Computational Fluid Dynamics

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Lecture – 43
Tutorial on Pressure Correction Method Contd

In the previous lecture we have seen how to discretize the momentum equation and converted into an linear algebraic equation, taking account of the non-linearity coupling involving for example for the x-momentum equation, the velocity in the vertical direction in the y-direction and also the pressure at a different points within the flow field.

And we are able to take the x-momentum equation in terms of the partial differential equation for a steady two-dimensional case, and we converted that at a particular point at the phase of the control volume of a point center around the point where pressure is evaluated. We are going to draw the diagram.

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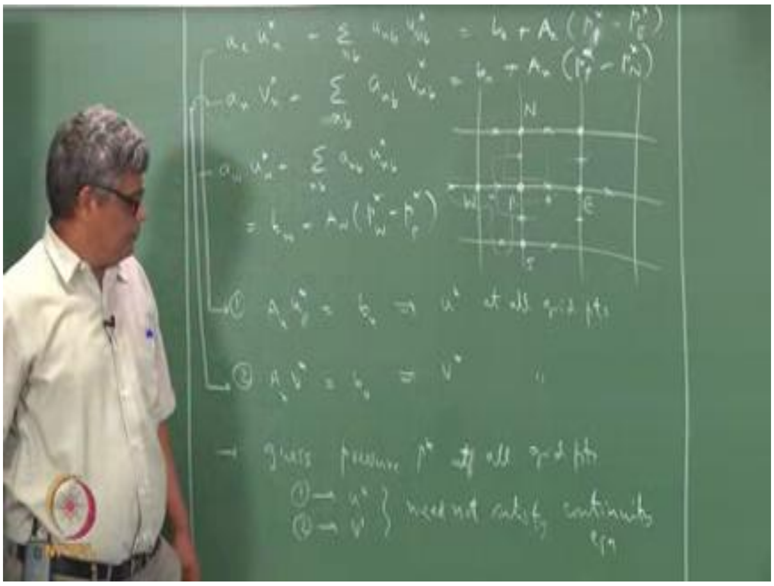

And we said that at this point we could approximate the x-momentum equation and convert that into an equation for the discretize momentum equation, which we wrote as sum over the neighboring locations where u velocity is evaluated times their respective coefficients times the neighboring velocity component equal to a constant b plus to be consistent with our notation on the power point slide, we have a coefficient times p dash

E minus p dash capital P minus p dash capital E, because we are put a plus here assuming A is positive quantity. This is the discretized momentum equation x-momentum equation on a staggered grid, which we identified like this.

These are the major grid lines the equivalent of i, i plus 1, i minus 1, i plus 2, j, j plus 1, j minus 1, and intersection of these were pressure is evaluated and vertically displaced half points is where velocity is v velocity component is evaluated. Horizontally displaced midpoints is where u velocity is determined, so that the x-momentum balance equation in it is discretized form involving if we call this is as capital P east point, the west point the north point and the south point at which pressure is evaluated.

Then x velocity component is a evaluated at the east phase of a control volume centered around capital P, and it is also evaluated at the west phase; v velocity is evaluated at the south phase and the north phase of this control volume. So, this how we have staggered grid arrangement on which we discretized the x-momentum equation to get the velocities at points midway between pressure evaluations.

And similarly we can write down from the y-momentum equation v star n plus n b a n b v star n b equal to if you call this as b e and b n, these are all coefficients plus A north times p star capital P minus p star capital N. This equation can also be written for example, a west u star w west plus sum over neighboring values of a and b u star n b equal to b west plus A west times p dash w minus p dash p. So, the same template when it used for the east phase point that is velocity at this point looks like this. When the bringing the same template to a point at the west phase here small west, then the neighboring points are this one here, this one here, and half a point here, and this point here. So, it involves the same competition molecule just is displaced to the left here.

So, we can write down similar discretized equations for every point at which we need to evaluate u component, velocity, and like this for every point where we need to evaluate the v component velocity. And as we have seen in the previous lecture these coefficients can be computed from using previous guess values, previous iteration values, and geometrical parameters like delta x, delta y, and physical parameters like a rho, and viscosity and other things.

