Computational Fluid Dynamics
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Lecture – 51
Alternating Direction Implicit (ADI) method

In the last lecture we have seen the successive over relaxation method as a simple means of increasing the rate of convergence over Gauss-Seidel method for the specific case, where A is diagonally dominant.

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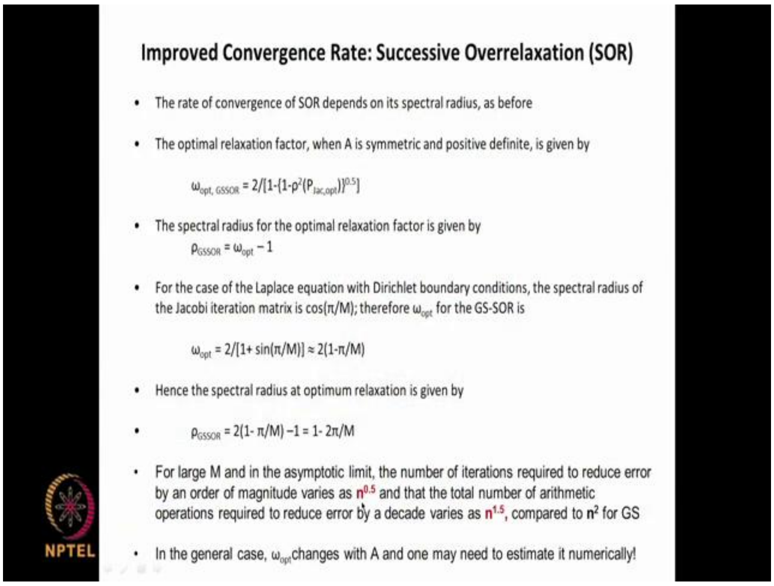

We made the point that the optimal SOR can have tremendous increase over the Gauss-Seidel method because it would take only square root of n number of arithmetic operations for in terms of the asymptotic rate of convergence. As opposed to n for the Gauss-Seidel method, therefore the total number of arithmetic operations varies as n rise for 1.5compared to n square for the Gauss-Seidel method.

When n is large, that is, when the number of grid points is large, then the variation of n to the power one point five or n square can make a huge difference. But, this kind of great improvement is possible only for the case where we have the optimal SOR parameter, optimal value of the SOR parameter, and that something that is not known in the general case. So, we have to keep that in mind and then try to estimate the value of the optimal value. And then, see how it can be how the rate improves.

There are; there is a different approach for acceleration of the solution of A x equal to b. And that method, one such method which is known as the alternating direction implicit method is what we will examine in this lecture.

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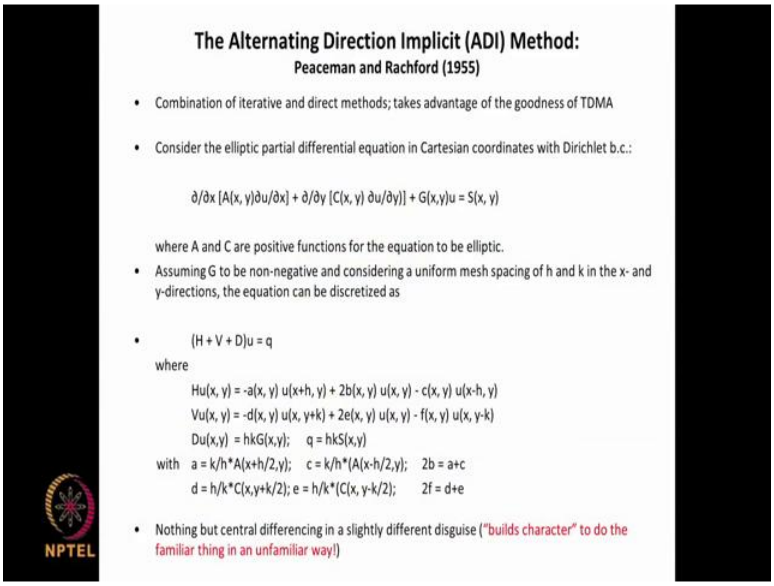

The original formula is proposed by Paceman and Rachford in 1955. It is a combination of the iterative method and the direct method. Specifically, the, it takes advantage of the goodness of the tridiagonal matrix algorithm, goodness in the sense that it takes only n number of k times n number of arithmetic operations to get a solution of A x equal to b, where A is tridiagonal. So, it takes advantage of that. But, it puts that in the context of an iterative method.

