Computational Fluid Dynamics
Prof. Sreenivas Jayanti
Department of Computer Science and Engineering
Indian Institute of Technology, Madras

Lecture – 40
Pressure equation method, staggered grid system

We have seen the difficulties that we face when we are dealing with incompressible flow equations in the sense that the continuity equation is not very useful to determine the pressure. And we saw three methods to somehow derive the pressure information from the continuity equation in the sense of introducing an artificial comprehensibility, so as to link an artificial density with the pressure or to complete the eliminate pressure through the use of stream function idea.

And also to derive a separate pressure equation and then incorporate an iterative method or a sequential solution of the momentum equation, and the pressure equation alternately in order to get an overall equation. In apart from this determination of pressure, we also have another problem with the evaluation of pressure in incomprehensible flows and this is known as a Chequerboard oscillation in pressure.

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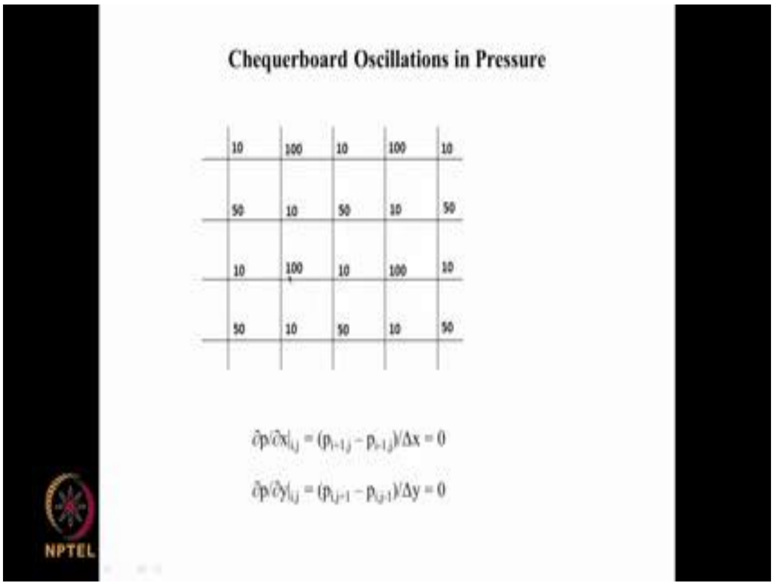

In the screen here, we have Cartesian grid, and then you have the grid nodes at intersection of this i equal to constant line, and j equal to constant lines. And what I put here are some fictitious numbers for the pressure indicating how pressure is changing and these are put in a special way here. We have 100, 10, 100, 10 like this; and in the y-direction is 50, 10, 50, 10 like that. And this pressure field is such that we see that there is variation of pressure, with every grid point there is variation of pressure. But if you want to evaluate, for example, at a given i and given j at this point if you want to evaluate what is the pressure gradient in the x-direction, because it is a pressure gradient that is coming in the momentum equations to try with the u velocity and v velocity.

If you look at the x-momentum equation, you have dou p by dou x and dou p by dou x evaluated at i comma j here as p i plus 1 minus p i minus 1 divided by delta 2 delta x. So, there is a 2 missing here. You will see that this is 100 minus 100 divided by 2 delta x, so that is equal to 0. And similarly, if you are looking at dou p by dou y at i comma j, again you have p at i j plus 1 minus p at i j minus 1 divided by 2 delta y, again a two missing here; and that again is 0 here because it is becomes 50 minus 50 divided by 2 delta y. So, even though we see a change in pressure here, as for as the momentum equation are concerned the pressure gradient is 0.

So, a pressure field which is varying strongly with the grid here at the smallest level 100, 10, 100, 10, 100, 10 like this and 50, 10, 50, 10 in this direction it does not matter what the numbers are. So, if there are fluctuations like this in pressure, then the velocity field will not respond to this and it is impossible to get rid of these kind of pressure variations through changes in velocity and these pressure fluctuations, these pressure variation will be there in the final solution. And so that is that is one of the problems, and we need to get rid of this.

