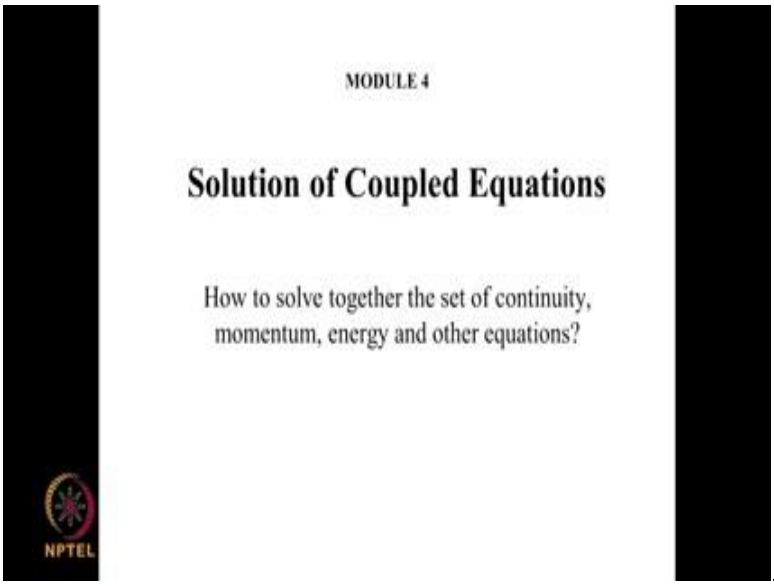
Computational Fluid Dynamics
Prof. Sreenivas Jayanti
Department of Computer Science and Engineering
Indian Institute of Technology, Madras

Module - 04
Lecture - 38
Solution of coupled equations: Incompressible flow

We are into the second part, part b of module 4, and we are concerned with the solutions of coupled equations, and that is question of how to solve together set of continuity momentum energy and other equations, but for the specific case of incompressible flow.

(Refer Slide Time: 00:32)



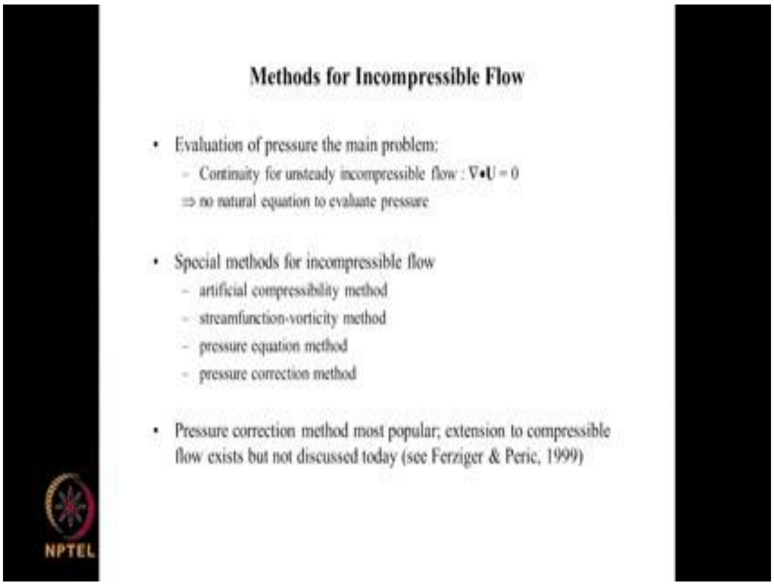
We know that incompressible flow is not a property of the fluid, but it is a property of the flow, if the velocity of the characteristic velocity of the fluid of the flow is such that, it is significantly less than the speed of sound in that medium. Then we could call it as incompressible flow irrespective. Whether density changes are there or not as a yardstick if the characteristic velocity is less than 0.3 of mark number.

So, that is 30 percent of mark number or less than you could effectively call that as incompressible flow, and any density variations any pressure variations within the flow which are induced by velocity changes are not going to make a significant change in the density. So, that is a kind of linkage that we have with between density pressure and velocity and this linkage gets broken to the linkage between the density changes and pressure changes a gets broken for low mark number flows, and those flows can be considered as incompressible flows.

So, for such flows the methods that we have discussed, so far the Mac Cormack method and the beam and warming method which are predicated on the linkage between density and the momentum equations through the equations of state. So, that kind of relation does not whole good for incompressible flows, because in the incompressible flow equations density does not come into picture it is a constant.

