Computational Fluid Dynamics

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Lecture – 42
Tutorial on Pressure Correction Method

In the last lecture, we have looked at the pressure correction equation method approach. In today's tutorial class, we will derive the elements of each of this. We will start the x-momentum balance equation, and we will try to come up with a discretized equation, which can be solved provided we know the pressure from the previous iteration and things like that. Let us start again. In the last lecture, we have seen the overall picture of how the pressure correction approach works for the solution of Navier-Stokes equations in incompressible flow.

In today's lecture, which we will take as a tutorial, we will work out the small details of this namely how to discretize the momentum equation, so as to convert it into a linearized algebraic equation, taking account of the non-linearity, and the coupling with other equations. We will also derive the pressure correction equation itself, which is a way of rewriting the continuity equation in incompressible flows. Once we do these two elements then we will try to put together the entire flow chart for the solution of Navier-Stokes equations. We will work with two-dimensional flows, and steady flows, and we will see how it can be extended to 3-D conceptually.

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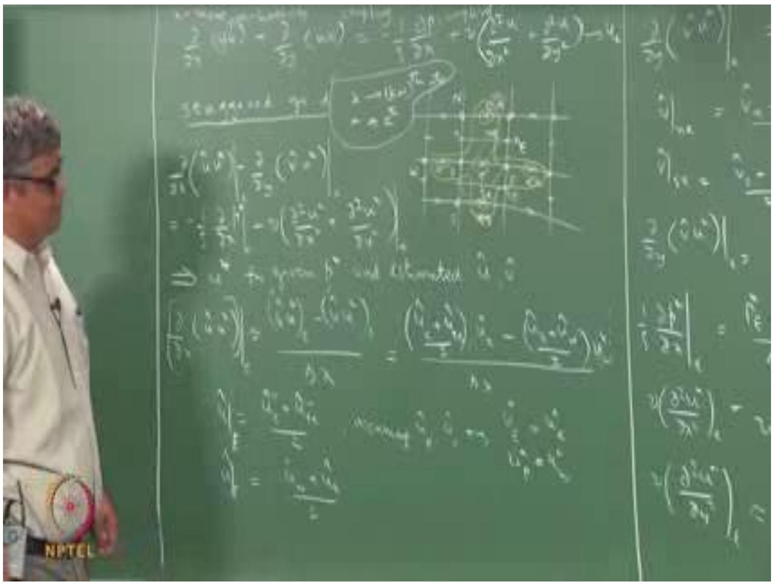

So, we will start with the 2-D form of the Navier-Stokes equation under steady flow conditions which we can write as dou by dou x of u square plus dou by dou y of u v equal to minus 1 by rho dou p by dou x plus nu dou square u by dou x square plus dou square u by dou y square. This is governing equation, and we would like to discretize this on a staggered grid.

The staggered grid that we have looked at these are the i, j, k lines. So, you have i, i plus 1, i minus 1, j, j plus 1, j minus 1 and it is at the centers of each of these pressure is evaluated, and so we can call this as p. And to the east of this we have capital E, and to the north of this we have capital N, and here you have capital S and capital W. So, these are the points at which pressure is evaluated. With respect to the pressure evaluation, the u velocity component that is the velocity component in the x-direction is evaluated at this point, and the v velocity is evaluated at these points, which are displaced by half a grid in the y-direction.

Now, when we are trying to solve the x-momentum equation when we are trying to discretize the x-momentum balance equation and try to get u velocity from this, then we are looking at this particular point here not we are looking at a control volume, a cell which is centered around this point, and this point, or this point, these points, but not centered around P and E - capital P and capital E. So, the control volume the cell over which we are trying to integrate this is centered around this point; and this point has our definition is small e, because that is a east face, and this is the west face, and this is the north face and south face, for a control volume which is centered around capital P.