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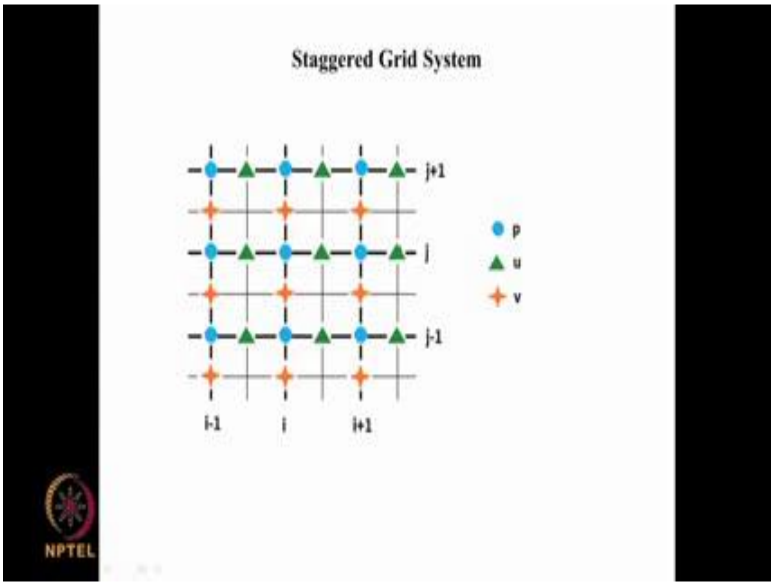

And the way that that is proposed quite early on in the mid sixties itself was to use a staggered grid system. We know that we are solving different variables using different equations. We have seen that the x-momentum equation is used for getting the u variation, and we solve the y-momentum equation for v, and we solve the continuity equation for density or pressure like that.

And we know that in CFD, we have many, many points at which these variables are evaluated. So, in the scalar transport equation, we have the variable phi and it is evaluated at points which are spread throughout the computational domain. And we also know that when we come to the Navier-Stokes equations, we are solving each equation separately. So that means, that it is not necessary that all the variables are evaluated at the same point, if this values are evaluated at close points also it is ok because the idea is that we make use of interpolation to get the velocities.

So, in that context, a staggered grid system like the way that it shown here is also permissible. And in this particular staggered grid system, you are evaluating the pressure p denoted with the blue circle here at intersection of i comma j. For example, this is i and this is j here at the intersection point you are evaluating p, again at intersection of i plus 1 j, i minus 1 j, j plus 1, and i j minus 1, so intersection of constant i and constant js are the locations at which pressure is evaluated. And in the case of 2-D, you still have u and v as the variables; and u is evaluated mid way in the between the two pressure evaluations in the horizontal directions.

So, this is at i plus half j, i plus half j minus 1, i plus half j plus 1 these are the triangles the green triangles are the locations at which u is evaluated. Similarly, v is evaluated at this star and these are displaced in the horizontal direction by half grid spacing by the vertical direction by half grid spacing. So, this is where that is at i j plus half, at i j minus half, and again i minus 1 j minus half and i minus 1 j plus half like that. So, in this staggered grid, pressure is evaluated at intersection of i comma j and u is evaluated at intersection of i plus 1 comma j, and v is evaluated at intersection of i j plus half, and these are spread throughout the domain.

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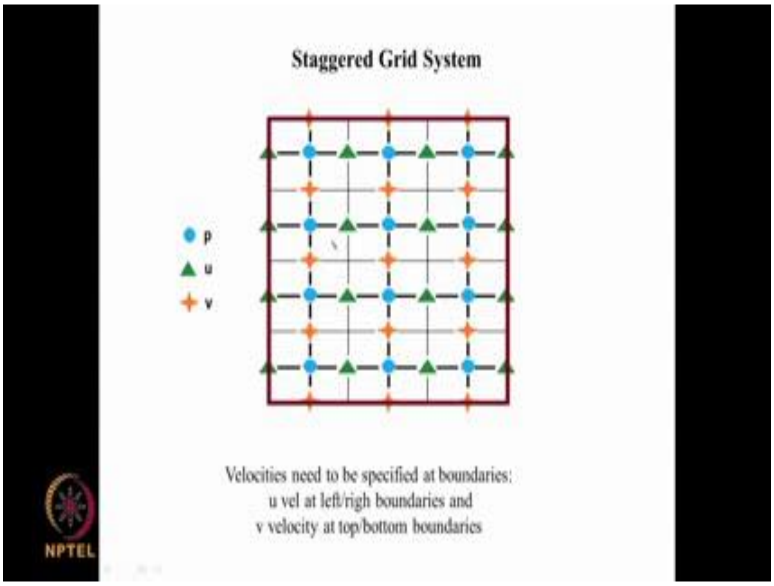

And if you are looking at for example, a rectangular domain like this, you can put a mesh here in which u velocity is evaluated at these points here; and these points are such that the left boundary, and the right boundary are the points at which you have u velocity is evaluated. And the top boundary at the bottom boundary is the points at which v velocity is evaluated, and pressure is evaluated at mid points here. So, in a way, we do not need to know what is the pressure we do not need to specify the pressure at these boundaries;

and pressure is evaluated only at the interior points.