It does not vary it is not there in the continuity equation and it can be factored out as we will see from the momentum equations. So, if density is not their then the linkage between continuity equations and the momentum equations through pressure density and equation of state is gone. In such a case we will not be able to use the density base methods that we have seen in incompressible flows.

(Refer Slide Time: 02:54)

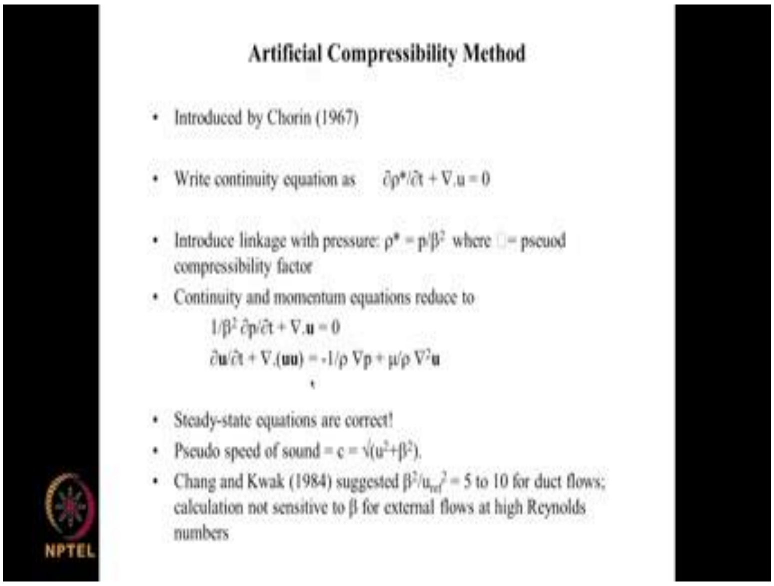


We also saw that there are special method that have been developed and we saw these things as artificial compressibility method, stream function vorticity method and pressure equation method these are the ones that we are going to learn in this part of the module. And I would like to say at this point that module 4 is difficult it is not easy, whether it is the solution of the compressible flows or the incompressible flows and you have seen the difficultly with which a solution method like that incorporating the beam warming concept has been put together and the idea of this course is not for you to know every bit of it.

It is more this lot more detail that would have to work out on your own, before you can fully understands all the details. Try cording it. So, the objective of module 4 is to expose you to the concepts and principles that have brought out in the solution of coupled equations. It is not to give you a recipe for the solution, and recipe for the solution requires lot more effect.

So, it is the flow of ideas that we looking at in this rather than the specifics of writing codes in algorithms. So, that will come with may be a more effect from each of you for your own specific problem area. So, with that thing let us go to what we mean by the artificial compressibility method.

(Refer Slide Time: 04:28)



It is a method which was introduced by Chorin in 1967 and the basic idea is that we know that the linkage between the continuity equation and the momentum equations is not there. So, we can introduce artificially such kind of linkage. For example, we can write the continuity equation as d rho star by dou t partial of rho star with respect to time, plus del dot u equal to 0 and this is an equation this is not which is not correct for incompressible flows because there is no density here.

And this density is not the density that we normally have for say water or air and those kinds of things this is a fictitious density it is brought in to create a linkage a bridge between the continuity equation and the momentum equations. How that bridge is built is shown here, we have dou rho star by dou t plus del dot u equal to 0. You define the rho star as p pressure divided by beta square, where beta which is missing here is the pseudo compressibility factor and for a given value of beta, if you know rho star you can get P. So, beta is a constant. So, if you substitute this expression here you are rewriting the continuity equation in this way. So, it becomes one by beta square dou p by dou t plus del dot u equal to 0.

And the momentum equation is written as dou u by dou t plus del dot u u equal to minus one by rho del rho del p plus mu by rho times del square u. Now, these are the equations that you are solving and we would like to keep in mind here, that in this form rho are the two densities and u here is a vector. So, this is the vector equation this is a dyadic product, this actually 3 equations, for the x momentum equation, y momentum equation and z momentum equation. So, we have to keep that in mind, but it is written in a compact notation in the vector form like this.