So, this particular for the x-momentum equation we have a control volume centered around the east face of a pressure evaluation, east face of a control cell for pressure evaluation. So, this extends half way this side, and half way this side, and half way north and half way south like this, and this is the control volume over which we would like to evaluate the x-momentum equation and get the velocity at the center of this control volume which is velocity at small e.

Similarly, we discretize over this volume to get an equation for this and this volume to get an equation for this. So, now we are trying to integrate this equation here. And in this equation, we have non-linearity we have coupling. And what we mean by non-linearity is that we have dou by dou x of u square, normally we have a constant times u. And here in solving this, we are trying to get u here and v is supposed to be a coefficient, and we do not know the value of this coefficient v, it comes from solution of the y-momentum equation.

So, this constitutes coupling, and this constitutes which we can write as u times u, this constitutes non-linearity. So, we write this equation in this form dou by dou x of u hat u star plus dou by dou y of v hat u star equal to minus 1 by rho dou by dou x of p star plus nu times dou square u star by dou x square plus dou square u star by dou y square; to emphasize the point that we would like to get u star from this, for given p star and estimated u hat and v hat.

So, we are writing this discretized this partial differential equation with non-linearity and coupling and pressure is also can be considered as coupling within here, because pressure is supposed to be coming from the continuity equation. We can write this in this way for given p star, and for estimated values of u hat and v hat, we can write this equation like this, and we solve this we discretized this and then solve it for u star. So, how do we do this here, we are doing this around this point here.

So, we can say that dou by dou x at point e plus dou by dou y of this at point e dou by dou x of dou p by star by dou x at point e and this whole thing at point e. So, this is what we are saying by integrating this or writing this for the central cell here. Now if you consider the first term here this is dou by dou x of u hat u star at e, so at this point. So, we can write this as because it is dou by dou x we can say that we can write dou by dou x of u hat u star this whole thing at point e as u hat u star at capital E minus u hat u star at capital P divided by delta x. So, dou by dou x of this quantity at small e at this point is the value of this at capital E minus the value of this at capital P divide by delta x, this an a approximation here.

Now, in this case, we need to know what is u hat at capital E, u hat at capital E is not known because we do not know what u at capital E, it is not evaluated. But we can say that u hat is the sum of this and this point which is we can call as e e to the next point of to the east of this. And similarly we can put this as e n and this is so this is e s like this, so by denoting it like this. We can write u hat at e at capital e as u e plus u e e divided by 2. Now this is an equation for u star. So, we can say that this is the current value of u star or if you make it u current value of u star then it becomes non-linear. So, we will put this as the previous values. So, and similarly, we have in this u hat at capital P, which is coming here. So, we can write u hat at capital P as this is u hat w plus u hat e by 2.

So, all the hat ones are from the previous iteration, we know that we going to start with some estimated pressure field and solve this somehow to get u and solve the y-momentum equation to get v and from that we get improved pressure and then we come back and substitute we are going to do it in a loop. So, this is star quantity is at k plus 1th iteration and hat quantity is at kth iteration. So, we can make use of this thing here. So, with this, we have tackled these two things. Now when we look at this term here, this is advection term. So, we know that advection term we would like to use using up wind.

So, assuming u prime at p and u prime u hat at p to be positive, we can say that u star at capital E, so that is this one, if the flow is in this direction, because your u p and u e are positive. So, flow is going in this direction. So, the advected velocity is equal to u e and similarly u star at capital P here, the advected velocity component is the velocity which is coming from the nearest neighbor here and this is equal to u u star w. So, in this way, we can fix this thing and we can write this as u hat e plus u hat e e by 2 times u star of e minus u hat e plus u hat w by 2 times u star w by delta x. And this is a known value, this is a known value, this is what we are trying to find out and this is the neighboring value that is coming here.