And the advantage of this particular way is that if you now evaluate dou p by dou x at i comma j, for example, at so when you are looking at in the x-momentum equation, you are solving for u at this point and this point here. So, at this point, you need to know what is dou p by dou x; and when you do that then pressure at this point minus pressure at this point divided by this distance will give you the pressure gradient at the midpoint here.

So, as in this example here, you have you have 100 here and 10 here, then the pressure gradient is not zero because pressure gradient is 100 minus 10 divided by this distance, so that will be minus 90 by this delta x. And pressure gradient at this point will be 10 and 100, 10 minus 100 divided by delta x here. And similarly, you had 10 here and 50 here and 50 here, earlier if you are evaluating the velocity at the same location then you would get this as 50 minus 50 by 2 delta x, but you are evaluating the v velocity at this point here.

And if you say what is the dou p by dou y at this point then the difference between the two neighboring points that is 50 minus 10 divided by this delta y will give you the pressure gradient here and here it is 10 minus 50 divided by delta y. So, in that sense, even if you had 100, 10, 100, and 50, 10, 50, you would not be getting pressure gradient evaluated as zero at the points, where the momentum equation is being evaluated, and therefore, the velocity field will be responsible to this pressure changes here.

So, in a grid like this, here the pressure gradient is computed to be 0 both in the x momentum and y momentum equation, but if you are evaluating velocity v velocity here then the pressure gradient is not 0. If you are evaluating the u velocity at mid point here then the pressure gradient is not 0. So, therefore, the velocity field will respond to this pressure changes, and it will eventually produce a smooth variation of pressure and this Chequerboard oscillations can be avoided, so that is the concept of the staggered grid system. And we look at the pressure correction equation or the pressure evaluation those type of things on this staggered grid system, and this is how it has come about historically.

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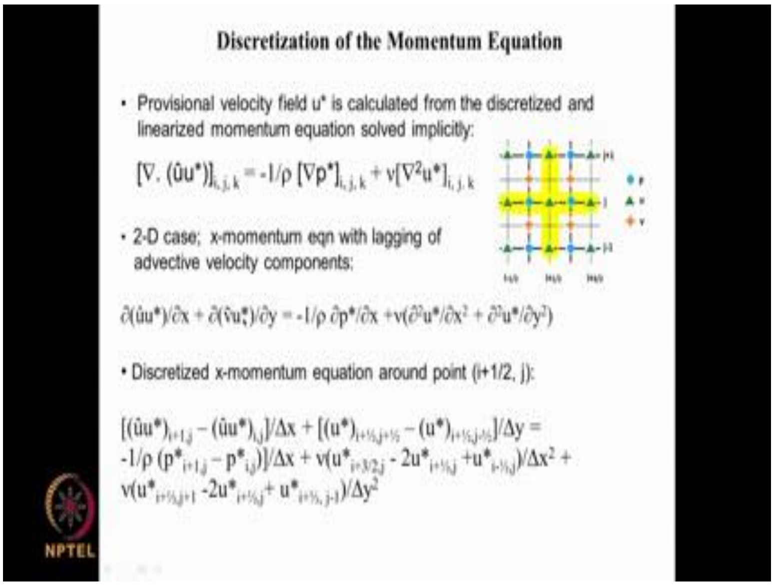

So, in this short lecture, we look at how to deal with the discretization of the momentum equation on the staggered grid, which is shown in the small figure here. Here, we are looking at discretization of the x-momentum equation and so as to understand the inter cases involved in this. And when we are looking at an x-momentum equation, it is evaluated the u is evaluated at i plus half j. So, we are looking at discretization of the x-momentum equation at this particular point here, and so we notice that pressure is evaluated at half distance in the x-direction here and it is not evaluated at these points here, it is evaluated in the y-direction only displaced by these points.

And the immediate neighbors for the velocity u velocity are i plus half j plus 1 and i plus half j minus 1 and i plus 3 by 2 j and i minus half j. So, those are the immediate points at which u velocity is known. V velocity is known at these points which is i j plus half and i j minus half like that, and pressures like this. And our momentum equation is gives as del dot u u minus 1 by rho gradient of pressure plus nu times del square u. And you also have the time dependent term lets first look at the steady stay terms and see how we do this.

So, in this equation, we have the usual coupling and non-linearity associated with the Navier-Stokes equations. And we deal with those issues in this way. This should be actually del dot u u the same thing and but we are solving this equation this is the x momentum equation. So, for the 2-D case, the x-momentum equation can be written as dou by dou x of u u plus dou by dou x of v u minus 1 by rho dou p dou x plus nu dou square u by dou x square plus dou square u by dou x square that is a correct equation, without any of this extra stars and then tilde's and all that thing. So, that is the exact equation.

