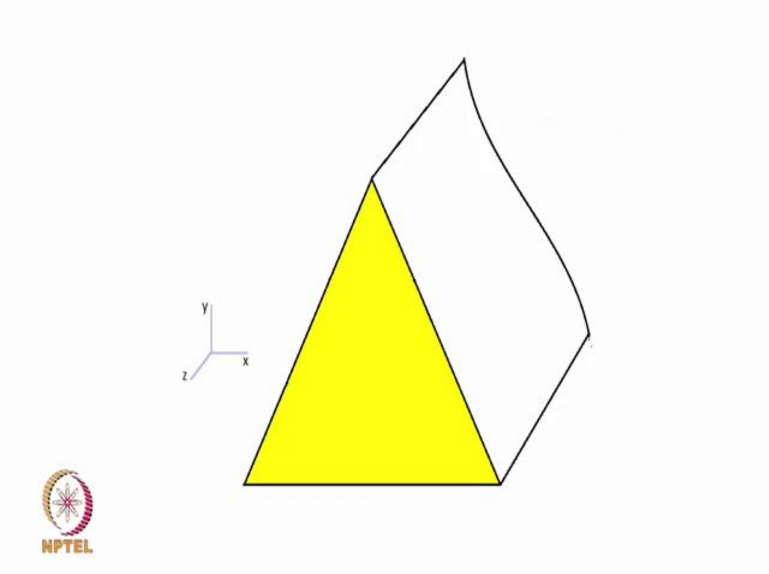
(Refer Slide Time: 17:32)



The points is that we can describe a face like this with part of the circle here and a flat face here and something like a triangle openings here and all these things and we can make the grid points closer together to make a fine grid. So, this is the power of a triangulation and we can represent any arbitrary geometry using this kind of approach.

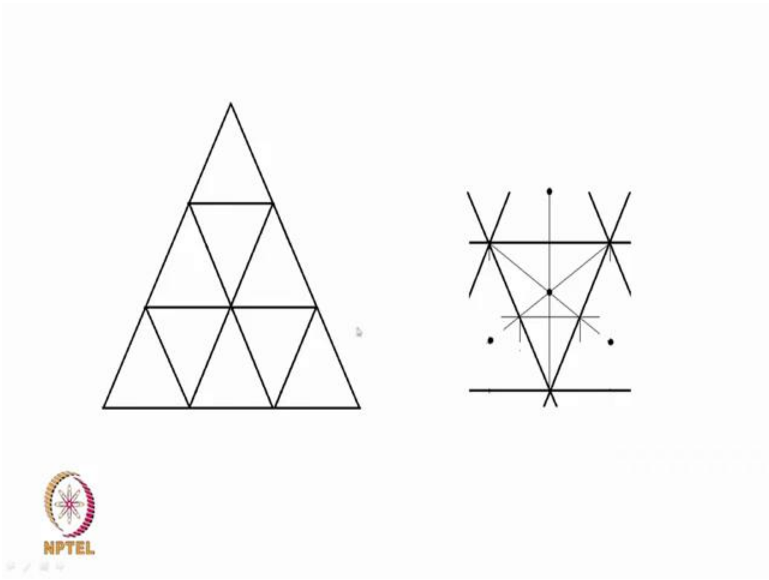
Now, we would like to see how having done this triangulation, how we can then use convert the partial differential equation into approximations, how to derive the approximate form of the partial differential equation at the centre of each of these nodes. So, that is the objective of the lesson that we are going to do today.

(Refer Slide Time: 18:25)



Let us go back to our triangle. This is a triangle and the yellow is what we are going to look at it is a two-dimensional domain and we would like to know the velocity at any point within this. That is in theory, but in CFD we do not give at any point. We give only at selected points in the case of finite difference approach that we have seen in the first example. There we locate at the grid points at intersection of the coordinate lines and we do not have that luxury here. So, we will locate them.