And, we will see that for the specific case of elliptic partial differential equation in Cartesian coordinates with Dirichlet boundary conditions and two dimension problem, which is written in this. Rather unfamiliar, but more generic form, where you are writing it as dou by dou x of a which can be function of x and y dou u by dou x plus dou by dou y of another function C, which can again be a function of x and y times dou u by dou y plus G times u, u is the unknown variable here, like our phi, G which is again a function of x and y equal to S, which is a function of x and y. So, this is a generic elliptic partial differential equation in Cartesian coordinates in two dimension. And, a and c have to be positive functions for the equation to be elliptic

So, if G is assumed to be non-negative, and considering uniform mesh spacing of h and k in the x and y directions, this equation can be discretized as H plus V plus D times u equal to q, where H, V and D are operators. So, we have H V equal to something; is given by minus a x y times u at x plus h times y plus two b times x y times u at x y minus c x y times u at x minus h y. Now, this is, this looks a bit complicated. Essentially what we are using is central differencing here. And, so this is, this becomes a u at i plus one and that is what this is x plus h; h is a delta x equal to delta x here. And, k is equal to delta y. It is just unfamiliar notation.

And, one would say that it is nothing but central differencing in a slightly different disguise. And, I would like to mention the point that sometimes we have to go in to unfamiliar territory. And, if we work it out by ourselves, then it actually enables us to look at what somebody else is saying and it helps us understanding what somebody else is saying, may be in a slightly different language. And, we should always be receptive to those kind of ideas from other people, who may not speak in the same languages as us.

So, in that sense that is why I put up so as to build character. And, so I would like you to just follow it up and see that this is nothing but central differencing. And, applied where, since you have A and C and also G and x are functions of x and y. They take, they can vary here. And that is what this is saying And, but what you have in H u is variation with respect to x. So, you have u at x plus delta x, u at x and u at x minus delta x. Similarly V u, H is for the horizontal and V is for the vertical, horizontal meaning x variation and V vertical meaning y variation.

And, similarly we can see that V u is u at x y plus k. So, that is y plus delta y. And, again we have y minus delta y here and then you have u at y. So, this is u at i plus 1, i j plus 1, u at i j and u at i j minus 1. And, you have corresponding coefficients here because this coefficients of variations of x and y. You also have this. And, D u is h k times G x y. So, that is coming from here, and q is coming from here. So, q is right hand side term and D is a diagonal term, which involves value of u only at i j point. And, so that is given by this one here. So, when you put all these things together, then you can put it in this particular form. So, a is k times k divide by h times A at x plus h by two. So, that is a at i plus half j and c is i minus half j and d is i j plus half and e is i j minus half. So, in our notation these are all fairly simple things to follow. And, we can write it like this, but we would like to have the practice of doing it in a different way and that is how we have put it.

There is also another reason for doing it, that is, we have the general case. And, we have the general case and we have not assume this to be constant. And, a general case can give us, can extend our range of applicability. So, what we have done so far is nothing but discretization of our elliptic equation, general form of elliptic equation in Cartesian coordinates.

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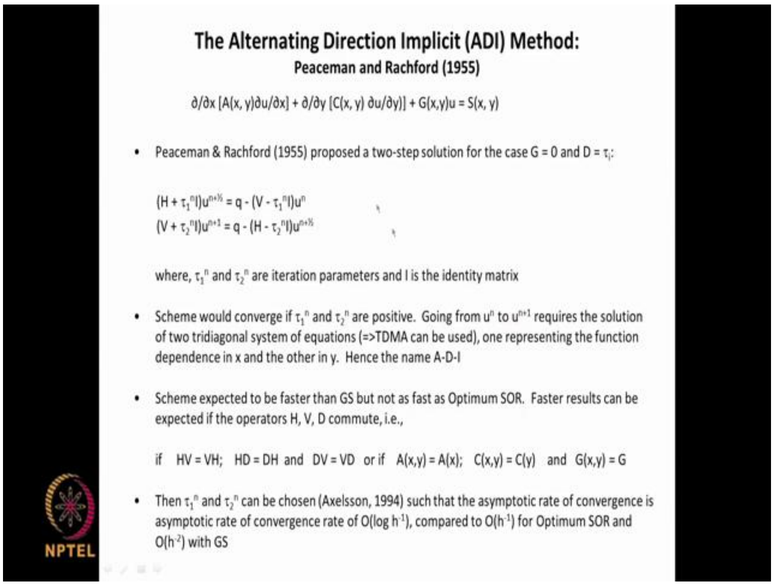