The discretized form of this we are solving this for u and we are also solving this currently in the pressure correction of pressure equation approach we then assumed a pressure. So, the star quantity here means that it is as of now it is a provisional value, and this equation is such that we are using this equation to get u star for a specified p star.

So, we assume that we know the pressure field and using if the pressure field is known then this equation can be used to get the velocity. And in the case of the x-momentum equation, if the pressure field is known, so that we can evaluate this dou p star by dou x at any point then we can discretize this rest of the equation, so as to get u star. Now in this you have non-linearity. So, we say that we make use of an estimated velocity here and then the two velocities which is needed to be determined. And here also you have v tilde this is the velocity that is coming from somewhere else; this is assumed to be known may be from the previous iteration of previous time step like that. So, this is an assumed value, this is an assumed value, and this is u star is a one which we are going to determine.

And here in this case the viscosity is supposed to be known and so we have only this quantity here. So, the exact momentum equation is discretized is linearlized and discretized in the following way. It is linearized to the extent that the coefficient one of the use that is coming in this term is taken to be the previous iteration value it is longed, it is substituted using the Picard iteration. And similarly this is that is coming in this term is assumed to be known from the previous iteration. And p star we are assuming that when we are evaluating the x-momentum equation, then p is already known and the diffusion part is treated correctly. And we also are looking at evaluating this whole thing in an implicit way, so that is it is a same u star that is coming on the left hand side as this is coming on the right hand side.

So, we can write this as u u star at i plus 1 j minus u u star at i j divided by delta x plus this should be end. So, we take this point here this is a type less half j and then we make a control volume which is spreading over this half. So, it spreads between half ways between the next evaluations.

So, it goes up to this in the north direction; and in the horizontal direction, it goes up to the midpoint here, because that is a midpoint between this evaluation, this evaluation and; on the horizontal side, in the right direction, it goes up to this. So, the control volume that we are looking at is enclosed by these 4 stars here, with centered around i plus half j. So, we are evaluating this term as at u hat u star at i plus 1 j which is at this point minus u tilde u star at i j which is this point. So, if this is i plus half this is i here. So, this value minus this value divided by delta x and that is what we have written here.

And similarly, we should be looking at this term here as this value minus this value, this value minus this value divided by delta y here. So, there is a v that is missing here v hat v tilde at i plus half j plus half so that is this 1 minus v tilde i plus half j minus half. So, that is this 1 divided by delta y, we are writing it like this. And pressure gradient at this point is written as p i plus half minus p i minus half divided by delta x. So, this one is evaluated as p at this point minus p at this point divided by delta x. And this is done using central differencing, so that is given as u i plus 3 by 2, which is this one it is to the right hand side. So, we reevaluating this one using central differencing centered around this. So, this is u at i plus 3 by 2 j minus plus u at i minus half j, so that is this one that is this one minus 2 u i plus half j.

So, this is the approximation for dou square u by dou x square at i plus half, because this is a derivative in the x-direction, we take this value minus 2 times this value plus this value that is what we have here. This term is again using central differencing around this point here, because it is a second derivative with respect to y, we take the immediate up neighbor plus the immediate down neighbor minus 2 of this value. So, that is why you have this as u at i plus half j plus 1 - this 1 minus 2 u i plus half j plus u i plus half j minus 1 this divided by delta y square.

So, this is the discretized x-momentum equation at i plus half j at this point here. So, we have a computation molecule which brings in the u velocity at here i plus 3 by 2 j which is this one, and here you have i minus 2 j which is this one, and then you have i plus half j minus 1 which is this one and i plus half j plus 1 this.

So, these are the four neighboring points which are being added to this to give you u star, but the coefficients themselves are that need to be evaluated are this u we need to have this one u star evaluated at i plus half and then u hat evaluated at i plus half like that. So, if you are able to come up with prescriptions here then will be able to complete this discretization here.

Now in this discretization, where are we, if you look at the diffusion term, diffusion term is not posing as any problem because we have exactly the four neighboring points plus the central point. So, these are the points at which velocity is being evaluated u velocity is being evaluated. So, this diffusion term and this diffusion term is like the usual thing that we are dealing with. And pressure is now coming at p i plus 1 minus p i minus 1 by delta x and again this is not going be a problem for us because we are going to get pressure from somewhere else, and we are making use of the known pressure or to be known pressure at the points where pressure is being evaluated.

