Computational Fluid Dynamics
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Lecture – 36
Implicit Beam – Warming Scheme

In the last lecture, we saw a new method Implicit Beam Warming method for the case of 1 dimensional advection equation of the form dou u by dou t plus dou f by dou x is equal to 0 through some manipulations and evaluations of terms.

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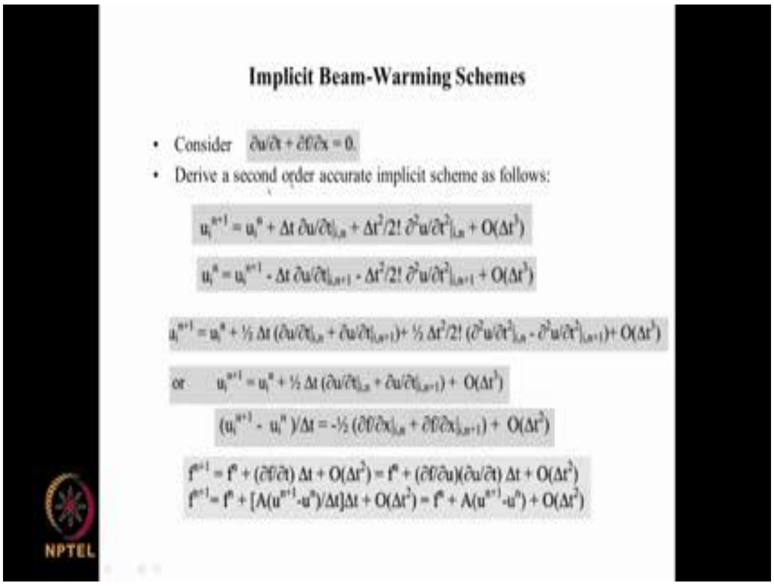

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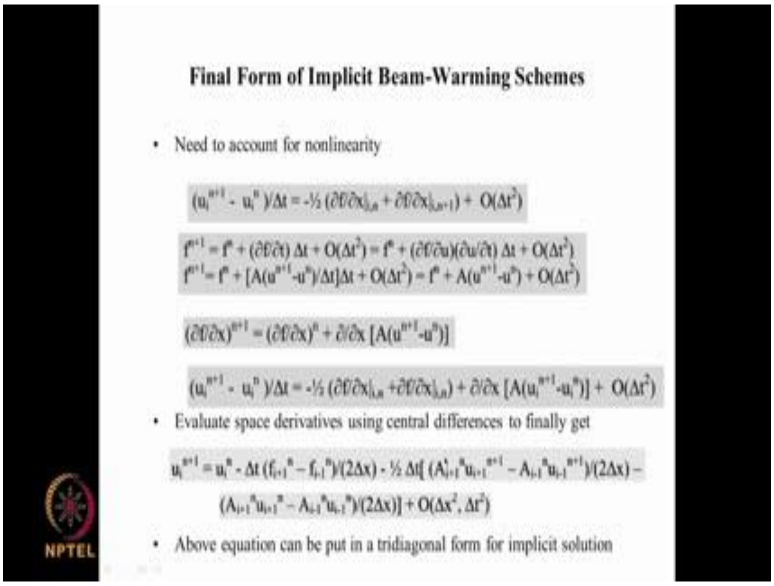

We finally, got an expression for advancing from time step from n to time step n plus 1, in this particular form. Where we substituted space derivatives, where we evaluated the space derivatives using the central differences to finally, get an equation for u i n plus 1 in this from it is implicit. Therefore, we had to evaluate u i n plus 1 in terms of u i plus 1 n plus 1 and u i minus 1 n n plus 1. These can be put in the form of a tri-diagonal matrix equation. As we noted at that time, this equation has a capability of take account of non-linearity since by dealing with the single equation the coupling aspect does not arise here and this particular form can be put in a special form.

We finally, get an expression for u i plus 1 in terms of u i plus 1 n plus 1 and u i minus 1 n plus 1, and as you remarked it in that lecture the resulting equations are in the form of a tri-diagonal equation and we have to solve at time step of n plus 1 for all the value of u i for i equal to 1, 2 n point where we need to find. So, this equation is of second order accuracy in both time and space it is also implicit and it is also a stable equation.