So, everything about this coefficient is known. The only problem is that in order to solve this if we write down for all the points at which we have a velocity to be determined then this equation, and similarly this equation and all these equations can be converted into a set of equations like b u where this is set of linearized algebraic equations. And if you solve this we get u star at all grid points.

And similarly, we can take this one and convert this into A n v star equal to b v, and solving this will give us v star at all grid points at which is v is defect. So, solution of these equations will give us the u velocity field and v velocity field. The only problem is that in order to find this u and v, we need to know the p, we need to know p at every point. We can see that in order to evaluate u e at this point, we need P at this point and this point and in order to evaluate u velocity at this point we need P p and P w. So, in order to solve this equation we need to know pressure at every point, and similarly we need to know pressure at every point. So, we can solve these things provided pressure is known.

So, what we do we guess pressure, we call this as p star, because it is a guess pressure at all grid points so that means, that we know pressure at capital P, capital E, capital W, capital S, capital N for every point at which pressure needs to be evaluated. So, with this pressure field, we can solve for example, this equation 1 here, and 2 here; and from these from 1, we get u star; and from 2 we get v star.

But these are with the guess field so that means, that although we have made use of the Navier-Stokes equations, the momentum equations to get u here the crucial pressure here is an assumed pressure field, so that means, that these are not necessarily correct solutions to the Navier-Stokes equations. And that we felt when you substitute this in the continuity equation, so these did not satisfied continuity equation because it has not been used and these are with an arbitrary guess pressure field.

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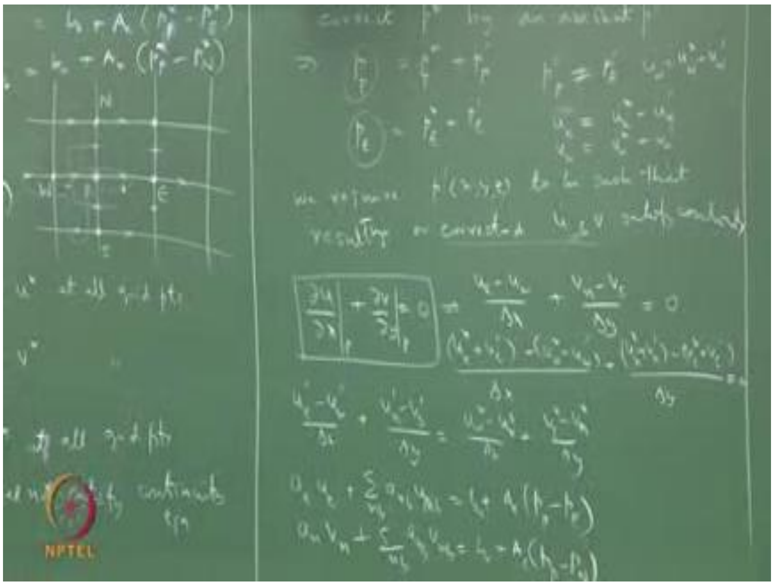

So, we say that let us correct pressure, therefore, by an amount of p prime, so that is we write the true special value is equal to a pressure value the estimated pressure plus a pressure correction p prime. So, this pressure correction will vary from point to point. For example, at capital P, we have the guess pressure field plus pressure correction. Similarly, the true pressure at capital E is p star capital E, which we have used for the velocity determination plus a pressure correction which is capital E. And we are saying that in the general case p prime p is not equal to p prime E, so that means that pressure correction needs to be evaluated at every point.

And we are saying that let us make this pressure correction such that if we were to substitute p and p E in the momentum equations, not p star P and p star E, but the what we believe are the true values then you get a velocity corresponding u velocity and v velocity from these discretized equations, which are otherwise unchanged. So, you have the same coefficients here; only instead of putting P star here, we put actual corrected pressure what we believe are true pressures. So, if get this then we get new values of u and v; and the new values of u and v it should be such that they satisfy the continuity equation. So, we require p prime let us call this as x, y, z because these are field values to be such that the resulting or corrected u and v satisfy continuity.

