**Computational Fluid Dynamics**
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**Lecture – 52**
**Strongly implicit Procedure (ILU) method**

So, let us see where we are at this stage we are looking at solution methods for Ax equal to b type of problems. We looked at some direct methods like Gaussian elimination method and Lu decomposition method and also the TDMA method. We also looked at some very basic iterative methods like Jacobi method and the Gauss-Seidel method. We also did conversion analysis for this.

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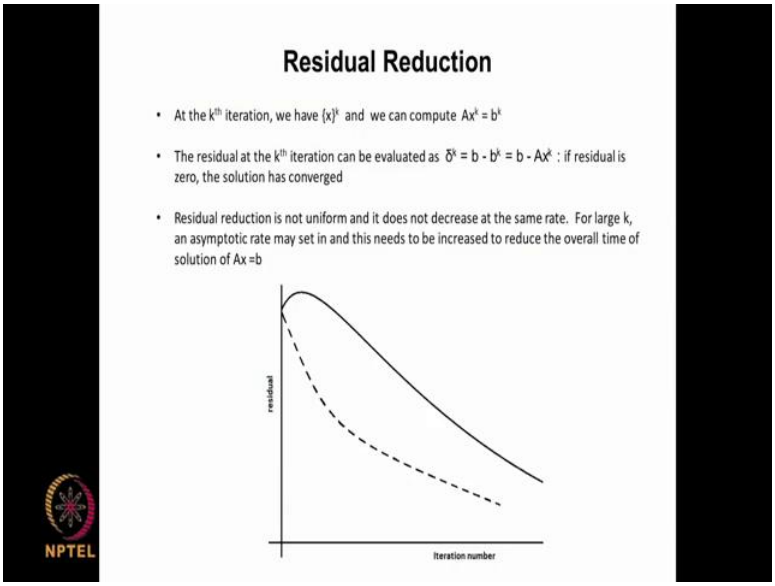


We found that we have to reduce the residual by or it is a magnitude to get a proper accurate solution, and we found that as you increase the number of grid points then the residual reduction becomes slow and slower with the conventional Jacobi and Gauss-Seidel methods, and we started looking at in the second part of this module, at methods which can improve upon this rate of convergence. We looked at the successive over relaxation method.

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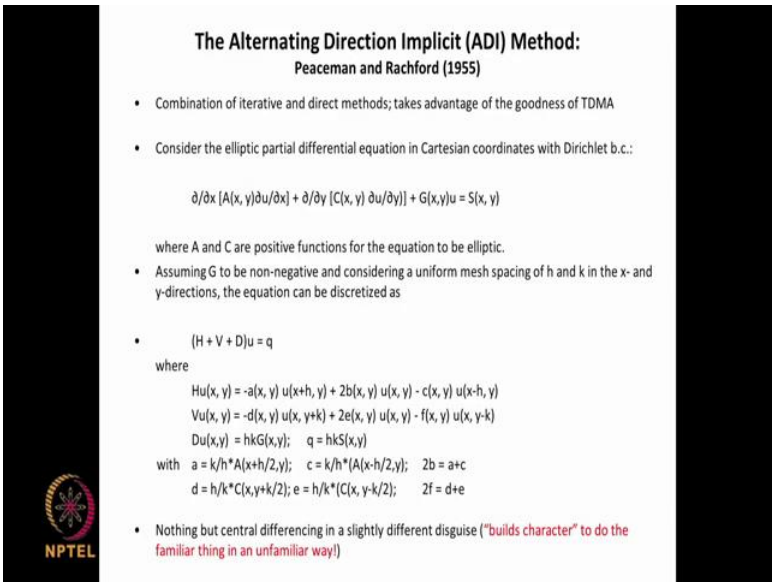


Then we also introduced in the last lecture the alternating direction implicit method, which tries to take advantage of the efficiency of the TDMA method for as the solution of Tri-diagonal type of equations.

In this lecture we look at a different approach, yet different approach for a getting a method which works better than the basic Gauss-Seidel method.