When you look at this equation are these equations correct? They are not correct for incompressible flow? Because of this term here one by beta square dou p by dou t. So, that is something that we have artificially introduced, but assuming that this is we can now see the how we can proceed with the solution, we solve the continuity equation for p it is like the previous continuity equation where you have density variation dou rho by dou t.

So, in a similar way you can solve this equation for p, or we can solve this equation for rho star and then once we have rho star we can get a p provided we, we have beta. So, we can solve this for p here or solve this for rho use this to get p and substitute that p in this and here we have an equation for u x momentum equation or y momentum equation in which pressure is no longer and unknown thing. It is already deduced from your continuity equation.

So, by rewriting the equations in this way through the addition of this artificial compressibility term here, we are creating a method of finding p and putting in that in the momentum equation. So, that when we get to the momentum equations it is no longer and a nuisance variable it is say non variable and it varies from point to point because it is given by this equation here. So, this makes pressure of function of x y z t and that variation is obtained from this equation.

So, that same variation is put here to get u. So, these equations can be solved using the methods that we have discussed in the previous part of the module 4. That enables us to extend those methods here, but when you look at these equations there is a problem in the sense that these are not correct, but these are correct for steady flow conditions because under steady flow conditions, any variation with respect to 0 goes to 0.

So, this term will go to 0 and this term will go to 0. You have del dot u equal to 0 plus del dot u u equal to minus one by rho del p plus mu by rho del square u. So, the steady state form of these equations is the same as the steady state form of the 2 Navier stokes equations with the true density and two true properties. So, under steady state conditions these equations can be considered to be accurate. So, this provides us a means of getting a steady state solution, even for incompressible flow using compressible flow methods through the use of this artificial compressibility here.

So, we solve these equations with appropriate initial conditions and boundary conditions. Then much forward in time and if we hold the boundary conditions steady unchanging with respect to time. Eventually under most cases we accept to reach a steady condition. So, there may be some unsteady periodic kind of flows which may be coming through, but if you are looking at flow which is going to be steady and if you can drive this equations to steady state where the contribution of these time dependent terms becomes negligible then, we have a velocity field and pressure field which satisfies the steady form of the momentum equation. The continuity equation and.

So, you have velocity field which satisfies this equation and this equation and pressure which is also coming from these two equations. So, we will be able to say that the steady state solution must be the steady state solution of the 2 Navier strokes equation for incompressible flow although we have got into the steady state using methods which are not fully correct. So, the transient solution that we getting here is not dependable it is not correct.

But the steady solution is correct and these equations can be used for 3-dimensional flows, because we have written here in the vector form and you have a this de l dot u will become equal to dou u by dou x plus dou v by dou y plus dou w by dou z and these here you will have 3 equations x momentum, y momentum, z momentum equation in which the same pressure which is coming out to this equation is used. So, there is no difficultly in extending this to 3-dimensional flows the only thing is that it is correct only when you are considering the steady state solution.

If your numerical solution does not reach a steady state or if the solution that you are looking for is not the steady state solution, but transient solution then, you cannot use this method. But if you are looking at a steady flow the steady flow solution of Navier strokes equation for incompressible flow. Then this method can be used and in order to use this method we need to fix a value of beta without beta being specified this cannot be solved.

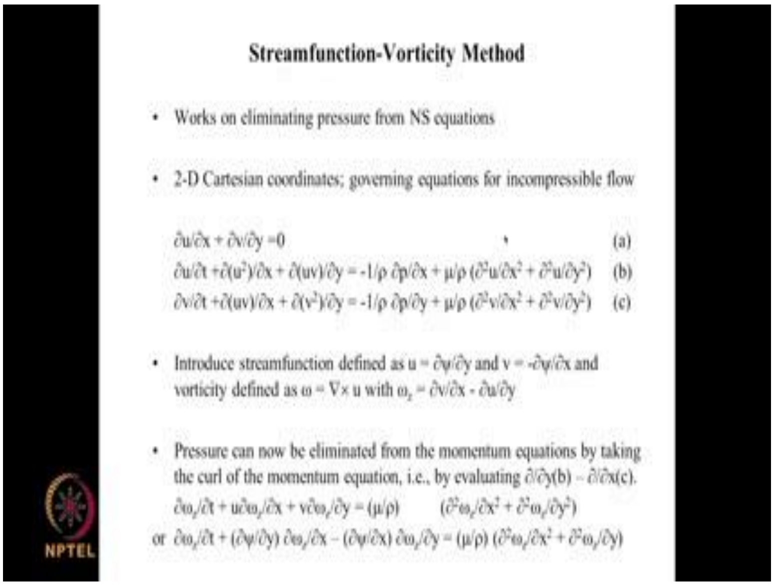
What value of beta can be used this formulation here rho star as p by beta square and all that will imply pseudo speed of sound which is given by square root of u square plus beta square. So, we call it as where u is the characteristic flow velocity for example, the average speed or average speed in a duct flow or the mean something like the u infinity in the case of a flow over external bodies, external flow over bodies. So, free stream velocity that is what it is. So, free stream velocity.