So, what we mean by this is we have the continuity equation which is for two-dimensional incompressible flows can be written as dou u by dou x plus dou v by dou y equal to 0. Now we discretize this around P location P not at small e not at small north not at the phases, but at the point where we need to evaluate pressure because we want to use this to get pressure. So, we want to have this equation approximated at capital P, so for that the staggered gird arrangement really helps us because dou u by dou x at this point it can be written as u e minus u w divide by delta x. So, this can be written as u e minus u w divided by delta x is this point, this approximation for this derivative. And dou u by dou y at P here can be written as v north minus v south divided by delta y.

We notice that here we are not putting any star quantities, because these are supposed to be the corrected velocities such that this is satisfy the continuity equation, and therefore when we substitute the correct velocities then this should be equal to zero. So, now here along with the pressure correction, we are falling we are talked about the velocity correction. Therefore, we are saying that just as we have true pressure is the stared quantity, the estimated quantity plus a pressure correction we can say that the true velocity is u star plus a corrected velocity component u prime, and the true v velocity is the estimated plus a corrected quantity. And specifically, we can say that at point small e this is given by the u e star plus u e prime and v n star v n is equal to v n star plus v n prime. And similarly, we can also write u w is u w star plus u w prime and so on.

So, what we mean by this is that corresponding to when we introduce the pressure correction, there is also a velocity correction; and this velocity correction is specific to each location at which the velocity is being evaluated. And in the general case, this velocity correction is not the same at all points. So, now these are supposed to be the corrected velocities, and they are supposed to be satisfying the continuity equation. So, we can substitute this expression here and then we can write this as u e star plus u e prime is what this is minus u w star minus u w prime divided by delta x plus v n star plus v n prime minus v s star plus v s prime by delta y is equal to 0.

So, the continuity equation tells us that this is the case. And in this equation, all the stared quantities here are already known, because we started with the pressure values guessed pressure p star and then we have the discretized x-momentum, y-momentum equation, we have solved this to get u star v start at every point. So, all the quantities that are stared here are known.

And we can take them to the right hand side, and write this as u e prime minus u w prime by delta x plus v n prime minus v s prime by delta y is equal to for example, u w star minus u e star by delta x plus v s star minus v n star by delta y. So, we have taken the stared quantities which are known onto the right hand side, we have this. So, here we have got a different form of the continuity equation, which is still not useful for us because this would not help us solve for v prime and u prime.

So, what we do is that we try to drive these velocity corrections in terms of the pressure corrections, because we know from the momentum equations that whenever we change the pressure, the velocity also changes. And how can we do that the truth velocities and velocities that we have got will be satisfying the discretized momentum equations with the true pressure fields.

So, for example, if you substitute the true pressure flied here, we can say that we can write the first equation as a e u e no star anymore plus sigma n b a n b u n b equal to b plus A e times p p minus p capital E. And similarly, we can write this one here, a naught v naught plus sum over all the neighboring quantities at which v n is being evaluated a n b v n b equal to b u and b v plus a e times pp minus p n. So, we have called this as you have not called this anything. So, let us subtract this from this and this from this one. So, these are linear equations so that becomes a e times u e minus u e star; and we know that u e minus u v star is u v prime.