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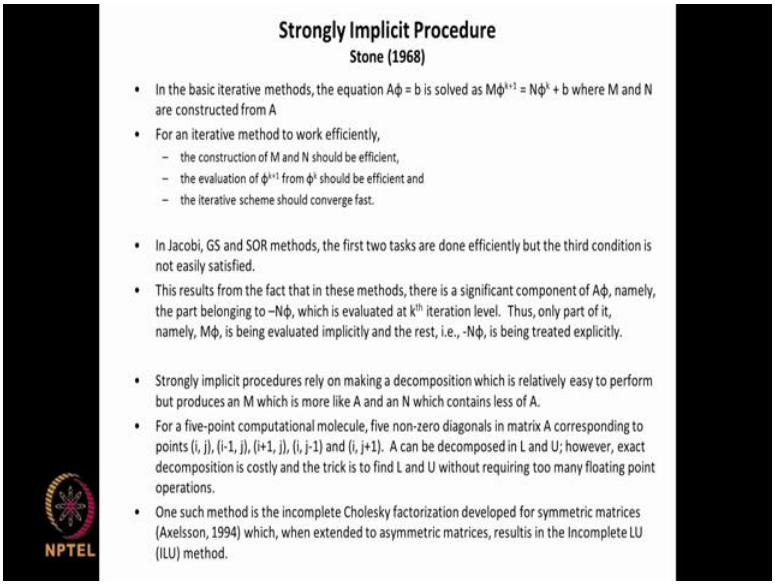


So, let us just this is again a method which is difficult to follow the idea is not for you to follow every step of this and be able to do this, but the idea is to look at what people

have done, what kind of approaches people have used to improve upon the rate of convergence. So, in the conventional methods you have a phi equal to b which not a x equal to b and this is solved as m phi k plus 1 equal to n phi k plus b. Where, m and n are constructed from a.

So, this is your another way of putting the same Gauss-Seidel and Jacobi methods and for this type of iterative methods to work efficiently the construction of m and n should be efficient the evaluation of phi k plus 1 from phi k should be efficient and iterative scheme should converge fast. These are the criteria that we laid out few lectures ago and we would like to recollect that and in the Jacobi Gauss-Seidel SOR methods the first 2 tasks so, that is decomposition of a to m and n and solution of ha phi k plus 1 from phi k is run efficiently.

But they suffer from the third condition the lack of the third condition that is for large matrices you have very slow convergence rate.

This kind of slow convergence of the iterative scheme comes from the fact that there is a significant component of a phi, which is especially the part which belongs to n phi is evaluated at kth level. Can we get the mouse operating?

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Not here now it is working (Refer Time: 03:53).

Student: you continue sir.

Shall I continue like this?

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So, here we are looking at the decomposition of a phi into m phi k plus 1 equal to n phi k and the part of the a phi which belongs to m phi is being term implicitly and the part of a phi which belongs to n phi is being done at kth level. So, that means on the part of it. Namely m phi is being evaluated implicitly and the rest is treated explicitly and this delays the rate of convergence.

Strongly implicit procedures relay on making decomposition, which is relatively easy to perform. So, that construction of m and n is efficient, but in which if m is more like a

then, that means more of a phi is been treated implicitly and less of it is remaining n phi. So, this is the basis for this strongly implicit procedure. So, how does it actually work? The idea is good, but how does it actually work? If you take 5 point computational molecule you have 5 non-zero diagonals matrix a corresponding to points i, j which the a diagonal i minus 1 j the 1 below the diagonal i plus 1 j which is 1 above the diagonal i j minus 1 which lies below i minus 1 j with some 0 between and then i j plus 1 which lies above this i plus 1 j with some 0 in between.

So, you have this kind of thing and 1 way of doing this is to decompose into L and U and once you decomposed into L and U, then the evaluation of the phi from the decomposed values quite efficient. So, it follows this is a pretty easy and exact decomposition of LU converges in 1 iteration. So, there is no need for convergence and everything is evaluated implicitly. So, if you write m equal to LU then n is 0. So, it is fully implicit, but the problem with that is that, the decomposition of a into L and L itself is computationally very costly.

So, if it is possible to make an approximate decomposition of a into LU, then we can take advantage of the efficient solution of LU phi equal to b to advantage and then we can improve an overall thing. So, the idea is to find a decomposition of L and U which is very close to the true solution and whatever is there in LU will go into m whatever the rest goes into n which is explicit. So, if you can make an approximate LU decomposition, then we can possibly put more of a into m and leave less for n phi.

One such method is incomplete Cholesky factorization method for symmetric matrices and for this you can look at this a excellent book of Axelsson 1994 and when it is extend to asymmetric matrices that we often find in a fluid flow problems, results in the incomplete LU method ILU method which is what will just see.