The characteristic velocity square plus beta square and square root of that is the pseudo speed of sound and we must make sure that this is reasonable and based on a some studies of sensitivity and all that Chang and Kwak suggested that beta should be taken in such a way that beta square by reference velocity square is at the order of 5 to 10 for duct flows. So, if you are looking at flow through a duct which is incompressible and if you know what is the estimated average velocity then, you choose beta such that beta square is 5 to 10 times reference square where reference velocity square. So, that gives you a velocity value of beta here and then that completes the specification the problem.

In case of external flows at high Reynolds numbers the value of beta is not so sensitive and you can take reasonable values for this and you will have a complete formulation of the problem here. So, the artificial compressibility method it tries to extend known method. Known and successful methods, which have been developed for compressible flows to the solution of incompressible flows at the same time the extension is such that the method will not give us time accurate transient solution.

So, when we say time accurate transient solution, we are starting with some initial condition because a transient flow that we are solving and so, for a initial condition is fictitious, the transient is fictitious. What is true is the steady state solution. So, when that is for you, then you can use this method and this method can be used for 3-dimensional flows, also and it has been used in for those kind flows and this is one specific approach there is a different philosophy in the stream function vorticity method.

(Refer Slide Time: 16:17)



The idea here is that that since pressure is the nuisance variable can we eliminate pressure from the Navier strokes equations.

So, it is possible under certain cases, and let us considers the case of 2-D Cartesian coordinates 2-D meaning two-dimensional flow Cartesian. So, we looking at two dimensional flows in which u and v are the non 0 velocities, and double u is 0 everywhere which makes it a 2-D flow. For this the governing equation in Cartesian coordinates we can write as dou u by dou x plus dou v by dou y equal to 0, which is a continuity equation will call this as equation a here the x momentum equation. We have written down the full equation here in the expanded for us to become familiar with it.

So, you have dou u by dou t dou by dou x of u square plus dou by dou y of u v equal to minus one by rho dou p by dou x plus mu by rho dou square u by dou x square plus dou square u by dou x square. Since we are looking at incompressible constant property flow, the second coefficient of viscosity term does not appear it is usually it is something plus lambda times dou u k by dou x k and dou u k by dou x k equal to 0 here. So, the second quotient term goes to 0 and this is the equation here.

Since we are looking at constant property, constant density, gravity does not come into picture it as no way of influencing the velocity field. So, we can subsume the pressure here or we can simply neglect it. The y momentum equation is dou v by dou t plus dou by dou x of u v plus dou by dou y of v square equal to minus 1 by rho dou p by dou y plus mu by rho times dou square v by dou x square plus dou square by by dou y square. So, you have equation a, b and c. So, we introduce a variable called stream function you must have heard about this in your fluid mechanics courses.

Lines of constant stream function values are the stream lines and stream lines are such that flow cannot cross through this stream lines. So, if you have a stream tube then, the flow that is inside will come out of the one of the end of tube it cannot go through the walls the fictitious walls which makes it a stream tube. I mean refer you to standard books of a fluid mechanics to learn more about the stream function that the stream function is defined for the 2-D Cartesian in case with x and y as the coordinate directions of interest.

We can define u as dou psi by dou dou y and v as minus dou psi by dou x where psi is a stream function and we can introduce another variable vorticity, which is given by omega a which is the vorticity equal to del cross u. In the case of 2-D flow we are interested in the z component the vorticity and that is given by dou v by dou x minus dou u by dou y. Now, using these two variables that are the stream function and vorticity in the z direction it is possible to rewrite these equations, the 3 equations into a set of two equations and how do we do that we can eliminate pressure from these two equations by taking the curl of the momentum equation.