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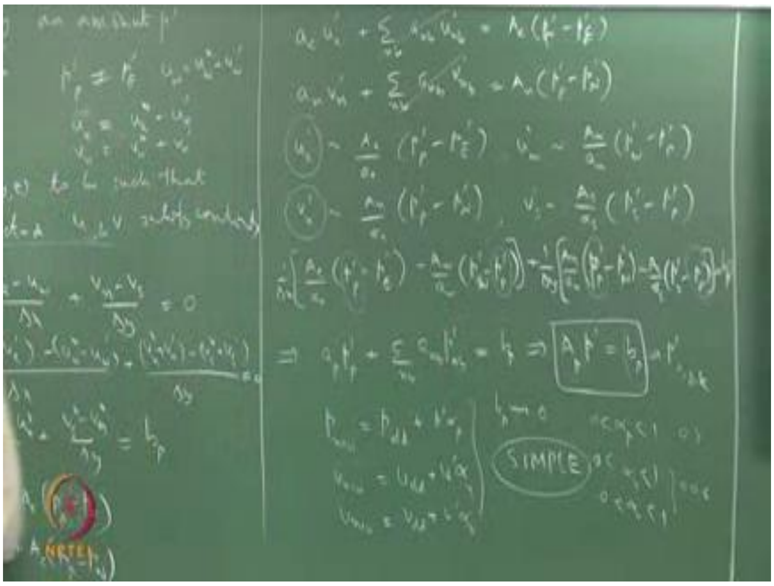

So, we can by subtracting this equation here from this equation, we can write this as a e times u e times plus sum over the neighboring quantities of a n b u prime n b equal to this b will go away. And here we have p p minus p p star so that gives us p p prime p E prime. Similarly, we can write a n v n prime plus sum over neighboring quantities a and b v prime n b equal to A n times p p prime minus p n prime. What these equations are telling us is that if we correct pressure filed by a certain amount, which are given by this these quantities here there is a corresponding change in the velocity correction and the velocity correction is given by this.

So, if you are able to solve this, then we can put those things here. But in order to do this, we need to know this, in order to know this; we need to know the velocity components. So, we can make an approximation, because we are only talking about corrections. So, there is a correction which is coming from the main point at which u is being evaluated; and a contribution from the neighboring quantities. All these things will balance the pressure correction. So, if you say that this is discretized momentum equation, there is a correction to the momentum equation coming from the pressure component, there is a correction which is coming from the main velocity here and also the neighboring velocity components are adding their own correction.

In the true case, where we have converge solution then the velocity corrections are zero, the pressure corrections as zero, because the converge cases is where the calculated velocity field will satisfy the continuity equation. So, if that is the case then there is no need for a velocity correction, and there is no need for pressure correction. So, since we are only talk about corrections, we can make an approximation.

We can say that let us just cancel this let us neglect this, and let us neglect this. Therefore, let us write that u e prime is roughly equal to A e by small a e times p prime minus p prime E. And similarly from this we can get v n prime is roughly equal to A n by a n small a times p prime p minus p prime N. And similarly, we can write the velocity correction at the west phase is given by A west by A west times p prime west minus p prime p; and v prime south as being roughly equal to A s by a s times p prime s minus p prime p.

Now what do we have, we have some estimated velocity corrections, and we substitute those in this equation, each of these velocity corrections induced velocity corrections are expressed in terms of pressure corrections, and if we do that, we get we have 1 by delta x times A e by a e times p prime p minus p prime E, so that is u e minus u w is A west by a west times p prime west minus p prime p plus 1 by delta y times A north by a north p prime p minus p prime north there is a minus A south by a south. And this whole thing is equal to zero. What is this? This is just the discretized velocity correction equation in which we have substituted systematically the velocity corrections by the corresponding approximate influence coming from the pressure correction.

So, if you look at this equation here we have delta x which is known, delta y which is known, all the coefficients capital A, small a, a west all these things are known. And we have these pressure corrections at the grid points here as the unknown quantities are here. So, this can be put when we write down for all the grid points. In this grid points, the variable that we are looking for is the pressure correction at capital P, because that is where we have written the discretized continuity equation.

So, this is the one that is being that is unknown here; and we have minus here minus times minus plus here and then plus. So, we have we can write this as a p p prime p plus sum over neighboring quantities of a and b p prime n b equal to sum it is not equal to b 0. It is equal to this whole thing, which we can call as b p, so this equal to b p here. So, just as we have put here we can write this also as A p p prime equal to b p. When we put this as system of equations by solving this system of equations, we can get p prime, p prime for at all i, j, k.