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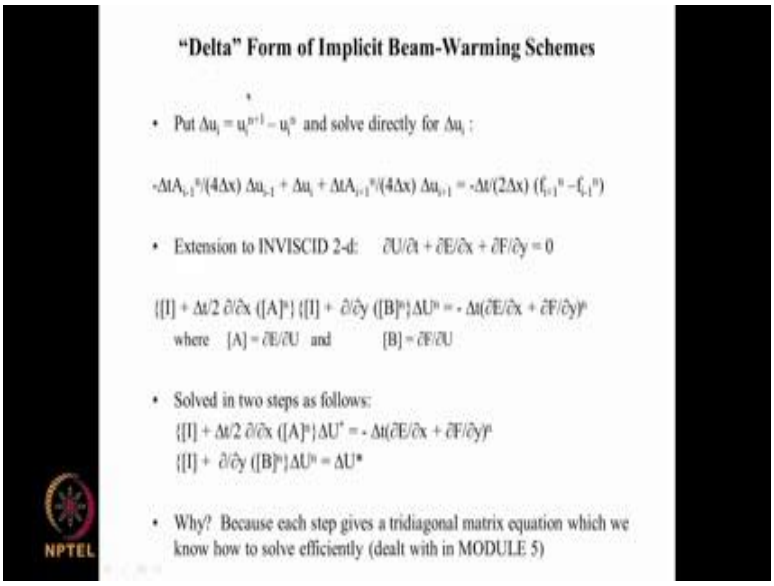

Now, there is a special form of this, called the delta form, which is actually more convenient to write and evaluate when we look at extension of this for 2-d and 3-d problems. So, we define delta u i as u i plus u i n plus 1 minus u i n here, this is at delta u i n is we can also keep like this, and rewrite this equation here in this particular form. So, at n plus 1 time step after we have evaluated u i n this is an equation that we solve to get delta u i and. So, this equation has variables not as u i n plus 1 u i plus 1 n plus 1 u i minus 1 n plus 1, but it has delta u i minus 1 delta u i, and delta u i plus 1 as the variables and this we can see is a tri-diagonal equation, this is the tri-diagonal form with the co-efficient of delta t a i minus 1 n by four delta x and a i plus 1 n divided by 4 delta x here and 1 here. So, this gives us a neat little form for the equations and you can solve not for u i n plus 1, but delta u i here

So, this is known as the delta form of the Implicit Beam Warming scheme it is not any different, but it allows us to put in a concise form. Now if you want to extend it to 2 dimensional cases, then we have a governing equation like dou u by dou t plus dou e by dou x plus dou f by dou y equal to 0, it is like the previous case that we have seen for the Mac Cormack scheme, but before we look at the full 2-d viscous case, let us consider how we do it for the INVISCID case, where we do not have the viscous stresses terms here and. So, e consists of only the conductive or the advective terms. So, in such a case we have u e and f are matrices containing for the 3 equations we have the continuity equation, x momentum and y momentum equation. So, have those specific forms here.

So, this becomes a matrix equation and this needs to be done re written in a matrix form and that is what we have essentially done here, and this is written as i identity matrix plus delta t by 2 dou by dou x a and we will see what this a is times i plus dou by dou y of b times delta u n equal to minus delta t dou e by dou x plus dou f by dou y n. Where earlier we had a here which is dou e by dou u we had this a has been given as dou f by dou u here this a here, is essentially dou f by dou u. Now we have the same thing given in this becomes matrix type of things. So, this becomes a Jacobin and this also Jacobin, where as a represents variations term which are coming within the x derivative b represents the terms coming in the y derivative here and both are advective terms.

So, these are non-linear terms. So, the non-linear part is linearised using in this particular way. This equation is something plus dou by v dou x times i plus dou by dou y times delta u i delta u i n equal to this, And this is actually solved in 2 steps like i plus delta t by 2 dou by dou of a n delta u star equal to minus delta t dou e by dou x plus dou f by dou y n. And then once we get delta u star here we put this here and then we solve this as i plus dou by dou y of b n times delta u n equal to delta u star. So, this is what is known as a operator splitting kind of thing, the we will see we will come back to this in module five ah, but the advantage of this is here when we do dou by dou x of delta u i; you obviously, get delta u i minus 1 delta u i, i plus 1 and delta u i and when we put dou by dou y of delta u again we get a delta u j plus 1 j minus 1 and j.