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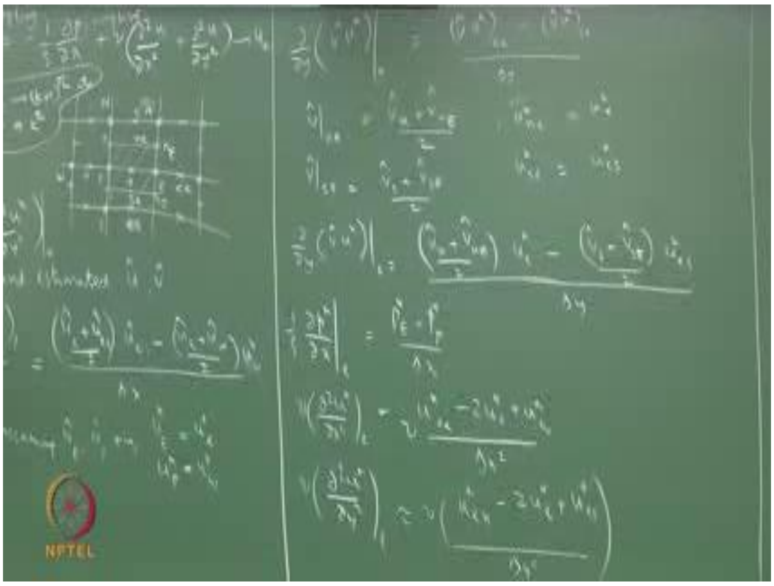

So, in the same way, we can come to this equation here we can this term dou by dou y at v hat u star evaluated at the same point here can be written as the value at this face minus value at this face. So, this is and let us put some nomenclature here. So, let us call this as the north face for point small e, and then the south face for point small e.

So, we can put this as v hat u star at n e minus v hat u star at s e divided by delta y. So, now, we have to evaluate what is u star and at this point and u star u star at this point here. So, we can say that v hat at n e is the average of v at this point and this point. So, we can call this as v hat at n which is being evaluated here plus v hat a n n and we will call this as n capital E, so that is v hat at the next point at which velocity is being evaluated this by 2. And similarly, here v hat at s e this point is the average of these two, and we will call this as again s e here. So, this is v hat at s plus v hat at s e by 2.

Now, what we are making use of here is that if we need to estimate the velocity v velocity at this point, we estimate it from known velocities at the neighboring points. So, here small v is evaluated at this plus point. So, these are the two immediate neighbors, the next immediate neighbors are here and here and here and here, so they are quite far off. So, we take this one to be average of these two. And again when we come here the nearest neighbors at which v is being evaluated is this thing here and then we can take this to be the average of the two. In case of non-uniform grid spacing, we have to take weighted average of the two based on the lengths in each direction or areas, or volumes in each direction, in that way we can get an estimate for this component at n e and this component as s e.

Now the u star value is based on upwinding because flow is coming in this direction, and the specific value of u that this is bringing assuming that v v hat is positive then at this point this is will be equal to u e. And u star at s e will be equal to the value at this point here, so these equals to u star e s. So, if the flow is coming in this direction, the value of u star that is bringing is from the immediate downward neighbor which is u star at e s. And similarly this point here the flow is coming in this direction, so it cannot be bringing the value in this direction it only bring in the upwind direction that is u e star here.

So, with this, we can write dou by dou y of v prime u star at e s v n plus v n capital E by two times u e star minus v s plus v s capital E divided by 2 times u e s star divided by delta y. So, now, we have 1 by rho dou p star by dou x at small e, this is variation in the x-direction. So, we can and the immediate two neighbors at which pressure is evaluated is p capital E minus p capital P divided by delta x, and here it is a star quantity, so we have this. And now we have this quantity here and nu dou square u star by dou x square at e can be written, here there is no coupling there is no non-linearity this is just a simple derivative second order we can use second order central differencing to write this as u star e e minus 2 u star e plus u star west divided by delta x square times nu.

