Computational Fluid Dynamics
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Lecture - 39
Artificial compressibility method, Stream function vorticity method

We have seen two methods for in compressible flows in the artificial compressibility method and the stream function vorticity method. Both of these have their own specific limitations the artificial compressibility method can be used for 3-D, but not for time dependent flows the stream function method can be used for time dependent flows, but not for 3-D. We look at pressure equation method which can be use for both 3-D and time dependent flows and, but will explain this method in a specific way and the idea again is to follow the method and work out the details at a later stage.

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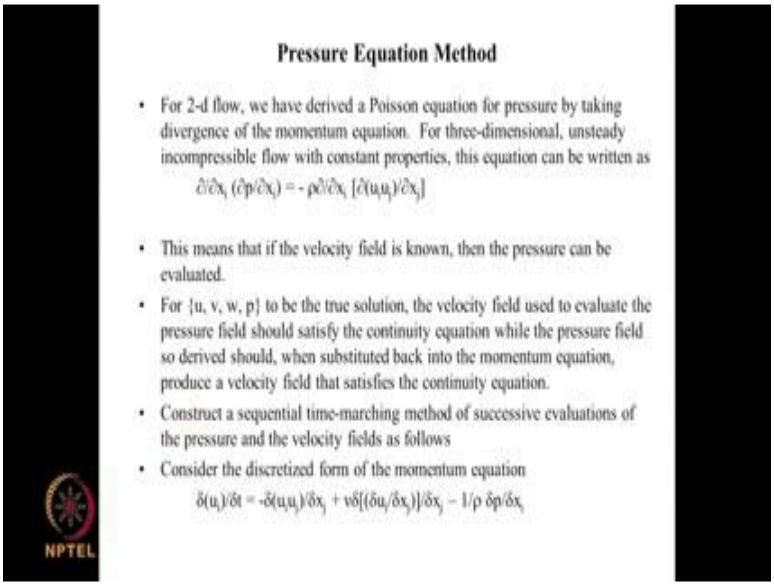

So, the idea of the pressure equation method is already the germination is there in the stream function vorticity method where, we derived an equation for pressure.

So, we look at a method which is applicable both for time dependent flows and 3-Dimension flows at the same time it is not either for this or for that. In this is based on the idea of solving an equation for pressure together with the momentum equations and how this equation for pressure is constructed we have already seen, when we discussed stream function vorticity method, in which we said that it is possible to manipulate by taking the divergence of the momentum equation we can derive an equation for pressure and that we showed how to do for 2-D flow for 3-Dimensional unsteady incompressible flow with constant properties the same Poisson equation for pressure can be written in this compact way using the index notation format it is dou by dou x i of dou p by dou x i in this term the index i is repeating therefore, this implies sum over the 3 values of i that it can take. So, that that makes it dou by dou x of dou p by dou x plus dou by dou y of dou p by dou y plus dou by dou z of dou p by dou z.

So, that is a Laplace in of p here and on the right hand side we have minus rho dou by dou x i of dou by dou x j of u i u j. So, what do we have on the right hand side density is known property and the velocity field. So, this equation here pressure equation implies as we have seen earlier that, if the velocity field is known then pressure can be evaluated. So, and we also know that for u, v, w, p to be a true solution of fluid flow then the velocity field that is u, v, w which are functions of x, y, z, t used to evaluate the pressure field should satisfy the continuity equation while the pressure field, derived when substituted back into the momentum equation should produce a velocity field that satisfies the continuity equation.

So, we would like to we would need to solve both the continuity equation and the momentum equation. So, if we are evaluating pressure from a velocity field this pressure when substituted in to the momentum equations to get a velocity field, that velocity fields should also satisfy their continuity equation. So, that kind of necessary condition is there for u, v, w, p field to be considered as a true solution of the Navier stokes equations. So, using this idea it is possible to construct sequential time marching method will see and will start with an explicit method of successive evaluations of pressure and velocity which is as follows.

So, you consider the discretized form of the momentum equation. So, we have dou u by dou t plus dou by x j of u i u j equal to nu times dou square u by dou x dou x j minus 1 by rho dou p by dou x i. So, that is the momentum equation you are writing this as discretized form. So, as to distinguish it from the continuous form is using this delta here and we have taken this term on to the right hand side. So, that we have dou u i by dou t equal to del u i by del t equal to del by del x x j of u i u j plus mu the kinematic viscosity times; del by del x j of del u i by del x j minus 1 by rho del t by del x i these del c are difference operators you could be using upwind scheme here, we could be using the centers scheme for this, you could be using central scheme or whatever other scheme that you want to put here.