So, that is our equation here. And Peaceman, Rachford proposed A 2-step solution for the case where G is equal to zero and D is a constant tau i. So, this discretized equation H plus V plus D times u equal to q. They proposed to solve as two separate equations in A 2 step process. As you go from n to n plus one, you first evaluate u at n plus half by taking the horizontal derivative and constant tau 1, and vertical derivative is evaluated at n th value, and horizontal derivative is evaluated n plus half. So, this will involve, for example, the first equation will involve u at i plus one j, u at i j and u at i minus one j. So, that makes it implicit in x direction. All values given was at n plus half. And, this part which is V u n are the y derivative terms are coming in this. And, these are being evaluated explicitly.

So, the first equation is implicit in x, but explicit in y. And having done this, having evaluated the x derivatives at implicitly at x plus half at n plus half here, you now come to the vertical derivatives. So, these are evaluated as implicitly because V u n plus one will be u i j plus one, u i j and u i j minus one. All these things are evaluated implicitly because that is how we have put as u n plus one. And here, we have H derivative also contributing to this. And, these are evaluated at the latest values that are available here. So, that is H u n plus y, n plus r. And, what we notice that the right hand side terms can be evaluated because these are in the previous time step of previous half time step values. And, only the left hand side terms are implicitly evaluated.

And, the first equation evaluates implicitly the derivatives and variations coming in the x direction. And second one, takes care of the variations in the y direction. But, since each time we are only considering one dimensional variation, so that is at best dou square u by dou x square here and dou square u by dou x square here. We have a tridiagonal matrix for each of this. The tridiagonal matrix represent in the first case, the second derivative and other variations in the x direction.

And in the second case, it represents variations in the y direction. And, this is also such that we can get diagonal dominants. So, we can make use of TDMA scheme; and, since as we go from iteration n to iteration n plus 1, we first evaluate the derivatives in the x direction implicitly. And then, we evaluate the derivatives in the y direction, again implicitly. This is called an implicit method and alternating direction implicit; because you alternating in x direction and alternating and then in y direction, before you complete the step from n to n plus one. So, that is why this is alternating direction implicit method.

This scheme is expected to be faster than Gauss-Seidel method, but not as fast as the optimal SOR method. So, in the specific case where the operators H, V and D commute, so that is H V equal to V H and H D equal to D H and D V equal to V D, which also essentially means that if A of x y is just a function of x and C of x y, which is coming in the y derivative term here. If that is only a function of y here and G of x y which is coming here is not a function of x and y. So, in this particular case we can get much faster results then approaching; in fact, even better than the optimal SOR methods. And, I would like to refer you to the book by Axelsson. It is a mathematical book. It is mathematical base book.

And, in this condition when you have this kind of commutativity of the H V D operators or if we have your equation in which A is a function x only and C is the function of y only and G is constant, then in such a case it is possible to generate a series of numbers tau 1, tau 2, such that you have an asymptotic rate of convergence, which is of the order of log h inverse one compared to h inverse one. So, it is one by delta x here and this one by log of one by delta x.

Since this variation is smaller than this variation, we can get much faster rate of convergence than in the case of Gauss-Seidel method or optimal SOR method. And, what this functional variation where A is function of x only and C is a function of y only, makes the equation here like a variable separable. The solution is like variable separable type of format.

So, if you have that kind of elliptic equation with an arbitrarily complex variation of A in terms of x, which makes the solution probably more difficult analytically, if A is function of x only, but it can be any complicated function. And, we are evaluating it numerically. So, there is no problem here. So, when we look at the numerical solution here, we are going to evaluate the small a in which a functional form is put in is evaluated numerically. So, it can be any kind of arbitrary complexity in x only. So, in such a case we can get tremendous rate of improvement of the alternating direction implicit over the Gauss-Seidel method or even optimal SOR method.