(Refer Slide Time: 19:20)

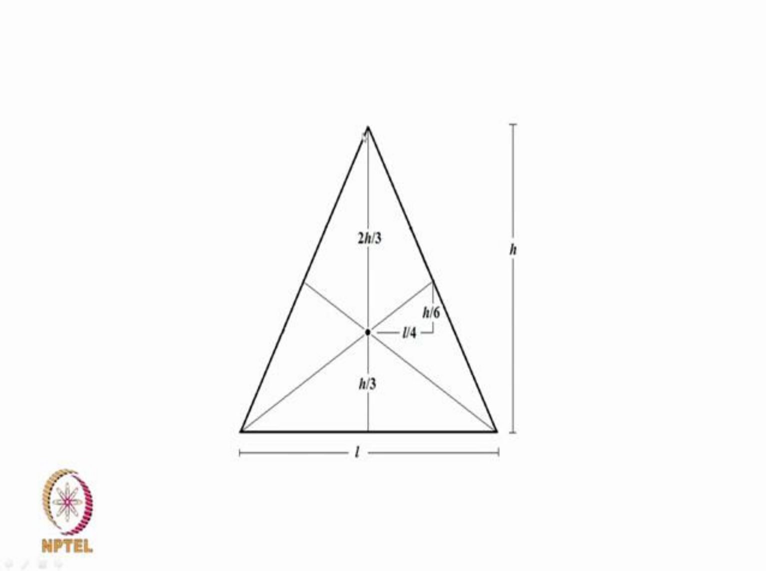


We will do a different kind of discretization. So, for this simple case especially in order to understand the principles and in order to do it by hand, we are divided this equilateral triangle which you have taken for simplicity into 1,2 , 3, 7, 9 equilateral triangles here. So, we just said we will divide this into three parts and you can easily make a triangulation here without too much of difficulty and what we would like to do is that we would like to say that in this case, we would like to know the velocity, the w velocity at the centroid of each of these triangles.

For examples if this is one triangle here, the centroid is given by the point of intersection of these lines here. So, this point is the midway point for this side and this point is midway point for this side and we locate the grid node at which we want to evaluate the velocity at the centroid of each triangle.

Similarly if you consider this adjacent triangle here for example, if this triangle is this triangle, you also have an adjacent triangle. So, for this also we located the centroid and for this also we located the centroid. So, at the centroid of each of these triangles, we would like to have the velocity. So, that means the first step of finding the solution which is find determine the nodes at which we would like to deliver the velocity is done in the following way that is we triangulate the domain, we split it into triangular cells. In this particular case, we have the luxury of making it into equilateral triangles which is not necessarily the case in general, but we can convert it into triangles and then, we say that at the centroid apex triangle is the point is the node at which we would like to have the velocity. So, in this particular case, we have nine triangles and we would like to have velocity at this centroid here, this centroid here, this centroid here, here and ends here. So, how to go about it is the next thing we will see.

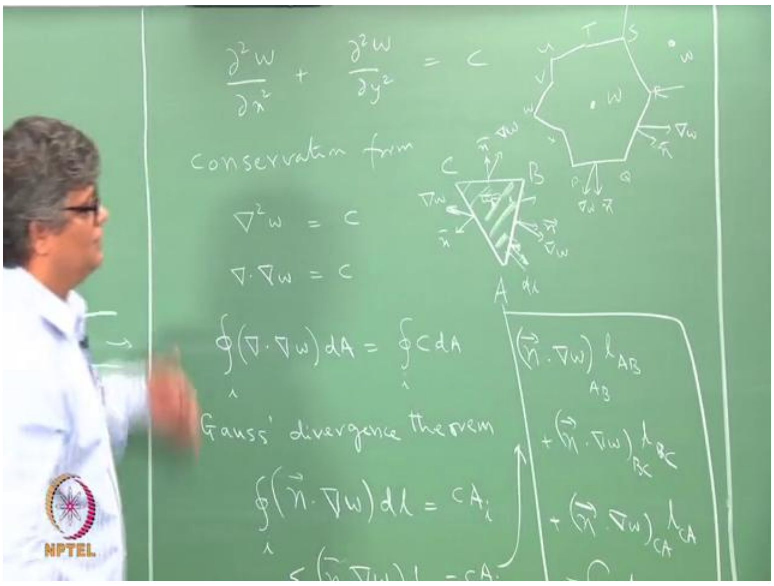
(Refer Slide Time: 22:19)



So, if you take a triangle here for an equilateral triangle, the centroid is located from the vertex at two-thirds of the distance where h is the height and l here is the length of the thing. So, this is located at height of h by 3 from the bottom and 2 h by 3 and it is also important for us when we do the conversion of the partial differential equation into an algebraic equation by writing an approximate form or the partial differential equation at the centroid. We need to know what is the nearest distance to the walls.

So, here we like to know what the centre of this wall here and centre of this side and centre of this side is. So, for this particular case, the centre of this side is located at midway point here and this again midway point and this midway point is such that it is a distance of h by 6, where h is a height and l by 4 from the centroid. So, these are the kind of geometric information which we can reduce for any triangle. So, this information will be needed for us in order to convert the partial differential equation. Now, how do we do this conversion? So, we will go back to the board, so that we have control over it. So, we have our governing equation like this. When we are doing the finite volume approach for discretizing this writing, an approximate form of this over an unstructured mesh having not necessarily a rectangular cross-section like this triangular or polygonal cross-section of an arbitrary shape, we would like to convert this equation, write this equation in a special form known as the conservation form.

