Computational Fluid Dynamics
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Lecture – 35
Stability limits of Mac-Cormack Scheme and the Intro to Beam-warming Scheme

In the last lecture we have seen the Mac-Cormack Scheme, a compressible flow; which consisted of a predictor step and a corrector step, which enables us to go from U ij n to U ij n plus 1.

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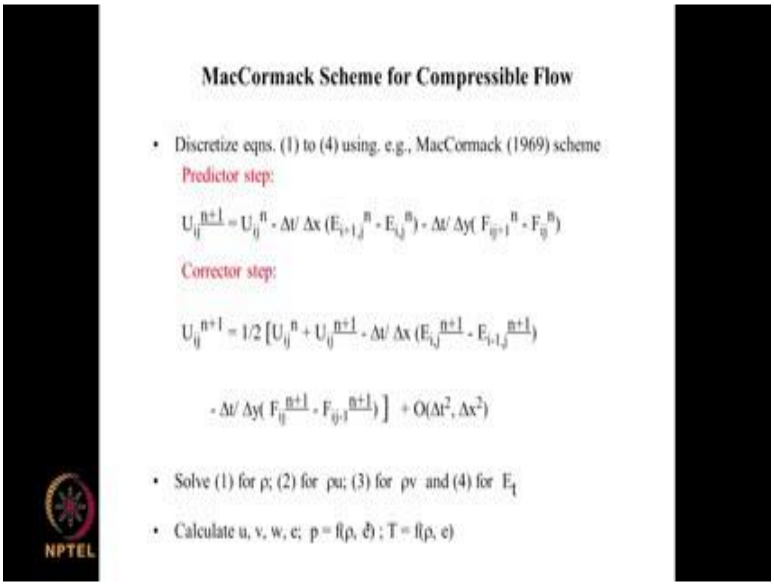

In this we have noted that this can be used for the coupled equation describing the fluid flow, and we also noted that each of this predictor and corrector step is explicit calculation and also that it is both second order accurate, both time and space. So, it is order of accuracy delta t square, delta x square and delta y square that is not mentioned here. This enables us to solve for rho u rho v and E t from which we can evaluate rho u v and (Refer Time: 01:12) energy from which we get using the equation of state we get the pressure and temperature.

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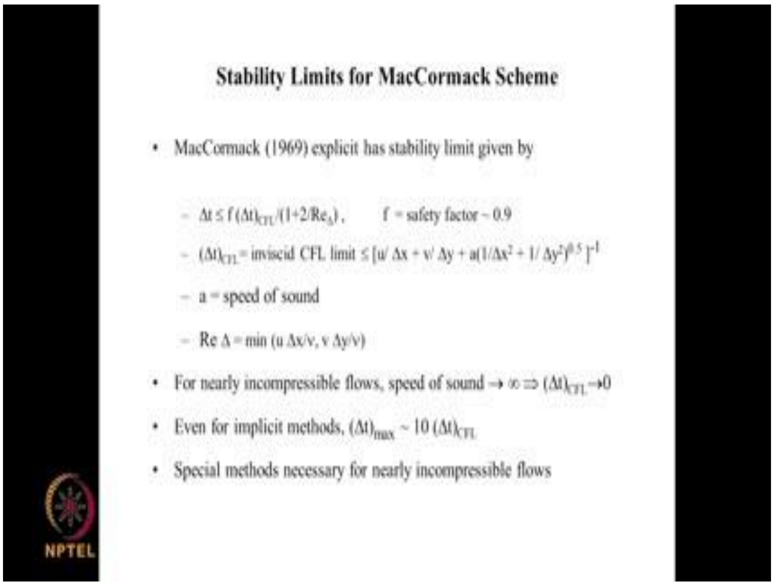

Now, because this is an explicit method it has stability limits potentially stability limits and when you have these coupled equations and when you have non-linear equations all these complications; when these complication arise it is not possible to get an exact analytical expression for stability and so the Mac-Cormack have estimated the stability and express this time step delta t that can be taken in using the factor of safety f. So, this is delta t must be less than f times delta t CFL.

CFL is obviously the courant friedrichs lewy limit and that is true for a inviscid fluid that that was we had dou u by dou t plus c dou u by dou x is equal to 0, no diffusion term. So, that is in inviscid fluid. So, delta CFL is the inviscid limit which is given by delta t must be less than u by delta x plus v by delta y plus a times 1 by delta x square plus one by delta y square root.

So, this is where is the speed of sound and this is the 2-Dimensional form of the inviscid CFL limit. So, it is a 2-Dimensional compressible form of the CFL limit is what this is and this does not include viscosity effects and therefore, they suggested that it is the delta t that is possible is the delta t divided by 1 plus 2 divided by mesh Reynolds and where the mesh Reynolds is defined as u delta x by nu or v delta y by nu whichever is the minimum is the one that is to be taken.

Let us examine this limit to understand what it is showing. It is saying that the delta t that you can take has an upper limit, it has to be less than some value; and what is that value?

In the case of inviscid flow, 1-D flow this is where the delta t must be less than the courant number of equal to 1. The delta t given by the inviscid courant number limit is one condition and that is value is not really correct because what we are dealing with are equations which includes viscous stresses. So, there must be a modification and here they are dividing by 1 plus some quantity here which is a positive quantity and because it is a positive quantities 1 plus something, so that reduces the delta t that is allowable here.

The fact that we are dealing with viscous stresses reduces the delta t to less than what is possible with inviscid condition. Now with this inviscid condition itself, we put courant number equal to 1, for the 1-D case and here we have a 2-D case and for 2-D case delta t is not must be less than u by delta x, it is u by delta x plus v by delta y.