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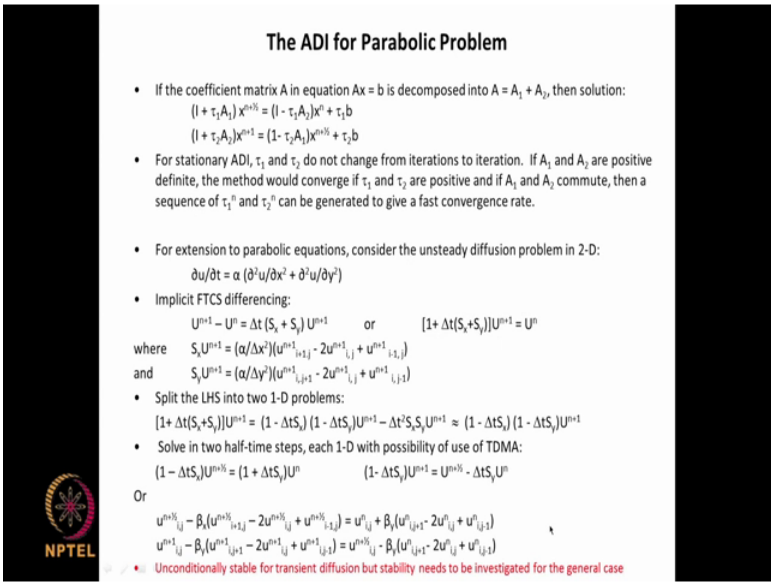

So, we can generalize this ADI method. If we take A x equal to b and then decompose it into A 1 plus A 2, then we can put it in in this particular form, where one plus tau 1 A 1 x n plus half equal to one minus tau A 2 x n plus tau 1 b and one plus tau 2 A 2 x n plus one equal to 1 minus tau 2 A 1 x n plus half b. Now, what this one is saying that this A 2 part of this is being evaluated explicitly and A 1 part of this is being evaluated implicitly. And having done this, how you come to the other part A 2, which is being done explicitly, in the implicitly in the second step of the equation.

And, A 1 part is now being evaluated x explicitly again. If you do this in the case of stationary ADI, so that is tau 1 and tau 2 are constant. They do not change from iteration to iteration, if this decomposition is such that A 1 and A 2 are positive definite. Then the method would converge, if tau 1 into tau 2 are positive and if A 1 and A 2 commute, then a sequence of tau 1 and tau 2 can be generated to give a fast convergence rate.

So, the idea here is to say that there is no one approaches to the solution of A x equal to b. So, the idea here is not to say that this is the method by which we can solve. There can be different approaches for different things. When a mathematician looks at A x equal to b, then he will see many possibilities. And, in some cases if we are aware of those kinds of possibilities, then we can take advantage of that and then get a solution. And, sometimes when we by looking for alternative ways of getting a solution, we may get some specialized method which will apply for special classic problems. So, it is in that sense we would like to have background to the solution A x equal to b. And, it is not a trivial problem. And, people have had spent their career, trying to look for various methods. And, this module is just it is an important part of CFD solution.

And, this module is looking at how many different ways or what are the different kinds of successful approaches that people have adopted for the solution A x equal to b. So, it is in that context we are looking at it, not in the sense of trying to solve our simple problem. We are trying to get good view of good prospective of the field of CFD from especially in the context of solution of A x equal to b, which we realize is very important problem when we have large number of grid points, which we need to have in order get confidence in a CFD solution. So, it is in that context I am putting this up; not from an examination point of view, not from a solution point of view for a specific problem, but, as a general matter of academic significance.

So, we can extend the ADI method, even for parabolic equations. But, we have to do in a careful way. So, let us consider this parabolic equation when you have dou by dou t equal to alpha dou square u by dou x square plus dou square by dou x dou y square. And, this is an unsteady two dimensional problem with constant alpha. Earlier, in this case we have put some diffusivity, which is especially varying here. And, here we are looking at constant alpha, which is the same in the two directions.

We can use over implicit FTCS as differencing for stability. We know that explicit means that there is conditional stability. But, implicit will make it unconditionally stable. But, in the process we get an equation which is, which has five diagonals. And, five diagonals mean that we cannot solve it using the TDMA method. So, now, in order to bring in the TDMA possibility, we would like to pose it in this particular form where we take only the x direction and only the y direction variables.

Now, how can this be done when we have this time derivative also. So, for that we write it symbolically as u n plus 1 minus u n equal to delta t times S x plus S y times u n plus 1. So, S x and S y are operators in the sense that you do not have S x without u. So, we define S x u n plus one as alpha by delta x square u n plus one which is a time index and which subscripts i plus 1 j, you i j and u i minus 1 j. So, this whole thing is representing dou square u by dou x square. And, the right hand side is being evaluated at n plus 1 time step. So, in a way we know the difference between explicit and implicit. And, this is an implicit way of dealing with the space derivative; because your time derivative involves this u n plus one here. And, when we put this, evaluate this set u n plus one, this makes it implicit. And, that is why we call this as FTCS implicit

And similarly, S y u n plus one is now taking care of the y derivative alpha by delta y square u i j plus one minus two u i j plus u i j minus one. So, in that sense this is again like what we have written previously in a straight forward way for a FTCS. Now, we are putting it in more symbolical language. And, once you put it like this, you can bring in the, we can club this operational u n plus 1 with this. And, write it as 1 plus, there must be a minus here, 1 minus delta t S x plus as x y u n plus y equal to u n. We take this on the other side. We bring this on to this side and we will be getting 1 minus here.