So, curl of the momentum equation is a fancy way of a saying it is a mathematical way of saying this, but what we are meaning is that by evaluating dou by dou y of this equation dou by dou x of this equation. Every time in this equation is done through this corresponding operator. So, you write this as dou by dou y of dou u by dout t plus dou by dou y of dou by dou x of u square plus dou by dou y of dou of u v by dou y equal to minus dou by dou y of one by rho dou p by dou x all that kind of thing.

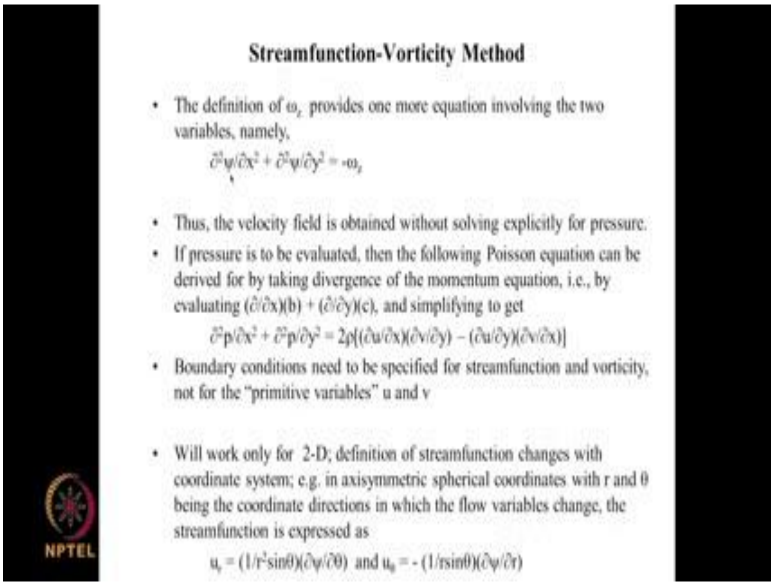
So, every before every one of these terms, you put the dou by dou y operator and similarly this and then you can go through some algebraic manipulations and finally, you can simplify and get a this particular equation here and we also make use of this definition of omega z to do this and finally, we get an equation like this. If we examine this equation it is dou by dou t of omega z omega z is a variable it is a function of x y and t and plus u times dou w by dou z dou w dou omega z by dou x plus v times dou omega z by dou y equal to mu by rho times dou square omega z by dou x square plus dou square omega z by dou y square and what is familiar about this equation this is like.

A regular scalar transport equation if you put omega z equal to phi you get a scalar transport equation. So, this dou phi by dou t plus u dou phi by dou x plus v dou phi by dou z equal to mu by rho or some diffusivity times dou square phi by dou x square plus dou square phi by dou y square. So, in that sense it is like a regular transport equation, and we already have developed some templates for solution of this type equation. So, the solution of this does not pose any specific problem accept for the non-linearity and the coupling that comes with this here.

We are trying to eliminate p and we can completely eliminate u by substituting this definition for u and this definition for v ones we do that we get this equation here. So, in this equation we see there is no u, there is no v, there is no p and it is obtained from rewriting the momentum equations and we make also use of a the continuity equation to come up to this here is an equation and this as omega z as a variable and size also variable.

So, we cannot have one equation and 2 variables. So, we construct another equation from the definition of this. So, we have omega z equal to dou v by dou x minus dou u by dou y and we know that v is given by this and u is given by this. So, if you substitute this you get dou by dou x of minus dou psi by dou y and you substitute this here, you get minus dou by dou y of dou psi by dou y. So, substituting that we can get a different expect different equations dou square psi by dou x square plus dou square psi by dou y square equal to minus omega z.

(Refer Slide Time: 23:52)



What is familiar about this equation? These also almost like the scalar transport equation except that we do not have the time dependent thing. So, this actually a Poisson equation with a right hand side which is a variable. Now, if we look at this equation and the previous equation here we have two equations involving two variables psi and omega z here. So, if we solve the previous equation for omega z and then, substitute that on the right hand side we solve for psi here we put back the psi in this thing here and then again solve this for omega z and then we come back here, and then we put this and then we can get a new value of psi.

So, we can solve these things together and then we can get a solution for psi and omega z and the psi and omega z; obviously, are obtained from a discretization in terms of psi and omega z and all that and we write finite difference approximations and for this term this term and. So, on and. So, using the regular c f d procedure we can get psi as a function of x and y that is we get psi i j and omega z i j.