(Refer Slide Time: 24:36)



Now, we do not want to know too much about it, but we see that this is Laplace equation, Poisson equation and the left hand side here is equivalent to vector form del square w equal to constant and we can also write del square as del dot gradient of w equal to constant. So, when we go through the equations and all that in the next module, we will be able to see how this whole thing comes about here, but for the moment we say that this equation can also be written like this in vector form and it can also be written like this, ok.

So, now we take any of the cells here and we integrate this equation over the cell. So, we write del dot del omega a del w integrated over the entire closed surface of cell i is also equal to the integration of the constant over the close surface or the same i of the same cell here. So, we have a cell i here and cell j cell k like this and we can do this. So, this is a governing equation. We can do any mathematical operation with that and we choose to have as integration over the closed boundary of that particular cell as a mathematical operation that we want to do with this. So far we have not done anything other than rewrite the equation and do a mathematical operation on it which does not compromise the solution in anyway. Now, this is a special type of integral here, this is a divergence of the gradient of w here which is being integrated over a closed surface and we can make use of gauss divergence theorem to convert this equation which is integral over the area into integral along the boundary, closed boundary of that particular cell. So, the left hand side is written like this, where n is the normal vector and this is a gradient and the right hand side is just c times the area of the cell.

So, the right hand side is just the constant value times the area of the cell. So, let us just see what we are at. Let us say we have a triangular cell like this. This is the centroid of the cell and we are integrating this whole quantity over this entire area and this is what this equation tells us and what this equation tells us is that you just evaluate the gradient of w at each of these phases because this area is covered by these surfaces here and then, you take the outward normal vector of each and then, you take the dot product and then sum over all these things. So, that is what we are instead of evaluating this whole thing over the one evaluation of this, we will do this over each area within this and then, submit up the other possibility which allows us, which gauss divergence theorem allows us to do is to evaluate the gradient along the three phases in this particular case and then, take dot product of the normal vector with this. So, that means instead of it becoming an area integral integrated over the entire area, we just have to evaluate the gradient only along the surfaces.

So, assuming that the cell is small and the gradient does not change like this along the length here, we can therefore write this as discretize this as sum over all the phases of the triangle n dot. So, this is a continuous integral along the length. You start from here and then you go along this and then, at each point you evaluate the surface normal vector and the gradient and then, you integrate over this. So, you take small d l here and then, you evaluate the local gradient and then, you take the local surface normal vector, take the dot product multiply with this and then, you move on this. So, if the cell size itself is small, then we can approximate this as at the midpoint of this side i value at the gradient and then, i value at the surface normal vector and then, I take the dot product here and then I multiply by the length of this side. So, we can write this by putting a here and b here and c here like this. We can write this as n dot gradient of w over side a b times length of side a b plus n dot gradient of w side b c times length b c plus n dot gradient of w over side c a times length c a is equal to the constant c times area of the cell a b c.

So, this summarises for us the way of converting a partial differential equation into an algebraic equation, not quite algebraic equation at this stage, but we are going through a discretization from a continuous partial differential equation into discrete values of the gradients evaluated over the midpoint of each of these things. So, we evaluate the gradient at the midpoint of this and then, we multiply by the surface normal vector here and then the length here in order to get this. So, we do that over this and this and this. If the shape is like this and if this is our cell, we can do the same thing over each of this p q r s t u v w x, so it will be a sum over. So, you evaluate the outward normal vector and then the gradient and then, the gradient outward normal vector like this and then, all this.

Now, what is this gradient here? It depends on you have a cell node and you have a w associated with this and you have another cell here and this is part of another cell here and if you know this value and this value, from this you can evaluate the gradient. So, you evaluate the gradient in terms of the differences change between the node value and the neighbouring node value across which you have this as a separation. So, in that sense although this is a gradient and at this stage, it is a partial derivative, we are going to evaluate this in terms of the difference between the values of w at between the neighbouring cells between the node, this node and then, the neighbouring node. So, in that sense we can convert, we can evaluate this at discrete points and then, convert in the process the partial differential equation into an algebraic equation valid at the centre of at the centroid of this particular triangle. So, this is a way that we can do that discretization. In the next lesson, we will actually work it out fully.