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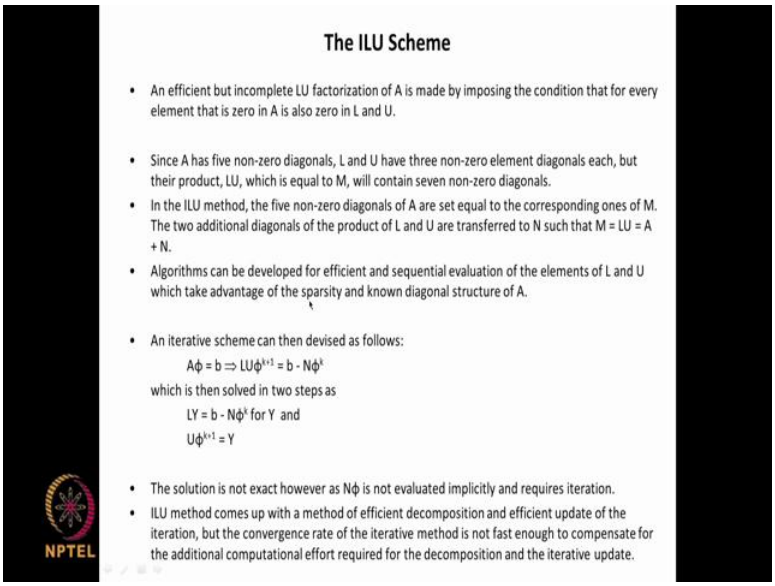


So, the idea of the ILU method is that we put the condition that for every element that is non that is 0 in a there is also a correspondence 0 in L and U and since a has 5 non-zero diagonals L and U also 3 non-zero element diagonals each. So, these are conditions that we are putting on L and U. If we can impose these conditions and get a solution then that is good, we can go through this method.

So, you have L and U which have 3 non-zero diagonals, but the product of L and U which is equal to m the implicit part, will contain 7 non-zero diagonals. Even though you have a has only 5 non-zero diagonals, since you put this condition here which makes it easy first to evaluate the elements L and U. We are putting this condition that only those non-zero diagonals of a must also be non-zero in L and U. That enables us to have an efficient way of getting this L and U systematically, but the product of this 3 by 3 here 3 of l and 3 of u will have 7 non-zero diagonals, that is 2 more than, in the incomplete LU method.

So, the 5 non-zero diagonals of a are set equal to the corresponding 1 s in m and. So, that m is equal to LU equal to a plus n. The 2 extra diagonals are put into n . So, you can develop algorithms for efficient and sequential evaluation of elements of L and U with these conditions and which take advantage of the sparsity and known diagonal structure of a. So, that is the key to decomposition of a in to m and n where, m is the incomplete L

U decomposition. So, it is not fully there and once you make that kind of decomposition, then you can evaluate a phi equal to LU phi k plus 1 equal to b minus n phi k.

So, this is the 1 which you have approximately decomposed and the remaining part which is not there in a is coming as n here, and then this can be solved as 2 step process that we are familiar with that is l y equal to b minus n phi k you solve this for y and then, for known y you can solve this for phi, and you have to do repeatedly because this is an incomplete LU decomposition. So, solution is not exact, because n phi is treated at k. So, you have to go through iterations and then you will ultimately get a converge solution. So, unlike the LU method which is exact and convergence in one iteration, ILU method which is incomplete LU decomposition and therefore, has a remnant of n phi which is evaluated explicitly will become an iterative method.

So, in the sense there are good some good elements of ILU which is that you have efficient decomposition of approximately into L and U and then you have efficient forward and backward substitution methods of the solution, but the ultimately the there is still a substantial amount of n which is equal to the two elements of L and U product such that the overall method although it converges faster it also requires significant overhead. The convergence rate of the iterative method is not fast enough to compensate for the additional computational effect required for the decomposition and the iterative update. So, the overall ILU scheme does not work very well not in a significant way.

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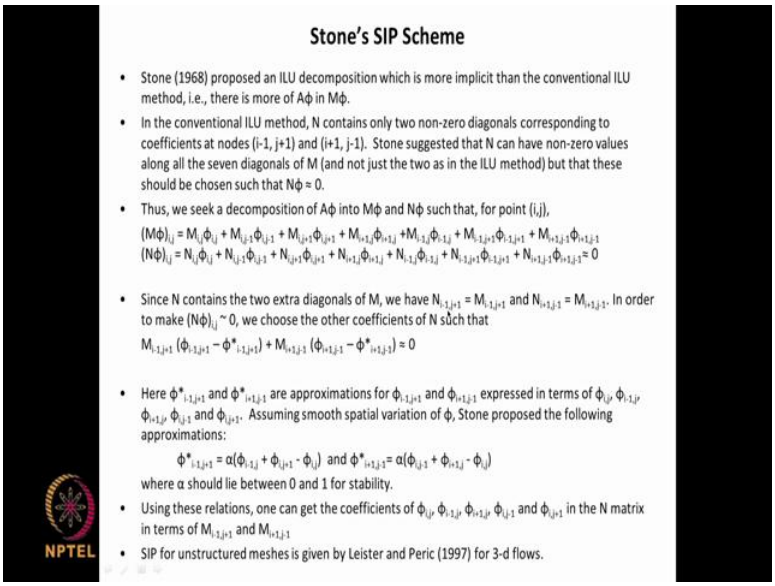