So, we and ones we know psi i j you can take the y derivative and get u and similarly we take minus of x derivative minus dou psi by dou x will give us v. So, we can reconstruct the velocity field by solving this equation a Poisson equation for psi, this stream function and a scalar transport equation for the scalar which is the z component of the vorticity vector. So, by doing this we can get directly the velocity field without ever solving for pressure.

If you want to know pressure then after solving for velocities after solving for u and v. FWe can solve a Possion equation for pressure which is obtained by taking the divergence of the momentum equation. Earlier we took curl to eliminate pressure and to derive an equation for pressure we take the divergence of the momentum equation. So, that is we take dou by dou x of b plus dou by dou y of c where; b and c are the x momentum and y momentum equations.

If you do this if you do that and then if you go through manipulations and all that you can finally, derive this equation for pressure. So, dou square p by dou x square plus dou square p by dou y square equal to 2 rho dou u by dou x times dou v by dou y minus dou u by dou y times dou v by dou x. What is good about this equation is it is like this equation, it is like a Possion equation and by the time you have come to solution of p you already know u and v. So, you can evaluate these things you can evaluate the right hand side, and then you use finite difference approximation for this and this and you can get a pentadiagonal matrix for p and then you can solve it to get a pressure.

So, in this way we can evaluate the velocity field u v and then p by solving a different set of equations. Which are essentially the same equations as the governing equations because these have not been derived with by making any further assumption, these are the exact equations, but written in a different way. So, as to enable a solution without having to worry about the pressure now the difficultly it is not as simple as it is sounds it is difficult and one of the difficulties is the boundary conditions for these for the solution of this a Poisson equation and this equation is not in terms of velocities.

Because, now only we have some velocity boundary conditions the no slip condition uniform velocity condition and the normal variant equal to 0. Those kind of things are known, but now we have to visualise what kind of boundary conditions will be there for psi which is given by this equation here, you may know velocity u and velocity v, but how can you construct a boundary condition for psi on a given boundary and similarly these things and one as to spent time and thought on rewriting the known boundary conditions. In terms of the boundary conditions that appear in terms of boundary condition for psi and a omega z.

So, in this context we introduce notion of primitive variables u v p are known as primitive variables. And stream function vorticity are the derived variables the new variables sophisticated variables, which enable this solution here. So, boundary conditions one needs to spent time to translate the known velocity and pressure boundary conditions into boundary conditions for psi and omega and another disadvantage is that this will work only for 2-D because, only for 2-D we can come up with definition of stream function which is given by this and this definition here is valid only for 2-D two d Cartesian coordinates in which u and v are non 0 and double u 0 if you go to a different 2-D coordinate 2-D problem for example, axisymmetric spherical coordinates with r and theta being the coordinate directions in which the flow variables change. So, I am looking at a case where u r the radial component of velocity and u theta is non 0, but u z is 0.

So, if you looking at that kind of 2-D case which is an axisymmetric 2-D case in spherical coordinates then, the stream function is defined in this way the 2 non 0 velocities, u r and u theta are linked to gradients derivatives of the stream function. In this particular way here we just define u is dou psi by dou y and v as minus dou psi by dou x and here you have to derive it like this. So, that ones you substitute into the 2-D continuity equation for this flow we have we can guarantee that the continuity equation is satisfied.

For example, here if you substitute this you get dou square psi by d x squares and if you substitute a dou square psi by dou x dou y here. If you substitute this one in this you get minus dou square psi by dou x dou y. So, dou square psi by dou x dou y minus dou square psi by dou x dou y will be equal to 0. So, the definition of stream function such that, the continuity equation is satisfied and if you want to do the same thing for axisymmetric spherical coordinates, you have to define u r and u theta in this way in order to satisfy the corresponding continuity equation in r theta spherical coordinate system to be valid.

So, the definition of stream function changes with coordinate system. So, if you write the stream function definition properly then, it is possible to solve for the velocity field and pressure field without having to explicitly solve for pressure write in the beginning we can get u and v without having to solve for pressure. But only applicability only a condition is that it is valid only for 2-D flows and not with 3 d flows.

So, in the next lecture we look at a method which is applicable for time dependent and 3-D flows and that is a pressure equation method.