So, it does not matter what it is you choose the appropriate once for the discretization operators for each of these even here you can use forward first order second order central all those kind of things can be used here, but this is the discretized form of continuity equation. Now we can write this simply as of forward differencing and write this as u i n plus 1 minus u i n by delta t.

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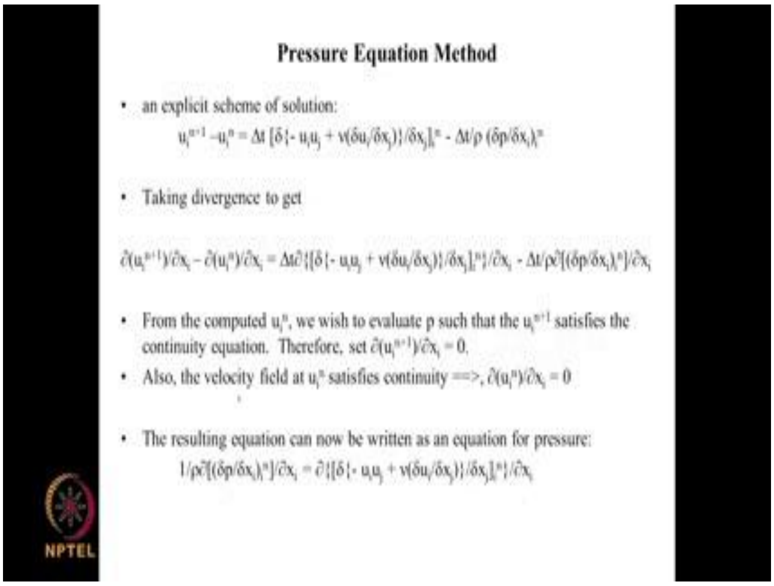

If we do that then we get an explicit solution scheme of solution can be written as u i n plus 1 minus u i divided by delta t we take the delta t to this side delta t times all the right hand side terms which are evaluated at nth time step. So, this is u i u j term del by del x j of u i u j with a minus n because it is taken to the right hand side and the diffusion term here. So, this is evaluated as at nth time step, and even the pressure gradient is evaluated at the nth time step.

So, here we are not doing any cheating we are just writing it like this. We are we are putting down a scheme of solution. So, that scheme of solution is saying that forward difference for the time derivative, and explicit differentiation as of now we are not specifying what these difference operative should be we can use for example, the up end differencing and central differencing and this will see.

So, now we take divergence of this equation. So, that is dou by dou x i because i is index here. So, when we take the divergence we get dou by dou x i of u i n plus 1 minus dou by dou x i of u i n. So, every term is put as dou by dou x i operator is put here and delta t is a constant, dou by dou x i of this whole thing minus delta t divide by rho times dou by dou x i of this particular term here. And in this is the modified momentum equation, and here we want to impose the idea that if we get the from the computed u i n n that is at the previous time step values, we want to evaluate p such that if this p is substituted into the momentum equation which then we get u i n plus 1 this u i n plus 1 which should satisfy the continuity equation and what is a continuity equation dou by dou x i of u i n plus 1. So, we want to we want to a velocity filed such that it is satisfies continuity equation. So, this term is 0.

We are starting we are we already have the previous times step value, the previous time step value must also satisfy the continuity equation. So, dou by dou x i of u i n must also be equal to 0. So, when you look at this equation here, if you are imposed the condition that u i n plus 1 must satisfy the bound the continuity equation then this is 0 since u i n satisfies the continuity equation because, you now moving on to u i n plus 1 and by then you have already got u i n and for u i n to be correct solution it must satisfy the continuity equation. So, dou by dou x i of u i n is also equal to 0.

So, if you said these things to 0 here; 0 this because this would have satisfy the continuity equation and this to 0 because we want to it satisfy the continuity equation. That gives us a condition for pressure. So, these two are 0 and we can now cancel out delta t here and will get an equation for pressure at nth time step like this.

So, this equation contains on the right hand side, velocity at the nth time step and these are also velocity set nth time steps. So, these are all explicit terms and since we have already got u i n and we moving on to u i n plus 1 you use this 1 to evaluate pressure, and this pressure is at p i n so; that means, that p i j k n. So, we put that into this equation here. If you put that p here and then all this right hand side is known and this is known. So, from this we can get u i n plus 1. So, we start with u i n and then we get p i n and is substitute in this and then we get u i n plus 1.