So, in the 2-d case we have this equation consists not of i minus 1 i plus 1 and i. It consists of i minus 1 j i, j i plus 1 j and also i j minus 1 i j and i j plus 1. So, it becomes not tri-diagonal matrix, but a penta diagonal matrix and 3 dimensions you will have seven diagonals here, and the penta diagonal matrices are different; obviously, from the tri-diagonal matrix and in this particular case there are 3 adjacent diagonals corresponding to i minus 1 j, i j and i plus 1 j followed by certain 0 diagonals followed by the i j minus 1 diagonal and i j plus 1 diagonal. So, in that sense we have zeroes in between some diagonals, and that is a typical penta diagonal form that we get when we discretize 2 dimensional equation. So, that something that to be keeping in mind and when we have a penta diagonal matrix then the tri diagonal matrix algorithms cannot be solved, but by first doing the x derivatives here, because in in this equation the right hand side is evaluated at nth time step.

So, this is all known here and you have dou by dou x of delta u star. So, this will involve i, i minus 1 and i and i plus one. So, that becomes a tri-diagonal matrix equation here and once this is done we get delta u star from this and if you come to this equation delta u star is now known from after solving the first equation, and then you have i plus dou by dou y of delta u here and this gives us another tri-diagonal matrix, but this time involving j minus 1 j and j plus one. So, you can solve this using as a tri diagonal matrix. So, this kind of decomposition is done because each step gives tri-diagonal matrix on equation which we know how to solve efficiently and we will deal with this in module five. So, when we are looking at 2 dimensional cases we can do it as a penta diagonal and then we can go ahead, but this operator splitting in this particular way helps us make use of the efficient tridiagonal matrix solution scheme that exists.

So, this is 1 particular thing. Now in this method this looks straight forward extension of the method that we have seen, and that is possible because we what we considered to begin with is just an advection equation without any viscous time, without any diffusion times, but our Navier stokes equations do have viscous times.

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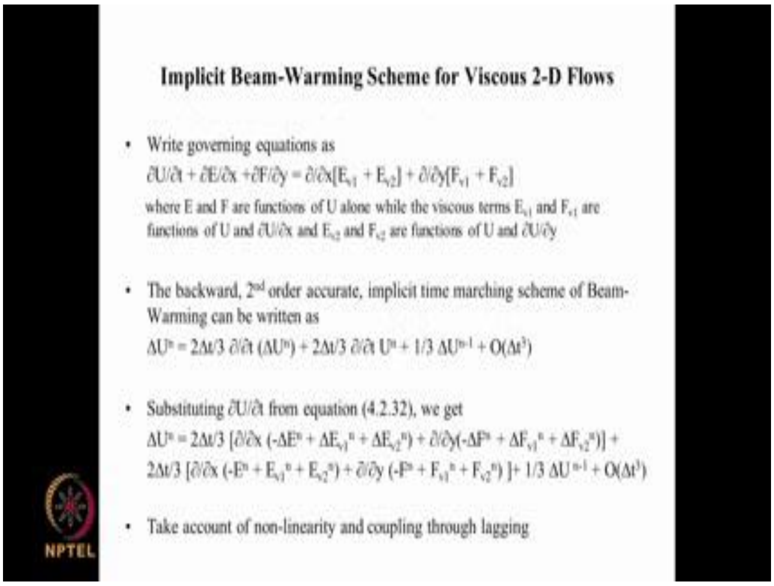

So, for viscous 2 d flows the governing equation is not the just left hand side equal to 0, but we also have viscosity terms here and here we are making a distinction between viscosity v 1 and v 2 and similarly f v 1 and f v 2. So, here we are writing the governing equations as dou u by dou t plus dou e by dou x plus dou f by dou y advection term and then the temporal term equal to dou by dou x of e e and f are functions of u alone and viscous, the e and f here are functions of u alone and we have seen that dou e by dou u is a and dou f by dou u is d. So, that is where this these things are, but here e f 1 and f 1 f v 1 are functions of u and dou u by dou x and e v 2 and f v 2 are functions of u and dou u by dou y. So, these are viscosities. So, you have a second derivative.