Now, we can split this thing here into one minus delta t S x times 1 minus delta t S y times u n plus 1. This is the full expansion of this, and because this is delta t square and these are delta t. We can neglect this as a higher order term. And then, we can write this as 1 minus delta t S x times 1 minus delta t S y times n plus 1. So, now we are writing this one with a minus sign here as 1 minus this and this.

So, now this equation, this being made equal to u n can be solved at two half time steps, each one dimension with the possibility of use of TDMA. So, we can first write this as one minus delta t S x u n plus half equal to one plus delta t S y times u n. And from this, we get u n plus half. And, we substitute this here. And then, we put, we solve for one minus delta t S y u n plus one equal to u n plus half minus delta t S y u n. Now, what we claim is that this solution here, this two-step solution here is the same as the solution of this, which is roughly the same as the solution of this within second order accuracy in time.

Now, we are using here this discretization is first order accurate in time. So, the second order terms can be neglected with respect to the first order terms here. And, that is the argument here. So, this decomposition into two independent steps of this one here is the same mathematical. And, how can we do that? We can do that by substitution, we can pre multiply this by one minus delta t S x. And then, we can substitute this expression here and show that this is same. So, it is a bit of exercise that I would like the interested reader to pursue, and show that this is equal to this one. So, there is premultiplication of this one by one minus delta t x delta t S x. And then, substitution of this thing here, this will actually give us this solution here. Ok.

Now, what is the advantage that we have gained in this? We can write this equation as u n plus half i j minus beta x, where beta x is alpha by delta x square alpha delta t by delta x square, which we have seen in simple differencing of this using FTCS implicit. So, this is minus beta x times u n plus half i plus one j minus two u n plus half i j plus u n plus half i minus 1 j equal to u n i j plus beta y times u i plus i j plus 1 n minus 2 u i j plus u n i minus 1 j. So, this is the term that is coming here and this is already known from the pears values.

So, all the right hand side is known. And, the left hand side involves i plus one j i j and i minus one j. And, you also have j i term coming here. So, gives us a tridiagonal matrix. And, this can be solve using TDMA scheme to get u n plus 1 at all i j. And, this will bring in here. And then, we write the second part of this equation here to get u n plus 1; now, at i j plus 1 and i j and i j minus 1.

So, once you solve this, this equation, then we gets u n plus 1 i j at all points. So, this is again a tridiagonal matrix, involving u values at n plus one times step at i, j, for all of them. And, this involves again a tridiagonal matrix, but involving u at n plus half time step, not at n. So, this two times step solution enables us to take, to break up this pentadiagonal matrix into two tridiagonal matrix solutions. Now, this pentadiagonal matrix is not 5 adjacent diagonals, but this with the 5 diagonals with some zeros in between.

So, this particular decomposition of this scheme into this is unconditionally stable for transient diffusion. Even in three dimension, we can extend this to three dimension. And, have three successive one dimension solutions. But, the stability is guaranteed for transient diffusion, but if you bring in some advection terms into this, then we have to look for stability and we have to see whether it is stable or not.

So, for the general case where you have other terms that are coming, then we have to look at this stability. But, otherwise this breaking up of essentially pentadiagonal matrix into two tridiagonal matrix or seven diagonal matrix. For the three dimension case of this transient diffusion problem into three one dimension problems is unconditionally stable. And, we can just go ahead and get solution for this. And, the overall scheme is expected to be faster than trying to use a Gauss-Seidel method for the same pentadiagonal or seven diagonal equation.

And so, this is a way, an innovative way of making use of the goodness of the tridiagonal matrix algorithm for the solution of matrix which is not tridiagonal. But, what this also tells us is that it is not trivial extension. We cannot just take it up into anything. We cannot factorize it and then take it up. This is part of operating splitting kind of approach and it has to be done properly.