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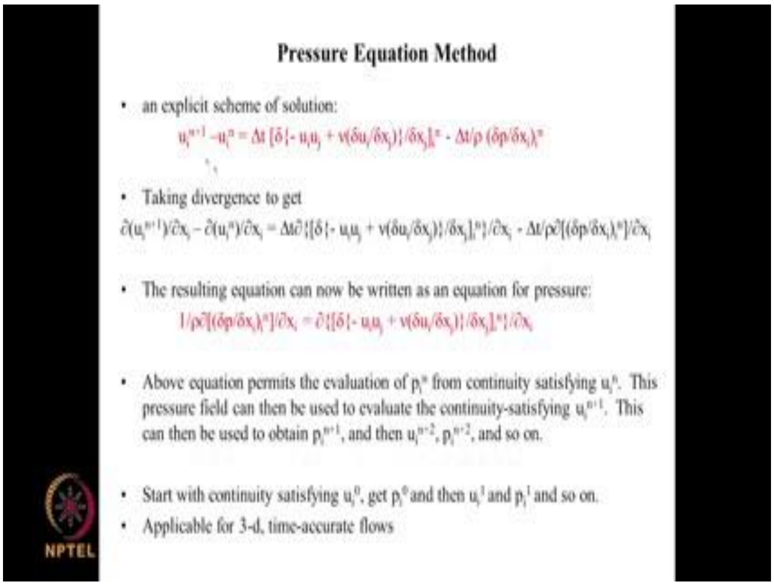

So, that is what is shown here. So, these equations permit evaluation of this equation here permit the evaluation of p i n from continuity satisfying u i n. So, this pressure field can be used to evaluate continuity satisfying u i n n plus 1 here. So, this can then be used to obtained p i n plus 1. So, once you get u i n plus 1 you can put this on to n plus 1 here, and then get p i n plus 1 and once you get p i n plus 1 you can put this on to this you can get u i n plus 2 and then you come here to get p i n plus 2. So, in that way you can use these 2 equations alternately, to get u i n plus 1 here from u i n you first solve this to get p i n and then, you put that p i n here to get u i n plus 1 and you put this here back to get p i n plus 1 and then you move on like this.

So, you are satisfying the momentum equation and pressure equation which is obtained by imposing the condition that the velocity field that we getting here satisfies this. Each of these is explicit method. So, we start with a continuity satisfying u i 0 get p i 0 from this and then put this p i 0 here and all the u i zeros here and get u i 1 and then you put u i 1 here, and then we get p i 1 and then move on like this. So, this method is applicable for 3-Dimensional flows and also time accurate flows and it is explicit.

So, in this way we are this method is differentiated from the two previous methods by the fact that we are solving an explicit equation for pressure, along with the velocity along with the momentum equations in the stream function vorticity method we completely eliminated pressure and it is not necessary for us to evaluate pressure in order to in order to get u n v.

In fact, we need not evaluate if you are not interested in p, then we need not evaluate u and you need not evaluate p you can get u i and v i n plus 1 without having to compute pressure, but here we have to compute pressure, but and we are doing in in a hopscotch manner first you get get this and then you get this from initial condition. Initial condition which is in which the initial condition is such that the velocity filed satisfies the continuity equations.

So, that this is 0 here, if it does not become 0 then the initial condition does not satisfy the continuity equation it is not a true velocity boundary condition. It is not a true initial solution. So, we have we need to adjust that here. So, hopscotch we can construct this.

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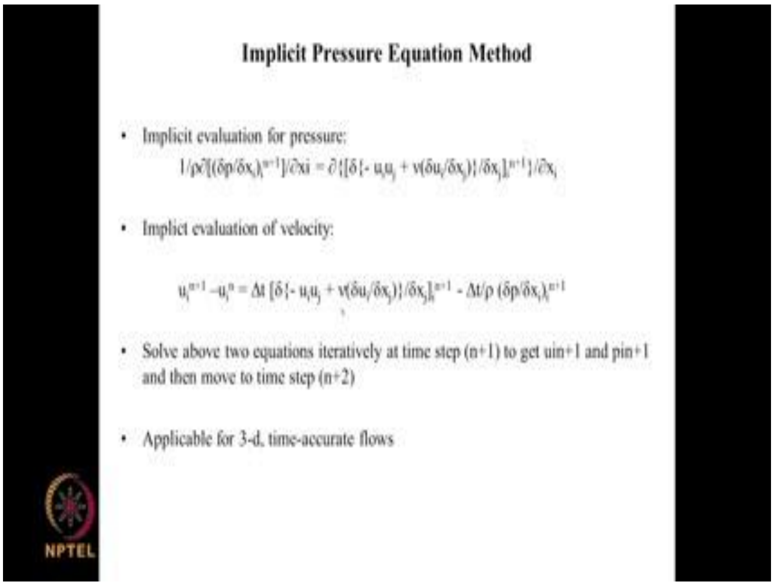

We can also construct an implicit pressure equation by the in a very similar concept. We got this equation here for the equation for pressure by writing this in as explicit form here, but we make this implicit. If you make the velocity field implicit here, then this is the equation, and we started by put in this to be equal to we made an explicit scheme like this if you make it implicit then this becomes n plus 1 and this becomes n plus 1.