So, when you have dou u by dou y here then you have the term here, and similarly we have dou u by dou x and that is also coming here. So, in that sense it just the same equation, but written in a special way because we need to evaluate cross derivatives in a certain way and normal derivatives in a certain way, as we have seen for the mac cormacks scheme. So, the same idea is being brought into this. So, having written this how do we make solution scheme which is the beam warming scheme. So, we can write the implicit time marching scheme, as per the beam warm method as delta u n equal to 2 delta t by 3 dou by dou t of delta u n plus 2 delta t by 3 dou by dou t of u i n u n plus 1 third of delta u n minus 1.

So, this is essentially the time marching method. So, what we mean by this is? You are going you are evaluating delta u n which is u i n plus 1 minus u i. So, you are going from u i n to u i n plus 1. So, in that sense you are marching forward in time and this is evaluated in the Beam Warming method using a third-order approximation which we have derived earlier for the 1-dimensional case. So, we are making use of that and we are we have dou by dou t of delta u n. So, dou by dou t of u i n plus 1 and dou by dou t of u i n those will be coming here and they will be coming here also. So, and you have dou by dou t being given by this just as we have done here. We have substituted dou u by dou t i n as dou f by dou x from this equation as minus dou f by dou x from this equation. So, we are extending the same logic, for a slightly for a vastly more complicated case. So, if you do that and then if you do some manipulations you finally get delta u n equal to 2 delta by 3 dou by dou t of delta u n.

So, you have dou by dou x of delta e n that is coming from here and then delta e v 1 n plus delta e v 2 n. So, these are the dou by dou x terms here, and then dou by dou y of delta f n and then you have f v 1 and f 2 n. So, these are coming here and then we can again substitute here, this is dou u by dou t. So, we can directly put dou by dou x of e n here. Where as in this case is dou by dou t of delta u n. So, we retained this delta form of this here it is just dou by dou t of u n. So, we are directly substituting here. So, you have minus e n plus e v 1 plus e v 2, at nth time step and that is what it is here and then you have 1 third of delta u n minus 1. So, this is one form of the implicit beam warming scheme, with the viscous terms it is not yet finished because we need to evaluate these terms here, and this is also system of equations and we know that the system of equations has non-linearity and you also have coupling and we have taken care of we can we have taken account of this and taken care of this using through lagging and what we have noted this down earlier and we can also note down here.

This coefficient matrix this Jacobian here is being evaluated at nth time step and therefore, where as this delta u here or the delta u here is involves u i n plus 1, and this part here is therefore, lagged it is based on all though it is a function of u it is making use of the u values at nth time step and not written n plus 1 time steps. So, in that sense this is an evaluation of the non-linearity and coupling based on the previous time step values and therefore, this is called lagging lagging. This particular contribution of the overall equation here, is coming based on the previous time step values where as this is based on the new time step values. So, this enable us to write the non-linear coupled equations in the form of linearised equations and which enable us to explicitly evaluate the coupling terms.

So, the same logic is being applied here it is more tedious.

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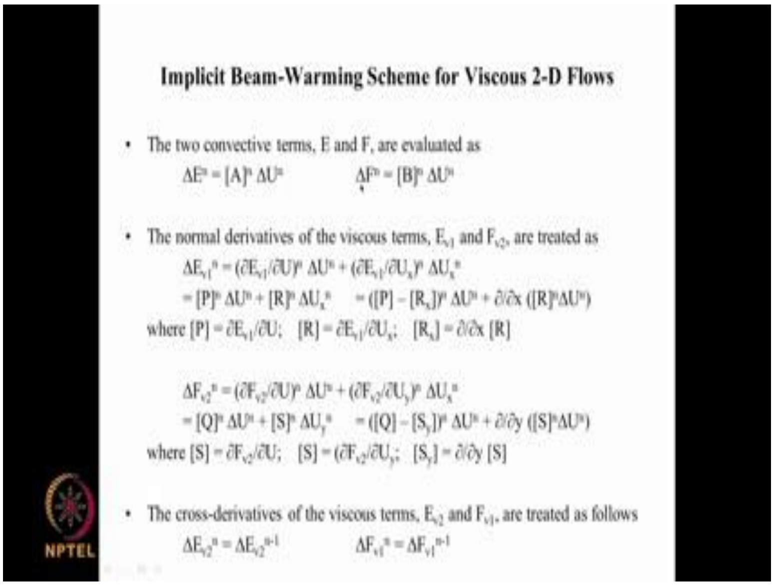