So, now, what do we do this p prime, once we have p prime here, then we can make use of this approximation, now we know p P and p E this is i j k and this is i plus 1 j k, so we can get the corresponding u prime e the velocity correction at the midpoint, so that is we can get the velocity correction at u prime u prime at i plus half j k. And similarly, we can substitute p i j k and p i j plus 1 k here to get the velocity correction v prime at i j plus half k so that means, that we know have a pressure correction and a velocity correction for the two velocity components.

And therefore, we can say that the new value of pressure is the old the value of pressure plus the pressure correction and new value of u is value old value plus velocity correction and new value of v is v all plus v prime. So, this why do we say that this is new and this is old here, because this p prime is not exactly evaluated, it is evaluated based on this approximation, based on neglecting this, before this is not an exact treatment.

So, we can only say that we have improved upon a pressure correction we have improved upon the value because we have taken into account contributions from the main thing, but not the neighboring things so this is already an improvement and therefore, we can have this. So, the new value of pressure will not be the exact value, but if you now put this back into this, and then we redo this whole thing get again new velocities and then new pressures and new velocities and new pressure.

If we do sufficient number of times then we will hopefully get a converge solution and progressively the pressure corrections and velocity corrections will decrease. And progressively we find that b p here, which is a u star w minus u star e divide by delta x plus v star s minus v star n by delta y this thing will become 0. Because when the star quantities that is when the whole quantities approach the exact solutions, then this is the discretized form of the continuity equation and discretized form of the continuity equation equal to 0, so at convergence b 2 to will go to zero. So, we will say that we will go until b p is sufficiently small and then we can we can stop at that point.

So, in the simple approach, we are making use of a semi implicit approximation here, because the velocity correction here coming from the pressure correction is not evaluated fully implicitly, the neighboring components are being neglected here. So, we have a semi implicit method for pressure linked equations is the acronym for this particular way. And what that particular way is that you start with a pressure estimated guess value of pressure at every point at which pressure is going to be determined. You substitute them in the discretized equations, and then you solve the all the x-momentum equations to get u star at all the grid points displays by half a grid cell in the x-direction. And then you put the same pressure field in the discretized y-momentum equation and get velocity at all the grid points displaced vertically by half a grid point from the pressure locations.

Now, you have you u star and v star for a given p star, and you can directly go to this equation here. In this equation b p is evaluated in terms of this u start and v star, so b p can be calculated; and all these coefficients are part of the derivation. So, all the coefficients are known. And we can assemble this equation and we can assemble this A p p prime equal to b p. So, this can be evaluated and these coefficients are already evaluated here, and so we can get this equation. We have everything in this to be able to solve for p prime. So, we solve for p prime here at all i, j, k's and using this p prime to be evaluate u prime and v prime and then we update our pressure guess pressure field and guess velocity field using some under relaxation factors.

So, we put here alpha p, and alpha u, and alpha v; ideally alpha all the alpha should be equal to 1. When alpha is equal to 1, then there is no under relaxation or over relaxation, but because we are looking at non-linearity, because we are looking at coupling and because we are looking at discretization, for all these reasons we would like to under relax, so we would like alpha p to be between 0 and 1; and similarly, alpha u to be between 0 and 1. Typically, you take these things to be around 0.6 and this to be around 0.4, so something like that is a is a good thing. And this plus this must be equal to 1 is a rough guideline is something that has been arrived it.

And so we with the specification of this under relaxation factors, now you have everything that you are required to complete one loop starting with a guessed or old value of pressure, we get u star and v star, old values of u star and v star which satisfy the discretized momentum equation with old pressure fields. And using this old u star and v star, we solve the pressure correction, and then we get pressure correction at every point. And using the pressure correction, we evaluate the velocity corrections.

And using this under relaxation formulas, we update and get the new values of p, u and v at every grid point. And using this new values we can go back to the discretized x-momentum and discretized y-momentum equation and then get new values of u star and v star and then new values of pressure correction new values of velocity corrections, and then new values updated values. So we keep on doing this in a loop and we should be getting a solution in many cases.