So, we are looking at if this is i this is i plus 1 and i minus 1. So, we take this as u at i plus 1 minus 2 u i plus i u at i minus 1, so that is u at double e minus 2 u at e plus u at w divided by delta x square. Similarly, we can write nu times dou square u star by dou y square at e s; in this particular case, we have y variation. So, we make use of this minus this minus two times this plus this. So, we can write this as u star we call it as en minus two u e star plus u star e s divided by delta y square. Now we can substitute all of these in this equation and get an overall discretized form of this equation.

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So, let us do that. So, we substitute this one here first in order to in order to emphasize the variable, I am writing all the variables here and the coefficients like this here. So, you have this u star minus this whole thing is equal to minus 1 by rho, here p is the variable coming from something other equation. So, this is assumed to be known when we are solving this equation plus nu by delta x square and here we have plus nu by delta y square, there is no equal to. So, this is the discretized equation of this partial differential equation for a cell which is centered around small e.

So, now let us look at this. We can now write this as for example, bring this u e star here, and the u e star here, and the u e star here, and here on to one side. So, we can write this as a e times u e star and then we have a w times u star w and then we have plus a e s times u star e s, and we have e n here plus a e n times u star e n plus a e e times u star e e equal to some b here. So, we have identified all the variables. So, we should have put this as a e w e s, let us call this as b e here.

Now, in this, we can now write what is a e. Let us write down what is e a w a w has a contribution coming from this, so this is u hat e plus u hat w by 2 delta x that is this one. And there is this one here this on the right hand side, so we bring it to the left hand side. So, minus nu by delta x square is u w here. And similarly, we can write a e s, there is no a e s here there is one a e s here. So, you have minus v hat s plus v hat s capital E by 2 delta y that is what we have from here and there is also plus nu by delta y square. So, that needs to be brought on to the left hand side. So, this will be minus nu by delta y square is the coefficient for this.

And similarly a e n is not here, not here, not here, not here, it is only in this and its coming on the right hand side. So, this will be minus nu by delta y square and a e e that is this one not here, not here, not here, it is only here, so that will be minus nu by delta x square. So, we have four neighbors west, east south, east north, east east.

So, we come back to this we have this is a point, we have east north, east south, east east and the west part. So, their coefficients are correspondingly evaluated here and then we can see that a e has contribution from here that is this one and this contribution is coming from v n here and let us just write it down. So, a has contribution coming from here which is u hat e plus u hat e by 2 delta x. And it is coming here plus v hat n plus v hat n capital E by 2 delta x.

And here we have minus 2 by nu delta x square, but when we bring it to other side, this becomes 2 nu by delta x square and here we have plus 2 nu by delta y square. So, we have u e star figuring in this, figuring in this, in this and this. So, you have contribution here this must be delta y here, and delta x square delta y square. So, we have and finally, b e is we have these this is here this is already included and the only thing that is is this. So, this is minus 1 by rho p star e minus p star p by delta x.

So, now we can say that we have discretized this x-momentum equation on a staggered grid for the u velocity components at the center of this node and at the center of the surrounding nodes at which u is evaluated. So, we have u at e, u at e, u at west, and u at e north and e south. So, this is our computational molecule. And these are the five variables; these are the 5 u velocity components which appear in the discretized equation with these coefficients here. And these coefficients are in the form of the values of u at the previous times step at the nodes at which they are evaluated, the values of v at the nodes at which they were evaluated, and geometry constants like delta x square and properties like viscosity.

So, using this way we can convert a discretize differential equation partial differential equation into an algebraic equation like this. And we can therefore, write the discretized x-momentum equation in this form that is u e star plus sum over neighboring values of a and b u star m b equal to minus 1 by rho p star capital E minus p star capital P by delta x. Where the neighboring points are the neighboring points for e which happened to be the west side, east, south, east north and east east. So, we can come back to this here that is what we have written here. And we have the coefficients and we call this as the corresponding coefficients can be written in this particular way.

In the next lecture, we look at how to derive the pressure correction equation by doing the corresponding substitutions like this.