So, here you have e here and f. You also have e v 1 and e v 2 and f v 1 and f v 2. So, each of these is treated in a specific way. So, we have delta e n is written as a n delta u n, and delta f n is written as b n delta u n and why are we writing like this because, a is dou u by dou u times delta u. So, that gives you delta u and delta f b is dou f by dou u times delta u. So, that again gives you delta f here. So, it is a re writing of these non-linear and coupled terms in this way, where the primary variable u i n plus 1 is coming out in this unsketched in an implicit form, but the coupled coefficients and the non-linear coefficients are evaluated based on the previous time step values which will know.

So, this enables us to take account of non-linearity and coupling and this is done for different terms convective terms, and the viscous terms in different ways and that is what to summarized here, for the convective terms. Where that is e and f which involve only u and not derivatives of of special derivatives of you like dou u by dou x dou u by dou y those terms are not there in e and f and that is way, that we have split this equation here e and f of functions u alone where as e v 1 and e v 2 of a functions can be functions of both u and dou u by dou x and these v 2 terms are functions of u and dou u by dou y.

So, we are making a distinction among the various terms that appear in these equations and these are being done in a different way. So, you have as we have noted for the Mac Cormack scheme the evaluation of the viscous terms, these terms brings in the idea of normal derivatives and cross derivatives it should be able to go back a here we have dou u by dou x dou u by dou y plus dou u by dou x.

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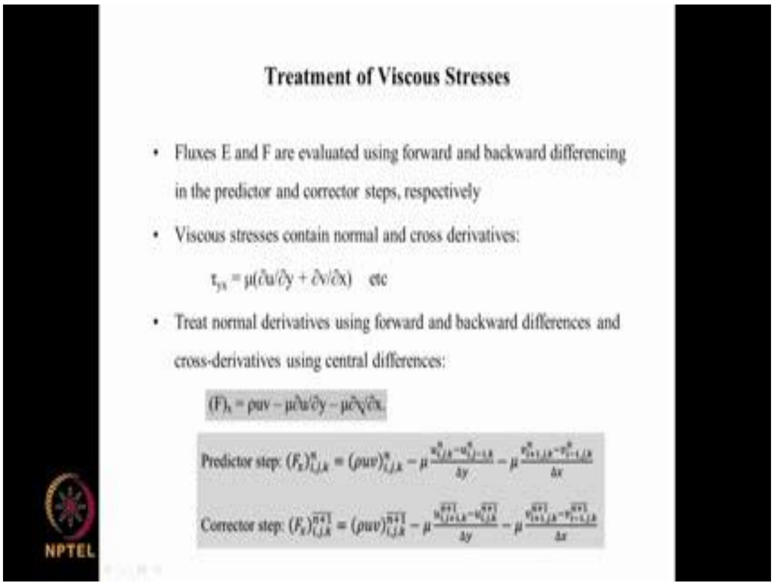

So, we have the once you put dou f by dou x here then that this term becomes dou square v by dou x square, but this this 1 becomes dou square u by dou x dou y. So, this is cross derivative and this is normal derivative. So, the same idea is being brought into play here, and the normal derivatives that is e v 1 and f v 2. So, because f v 2 is dou u by dou y dou f by dou y. So, this is dou u by dou y. That gives u dou square u by dou y square and e term is appearing with dou u by dou x. This will be this will give as the dou square u by dou x square term.