And in this method, we have not made any approximations about the governing equations although we have done it for a steady case; something similar can be done for unsteady case. And although we have done for 2-D, it can also be done for 3-D. And I would like you to refer to more advanced material to look at this or work it out yourselves for the 3-D case and the time dependent case.

So, this is a strategy which has worked very well for a number of incompressible flows, and not just a fluid flows, but also including energy, including reactions and including lots of other complications for multi phase flows and those type of things. Also a similar kind of strategy has been used and although this method has been developed primarily for incompressible flows, it is also possible to take account of compressibility compressible flows. And although we have explained this method for a staggered grid, it can be extended to non-staggered grid, but it requires some additional special way of evaluating the velocities.

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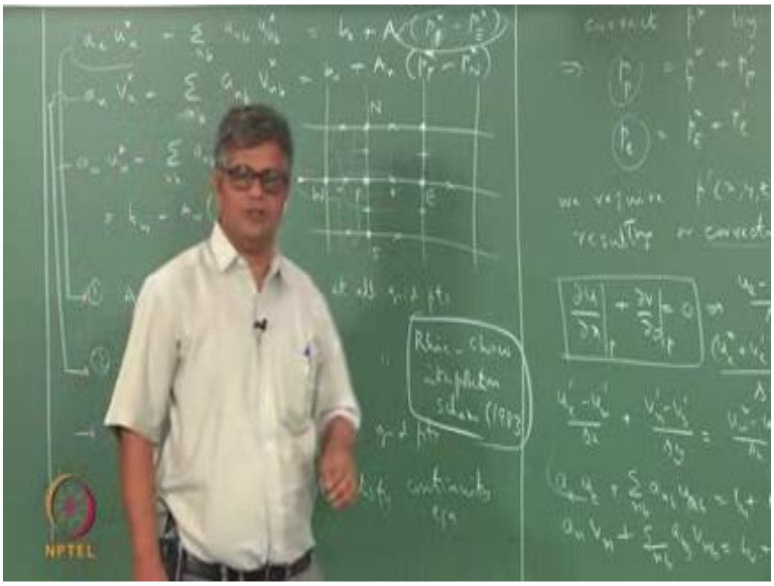

There is an interpolation scheme to get a velocity at this point which is known as the Rhie chow interpolation scheme. Using this interpolations scheme to evaluate the velocities it is possible to extend this method from staggered grid to non-staggered grid and avoid Chequerboard oscillations. So, in that form, this method has been used for all kinds of irregular flows using finite difference methods, irregular shapes using a finite different methods and finite volume methods.

So, with this we will we will conclude the discussion of how to solve the Navier-Stoke equations. We have seen methods which have been developed for compressible flows; essentially mimicking the template that we have developed for a scalar transport equation that adding extra features to take account of non-linearity and coupling. And we have seen a number of other methods for the solution of incompressible flows and which the linkage between the continuity and momentum equations provided through the density; density term is no longer there. And we have a range of methods in this way to solve the Navier-Stokes equations. And we can say that it takes a lot more effort to solve the Navier-Stokes equations than to solve the simple equations that we have tried out in a module three.

So, to this extent, we can say yes computational fluid dynamics requires some special techniques in order to get a solution. It is not like you have differential equations so you can just solve it using any method you need some specialized methods to solve fluid flow equations and that constitutes the field of CFD.

Now, in the next from next class onwards, we will go into another aspect of CFD which is how to solve this A equal to b type of equations. And we can see that in this simple method itself for the three equations we have three times we have to solve A x equal to b type of equations.

And this is only for one update of the pressure and velocities. And we have to do it many times before we can get a solution. So, we have to have an efficient method for the solution of A x equals to b even in the case of compressible flows using beam warming methods, and all those kind of things again you have to solve similar type of equations. So, in that sense solution of A x is equal to b is something that we will be cropping up again and again in CFD. And we need to have specialized method so that we can solve this for large number of equations, so that is what we are going to discuss in module 5.