So, this is where Stone stepped in 1968 proposed an ILU decomposition which is more implicit than the standard ILU method and thereby making the convergence faster. So, what is that how did he make this? So, we are looking at, and a phi we are looking at incomplete LU decomposition in such a way that there is more of m phi there is more of a phi in n phi. So, that is what it is. So, in the conventional ILU method n contains only 2 non-zero diagonals corresponding to the coefficients at i minus 1 j plus 1 and i plus 1 j minus 1.

So, you can work it out if you go through the whole process, but we mention d that 3 diagonals of L and 3 diagonals of U when multiplied will give us 7 diagonals and there are 2 extra diagonals and in the incomplete LU method we put the 2 extra diagonals as n. What Stone proposed is that, since anyway we are dealing with 7 diagonals let n have all 7 non-zero diagonals let n have elements of in all the 7 in not just in the 2, but choose the other coefficients in such a way that the total contribution of n phi is roughly equal to 0. So, if n phi is close to 0 and since a phi is written as m phi plus n phi then there is more of m phi there is more a phi into in m phi.

So, the key thing is that by relaxing the assumption that n should have only the 2 additional non-zero diagonals and relaxing it in such a way that you enable all the 7 non-zero diagonals to be present in n phi, but choose them in such a way that they cancel the effect of each other. You may be able to get an overall n phi which is equal to 0. So, what does it mean? We seek a decomposition of a phi into m phi and n phi such that for point i j m phi i j is m i j phi i j i j minus 1 phi i j minus 1 j plus 1 phi j plus 1 i plus 1 like this, and n phi i j is also having in all of them and this is roughly equal to0 and since n contains the 2 extra diagonals of m.

So, we have n i minus 1 j plus 1 is m minus. So, in order to make n phi equal to 0 we choose the other coefficients of n such that m i minus 1 j plus 1 times phi i minus 1 j plus 1 minus an estimated value of this plus m i plus 1 j minus 1 times phi i plus 1 j minus 1 minus phi star i plus 1 j minus 1 is roughly equal to 0. So, here we have the additional diagonals as we mentioned here are i minus 1 j plus 1 and i plus 1 j minus 1 . So, these quantities are here and here. So, we try to make use of these things minus the values that go into n in such a way that the sum of this m phi this additional diagonals cancels out with each other. So, that is what we are effectively saying here.

So, now the idea is what are these values of phi star? These are estimated values of phi at i minus 1 j plus 1. So, of these estimated values are pretty close to these then the result is 0 and the fact that we are multiplying by these coefficients of m minus 1 j minus 1 does not matter. So, the idea is to make these estimates close to these things and this is where Stone assumed smooth variation of phi. We are looking at phi x, y and if phi x, y varies smoothly, as the solution of a Poisson equation or Laplace equation might exhibit in such a case it is possible to make an estimate for phi i minus 1 j plus 1 in terms of neighboring values we have i minus 1 j here and i j plus 1 and i j.

So, these are some approximation something like the finite difference approximation for phi i minus 1 in terms of these neighboring quantities and similarly you have phi i plus 1 j minus 1 is expressed in terms of this and you have an alpha which is a factor something like a fridge factor which he needed to make sure that it is a stable. So, once you put this here, then you can get an overall expression for the coefficients here in terms of the coefficients for the other things using this relations 1 can get the coefficients of phi i j i minus 1 j i plus 1 j i j minus 1 in the n metrics in terms of phi m i minus 1 j plus 1 and i m i plus 1 j minus 1.

Let us try to go through this slide once again. So, that we follow the idea. We have some of the detail is quickly explained, but let us try to make a recap of this. So, that we understand the basic thought process. Thought process is that we want to divide a into decompose a into m minus n and m is treated implicitly and n is treated explicitly or m phi is treated implicitly and n phi is treated explicitly.

Now we want to have this decomposition as being done effectively. So, that is where the idea of incomplete LU decomposition comes into picture, but in the incomplete LU decomposition we have an efficient method if we impose a condition that only those non-zero diagonals of a must also appear in the incomplete L and U decomposition. So, that gives us the product of L and U decompose with this condition will have 2 extra diagonals which are then put into n. So, that you have m which contains 7 diagonals which contains 5 diagonals n which contains the other 2 diagonals. So, you have you can write m equal to a minus n or a plus n like that.