Now, what is remaining for us is the advection terms here, you have u star u tilde and u start at i plus 1 j and i plus 1 j is this point here. Now here we have a problem because we do not know what u value is at this point. So, we interpolate this value to be between this plus this divided by 2. So, the average of u i plus half and u at i plus half and i plus 3 by 2 is taken to be this value u star here. And now we have this one, if you are making use of central differencing that is what we would also use for this.

But if you are making use of upwind scheme, then this u star at i plus 1 here is taken to be this value that is a upwind value, if u is positive, and this value if it is taken to be negative. So, if you are using upwind scheme backward differencing scheme in the case of u being positive for this, there are two velocities u tilde which is the from the previous value and this is the current value.

So, when we make use of the upwind value we are trying to fix the value of the phi at this i plus 1, and we need to fix the value of phi at i minus 1. So, to be consistent with upwind we say that the value of phi at this location is equal to the upstream value if u is positive. So, assuming since we have an estimate of u from the previous value, previous iteration, we see whether that is positive. If that is positive then u star at i plus i plus 1 here is taken to be u i plus half j. And similarly u star at i which is this point here is taken to be u i minus half j, but the coefficient here is evaluated as average of these two and average of these two here. Similarly, here we have this v hat which this term needs to be evaluated at u i plus half j i plus half at this point here.

So, this brings in v hat. What is v hat here we do not know where the v hat we do not have v is not being evaluated here, but since we have the two neighboring points the v hat is evaluated as this plus this divided by 2. So, the v tilde here when we discretize this we write this as dou by dou y of v u star and we are writing this as v u star at j plus half minus v u star at j minus half divided by this delta y. So, we need to know what is this v tilde at j plus half and j minus half. So, we say that v tilde at j plus half are v at i j j plus 1 plus v at i plus 1 j plus n divided by 2, so the average of these 2. And similarly, we tilde at this point here is taken with average of these two. So, the coefficient that is coming in this which is missed out in this particular thing here is evaluated by taking the average.

Now, what about the u star at i plus half j plus half this is again based on up winding. So, if v is positive then u plus half j plus half here will be equal to this; if v is negative then this will be at u at i plus half j plus 1 here. So, the value of this star thing, when it appears in this term is dependent on whether v is positive or negative. If v is positive then we take the backward value, so that is this is j plus half, we take we take this value and if v is negative we take the this value, because when we say v is negative the flow is coming down in this direction, so the upwind value is this. If v is positive, we use the flow is going in this upward direction, so it brings in the upwind value which is this one.

So, this equation here is discretized as per our template and our template says that the advection terms must be done using upwind method and the diffusion term must be done using central method. So, the diffusion terms here dou square u by dou x square are being done using the upwind the central method. So, we take this as this value minus 2 this value plus this value here divided by delta x square. And this one is evaluated as this value minus 2 this value plus this value divided by delta y square, so that is being done here. The source term is being evaluated as making use of them to neighboring values of the pressure.

And the advection terms are the ones that we are looking at we which require some careful thinking because advection terms are evaluated at this point and this point and these are evaluated in a special way. There are two velocities here one is the advection velocity and this is the phi. So, you have v here and then phi here. The phi value is only is determined using up winding method. So, phi value at this particular thing is taken to be the upwind value always. So, if v is positive then the flow is going in this direction, so the upwind value is this. If v is negative then it is taken to be this. So, based on that we fix the value of this and the value of the coefficient v here is taken to be the average of these two.

Similarly, when we want to evaluate this v tilde u star at i plus half j minus half at this point here, then again we make use of the same upwinding principle. If v at this point is positive that is average of these two is positive, then the upwind direction means that it should be the u star at this point is taken to be the u star at this point that is i plus half j minus 1. If v the average velocity at this point is negative, then the upwind direction means that we have to take this, and the coefficient v tilde is taken to be the average of these two.

Now, if you come to this term here again there are two velocities it happens to be the same here, but within this the coefficient velocity is taken to be at this point is taken to be the average of these two. And this velocity this term when it needs to be evaluating at this point at this point here, the coefficient velocity is taken to be the average of these two. And the u star which is the equivalent of phi is done using upwinding. So, if u at this point is positive, then the flow is in this direction. So, the u star will be corresponding to this, and here this is going to be corresponding to this. So, in this way we evaluate a discretized momentum equation.

In the next class, which we are going to do on the board, we are going to revisit this and then we are going to write down the discretized x-momentum equation and discretized y-momentum equation and then we also write an equation for pressure. So, that we can evaluate pressure, and then from the discretized continuity equation, and u velocity from the discretized x-momentum equation, and v velocity from the discretized y-momentum equation using the staggered method, using the staggered grid system. So, we will work it out on the board, so that we can follow the argument easily and that will be done in the form of a tutorial.