So, dou x square u by dou x square and dou square u by dou y square are normal stresses, and these are evaluated as even these are functions of u. So, you have dou e v 1 by dou u at nth times steps. So, this is lagged non-linearity and coupling of this part which depends on u and then you have dou f by dou u x times delta u u x here, where x denotes the derivative with respect to x. So, we are taking care of the fact of the idea that e v 1 is a function of u and dou u by dou x and. So, you have function of u here, and function of dou u by dou x and. So, those two things are coming here and these are further written as. So, this is now written as p n and delta u n and this is written as r n and delta u x and. So, we can do further manipulations to get this particular form.

Similarly, delta f v 2 n is written as this again f v 2 the function of both u and dou u by dou y. So, you have this has dou f v 2 by dou u evaluated at the nth time step times delta u n. So, that gives us part of this delta f v 2 and that parts which depends on dou u by dou y is obtained in this way. So, dou f by dou dou f v 2 by dou u y evaluated at the nth times step. So, these quantities known here times delta u y it should be, so this is again written as q n delta u n plus s n delta u y. So, you can see and that gives us another equation here. So, s y is, these are as we mentioned s y is dou by dou y of s here. These are for terms which are v 1 and f v 2 e v 1 and v 2, but we also have e v 2 and f v 1 those will give us cross derivatives and those cross derivatives are evaluated based on lagged coefficients like this. So, once you do all these things substitute all these things then, we get Jambo expression for which is the beam warming scheme for viscous 2-d flows.

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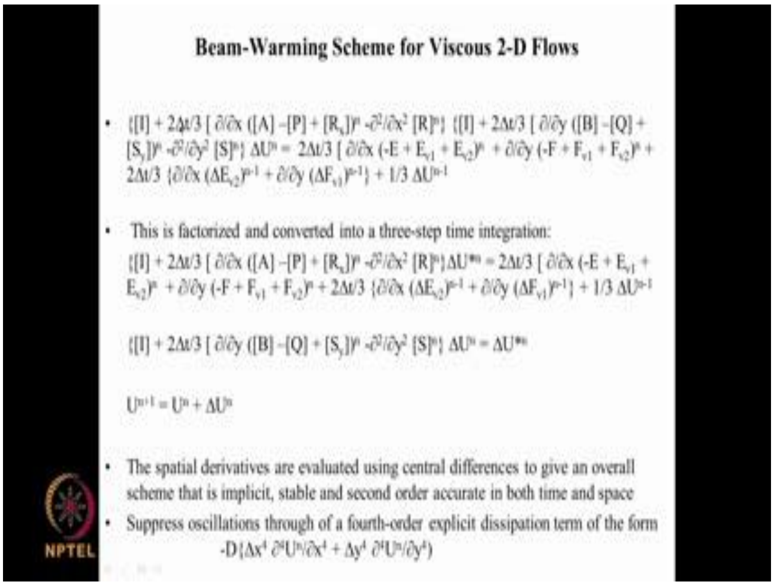

So, you have i plus 2 delta 3 by 2 delta t by 3 times dou dou by dou x of a minus p plus r x. So, these are all things that we have seen here with some approximations and the idea is that here you have this whole thing times. This whole thing times delta u n equal to the right hand side terms all of which are e s and e v once and the x derivatives y derivatives and all those things we are coming at nth time step and some of them are coming at n minus 1 time step, and all these things. So, all of the right hand side is known and the left hand side is a product of i plus dou by dou x and dou square x. So, essentially the x derivatives and similarly i plus the y derivatives dou by dou y and dou square s by dou y square.

So, this can again be decomposed evaluation of this will again give us a penta diagonal matrix and this can be factored into only the x derivatives and only the y derivatives. So, this enables us to solve this equation in two steps each involving tri-diagonal matrix. So, and that is our form here. So, we finally, get u n plus 1 equal to u n plus delta u n. Where delta u n is obtained by doing this and this is obtained by solving this equation. So, the essence of the beam warming method is now for viscous 2-d flows is given by these things, where the coefficients a p and r x r and all those things s s f all those things need to be evaluated. Either analytically if expressions are known and if necessary numerically these have to be evaluated which will examine in the next class, next lecture.