So, that you have the 7 diagonals here 5 plus 2; 7 diagonals there, but Stone proposed that n should have not just 2, but it should have all the 7 non-zero diagonals of m and we

should take the diagonal the additional diagonals that we are introducing, in such a way that they all cancel out. So, we have the additional diagonals of i minus 1 j plus 1 and i plus j minus 1. So, the effect of these should be such that they cancel out the effect of the other diagonals which are there in this. So, if you were to choose the coefficients here in of the m and n in such a way that this can happen then since m is equal to a minus n and if n phi is closed to 0 then we can have m close to a.

So, that is idea here. So, in order to derive an expression for this to become approximately 0 we can put these coefficients to be like this here and then we should say that this whole thing is equal to 0 and what we say is that this whole thing is these other 5 diagonals are such that, they are expressed in terms of this phi star i minus 1 j plus 1 and phi star i plus 1 j minus 1 and what are those expressions here? The example here you have phi star i minus 1 j plus 1 expressed in terms of i minus 1 j i j plus 1 and i j. So, you have i minus 1 j here and i j plus 1 this also be here i j plus 1 and; obviously, i j and i plus 1 j minus 1 expressed in terms of i j minus 1 that is this 1 and i plus 1 j i plus 1 j is also somewhere here yes.

So, that if you now put this into this then you have all the 7 things will appear here, but these estimated values are such that, all though these quantities are not 0 the once inside bracket has 0. So, if you choose that then the overall thing which is now this is now equal to the n phi j and that pretty it is close to 0 there. So, with these conditions it is possible for us to come up with again a sequential and efficient evaluation of elements of L and U.

So, it is possible to have a sequential and efficient evaluation of the elements L and U such that m is equal to LU and it is almost equal to a plus n, but n is close to 0 n phi is close to 0 because that is a condition that we have imposed in deriving this coefficients here. So, this lot of detail that goes into this and these can be derived by every 1 of you, but 1 has to study the original papers and by doing that you should be able to come up with first decomposition of a evaluation of L and U and once you have the L and U then you also know the n components and then we can go through this evaluation here.

So, what Stone proposed in1968 as a method for this 5 point diagonal was later on modified by a number of people including and they came up with something which is

even more efficient method for this and this strongly implicit procedure has also been extended for 3D flows for unstructured meshes.

So, when you are looking at complicated geometry irregular geometry like finite different methods, where there is a diagonal structure in a way we make use of the diagonal structure, but it is possible for us to have not a triangle at mesh, but if you have quadrilateral 2D or cuboidal 3D kind of thing then it is still possible to get some it is still possible to get diagonal structure and for unstructured meshes you have an extension of this method by Leister and Peric 1997. So, with these this kind of approach it is possible to have a solution of a phi equal to b which is faster than the basic Gauss-Seidel method.

So, let us just take a recap of different methods different approaches that we have looked at 1 is over relaxing the step size that correction delta x that we get from Gauss-Seidel method which is the successive over relaxation method and we also try to make use of the efficiency of the TDMA in the ADI method and this particular idea that we looked at in this lecture tries to take advantage of the efficiency of the LU method in getting a solution fast by successive forward and backward substitution especially which and this method is divides in such a way that it takes a advantage of the sparsity and known diagonal structure of a.

So, it is not a generic LU decomposition it is an LU decomposition which is in which it is known that there is a diagonal structure. So, with that kind of thing it is possible to come up with something which is more strongly coupled most strongly implicit than in the conventional ILU method.

And that gives us an incomplete LU decomposition which is strongly implicit and therefore, has a fast convergence rate and which can also be extended it to 3 dimensional flows. So, this is also 1 such method that is quite often used for 3 D computations and 2D computations this is not all, but we can see that as we as we desire more and more faster convergence rate then we have to put in more and more effort and we have to be more and more scheming and devious in terms of getting a solution and this is 1 such a idea which is a variant of the incomplete LU decomposition method which itself is a clever variation of the basic iterative method where in you are trying to do the decomposition quickly and fast and get more implicit solution.

So, in all these cases you have much more of overhead computation to do than in the case of simple Gauss-Seidel method. So, only if you are really pressed for efficiency of computations may be you would go for something like this. Otherwise you could do with Gauss-Seidel method.

In the next lecture we are going to look at something which follows a different approach for the same, but attacks the same problem that there is in Gauss-Seidel method, that is the slow convergence rate when you increase the number of grid points and that approach known as a multi grid approach. That has a different philosophy we look at that philosophy and with that will close the discussion on this advanced methods for the solution of Ax equal to b.