So, that is what we have here, and we take divergence of that and we have this as n plus 1 this will be still be there, and this whole thing is such that we put n plus 1 here and n plus 1 here. We use a same argument that the velocity if the new velocity of u i n plus 1 must satisfy the continuity equation. So, this equal to 0 the old velocity field anyway satisfies a continuity equation. So, this anyway equal to 0 and then you have an implicit equation for pressure which is given by this thing here. So, now, if you want to evaluate pressure at n plus 1 you need to know the velocity at n plus 1, but you do not know the velocity at n plus 1 you can evaluate it only if pressure is known.

So, you have an implicit evaluation of velocity here which requires p at n plus 1 and all the values u at n plus 1. So, what we do is that we start with some guess velocity field and then we can use this to get pressure field at n plus 1. Then we put this pressure filed here then we get new velocity field we put this back into this and then we get p n plus 1 you put this back here and then we get u i n plus 1.

So, just to evaluate the velocity and pressure field at n plus 1, we have to solve this and then we have to solve this and then go back and solve this and update, update, update we need to iteratively solve this things together to get u i n plus 1 and p i n plus 1 which satisfy the discretized momentum equation, the discretized implicit form of the momentum equation and discretized pressure equation derived subject to the condition that the velocity field at n plus 1 satisfies the continuity equation.

So, in this way we can solve implicitly for pressure this implicit method is differentiated from the explicit method that we have here, in which from starting from u i 0 we put that here and get p i 0 and we evaluate all p i zeros at over all i j ks here and then we immediately go on to this and solve this for all u i once u i j k at first time step and then, we come back here and then solve in 1 go the pressure at all i j ks and then we come back here. So, we are not solving repeatedly for u i n plus 1 and p i n plus 1 like that. You get this u n plus 1 immediately go on to get p i n plus 1 and the you are not coming back u i n plus 1 you are going u i n plus two in this case.

Where as in the case of implicit pressure equation you are solving this for you are solving this equation here for pressure at p i n plus 1 with an assumed velocity field it n plus 1 and once you get this p i n plus 1 we substitute here and then we solve this 1 to get u i n plus 1 and because, now u i n plus 1 exchanged the velocity pressure field also get exchanged.

So, you put this back here and then you do this you solve these iteratively together until convergence, and at that point you have p i n i j k n plus 1 and u i j k n plus 1. So, having got in a solution for these 2 equations simultaneously then you go back and then solve for u i n plus 2 and p i n plus 2 again by solving these 2 iterations, these 2 equations iteratively.

So, at each time step you are moving back and forth between this equation this equation several times, until you get converge solution and then only you move on to n plus 2.

Where as in the explicit method you solve this once and then you come here and then you solve this and then you solve this, and then you solve this, you are not coming back to both the equations in the same time step you solve for 1 time step you solve it once.

So, what is a difference? This is an explicit solution this is an implicit solution. When you are looking at coupled equations and all that implicit solution is better it is more stable. So, in that sense we can have probably much larger delta ts that a possible with this than with this. That is the adventures that we get this method is also applicable for 3-D time accurate flows just as this method is, where as in the explicit method we have forced to choose very small delta ts. So, that we do not to get into instability problems in the case of implicit methods we can drive this to study state and get a study state solution.

Just as we a if you a not interested in time accurate solution, we can take large times steps and then get to the steady state solution. So, that is 1 advantage of this. So, this pressure equation method is one which is applicable for 3-D time accurate flows. In the next lecture we look at pressure correction approach which is similar to this in some sense, but 1 particular form of the pressure equation method which is known as the which is a semi implicit method as proved to be very robust for range of incompressible flows both external flows and internal flows, and that is a method that we look at in the next lecture for the solution of incompressible flows.

So, that method is a semi implicit method whereas here we have seen an explicit method and an implicit method in both of which we are using a pressure equation, to be solved along with the momentum equation. In the pressure correction method that we going to look at, we are going to use a semi implicit method which is not fully implicit not fully explicit, it is a semi implicit method again we are solving some modified form of the pressure equation and momentum equations together and that a method that will discuss in the next lecture.