And, what we would like to show here is that that this problem broken up into, written up in this way and approximated at this way. And then, this is equal to u n. So, that problem is broken up into two steps which are given by this. And, this break up is such that this result in problem is given like this. Example, why we are writing it like this and why we are writing in a different way. Here is; part of the decomposition is a problem. Only doing this way will mean that this two-step solution is same as this one-step solution.

So, we would like to be aware of this that it is not to trivial solution. It requires some thought into the properly composition. And, in the general case it also requires stability analysis.

With this, we would like to conclude this presentation of this alternating direction implicit method. This is, as we can see is a different approach to accelerating the solution over the Gauss-Seidel method. And, the solution in this particular case is trying to take advantage of the fastness of the tridiagonal matrix algorithm. But this has, this works only in certain cases in a straight forward way. And even then, we have to be careful about the decomposition.

In the next lecture we will look at yet another way of accelerating over the Gauss-Seidel method. And, this again looks that it incorporates a different philosophy, different philosophy, in the sense that any iterative method like solving A x equal to b as an iterative method like this.

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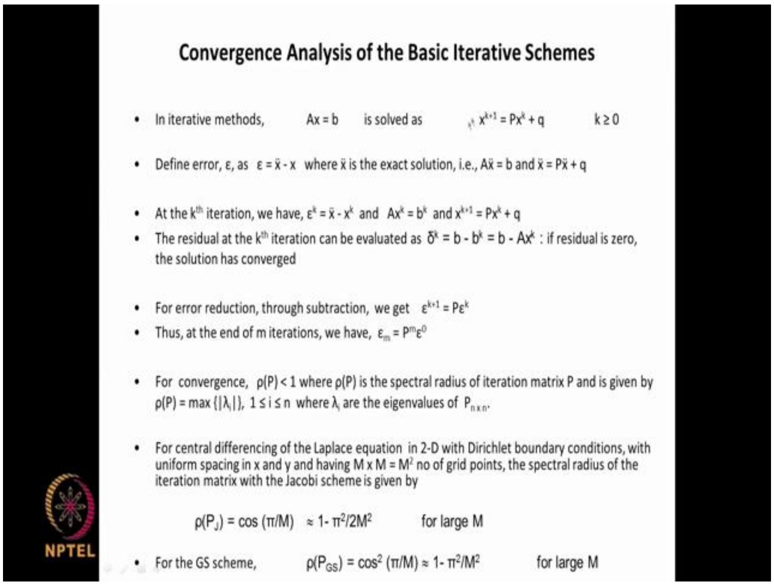

The convergence of the iterative method is faster if the breaking up of A into M and N.

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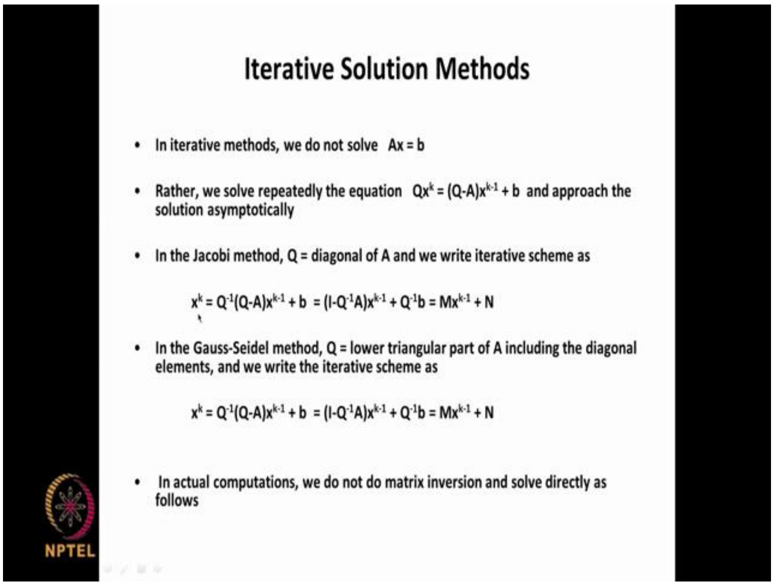

We are putting x equal to M and N here. And, part of this M is being evaluated in an implicit way. So, the idea we will introduce in the next class that if M is very close to A, then this becomes solution which is implicit. So, the idea of making; we cannot take fully implicit solution of A x equal to b because that would involve n (Refer Time: 32:00) number of mathematical operations. But if we can do it approximately, but as close to a fully implicit method, then we get a faster iteration, faster convergence; is the idea behind the strongly implicit procedure method, which we will see in the next class.