Now for a case with compressible flow which is what we are dealing with here, there is a speed of sound which also comes into picture. So, this is u by delta x plus v by delta y plus a which is also a speed, it is equal to this and we need to have a length scale. In this case it is delta y because it is v and delta x because of this and here you are taking square root of 1 by delta x square plus 1 by delta y square. So, all of this together fixes the delta t for the inviscid CFL limit and this value that delta t that is possible for the viscous fluid flow thing must be less than this value, by a factor of safety which is 0.9, so that means, that this in case 10 percent margin is here and in addition to that it is decreased by this much.

And what is this Re here that is a mesh Reynolds symbol, Reynolds symbol has a velocity scale length scale and kinematic viscosity here. Here you have two velocities u and v, which one to take? So, you evaluate the Reynolds symbol for the two cases taking the mesh dimension delta x, delta y as the length equal to the length dimension. So, it is called a mesh Reynolds symbol. So, this is u delta x divided by nu the kinematic viscosity is one Reynolds symbol and another estimate of Reynolds symbol is v delta y by nu, whichever is the minimum you take here and you take the minimum because you are dividing by Re and that makes it conservative.

If Reynolds symbol for example, if this is 5 and this is 10 here you take the minimum you take 5 here, so, 2 by 5 is 0.4. So, this is 1.4. So, you divide the inviscid limit by 1.4 and then you get the allowable value. So, if this a 24 minus 3 seconds 0.9 times 24 minus 3 divided by 1.4. So, we are getting some factor of safety here and some factor of safety here together we are modulating, you are modifying the inviscid limit for the viscous limit; viscous compressible limit is being evaluated in this way.

This is an estimated stability limit and if you are within that Mac-Cormack has found that it gives as a stable solution. Now this has a specific feature here, which means that firstly, that there is an upper limit which means that you cannot have two larger value of delta t and you cannot have any arbitrary value of delta t. So, it may require if conditions are such that your delta t is very small then you have to compute for so many times before you can get up to a target time and the delta t here depends both on u and v it depends on delta x and delta y and it also depends on the speed of sound.

If the speed of sound is very large then the delta t becomes small because this is delta t must be less than u by delta x to the whole inverse. So, it is equal to delta x by u in a way, that means, as u increases delta t will decrease. In the case where you have large speed of sound the allowable delta t CFL itself is limited and the actual delta t that is limited for the viscous coupled solution is less than the CFL limit. So, as the speed of sound increases, as the flow becomes more and more incompressible the allowable delta t decreases. And it can also decrease if your delta y is small or delta x is small.

For example, when you have high speed flows then you have boundary layer formation. So, in the boundary layer is very thin, you need to have small delta y when make a delta y small then delta t CFL will decrease and that will mean that your allowable delta t decreases. So, there are certain restrictions, limitations of the Mac-Cormack scheme first of all it has being explicit and like the conventional f t c s, f t b s kind of things.

It has an upper limit, it has something like an effective courant number limitation and that courant number limitation is crucially dependent on delta y and delta x and it is also dependent on the speed of sound. So, there are definitely conditions in which this particular method will have two low value of delta t and improvements are needed and one such improvement is the beam warming method and when we look at eliminating this delta t we would like to go from explicit implicit, so we are going to look as an extra example as a different method implicit beam warming schemes.

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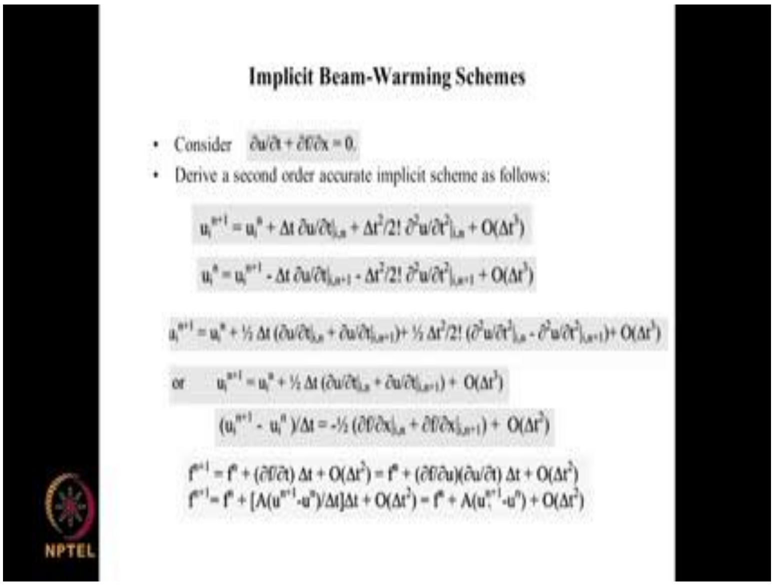

It is not a single method there are variants here and it is an implicit method. So, one could expect to have no stability condition, but in actual case we have coupled equations and when we say coupled equation we are saying that when we are solving for u we need to know v. And we are making some estimate of v it is not the exact estimate and if you take two large time steps and in the process v is changing then you could be wrong in the kind of values that you choose. So, even if you have an implicit scheme you may need to restrict the time limit to some values. So, that is something, but implicit scheme will be better than an explicit scheme in terms of stability limit.

So, that is what we are trying to do here we are looking at an alternative to the explicit Mac-Cormack scheme there is also an implicit Mac-Cormack scheme. But we would like to examine this implicit beam warming method which has also proved to be successful method and which employs a different way of taking account of non-linearity which enables us to get on this limitation of delta t arising in boundary layer type of calculations where delta y is small.

In this part of the lecture we look at the application of the implicit beam warming method or the derivation of the implicit beam warming method for the simple one dimensional case and so, we are looking at one dimensional wave equation type of thing - dou u by dou t plus dou f by dou x equal to 0. Where f is u square by 2 it is a usual thing and we have put it like in the previous cases in terms of the e and fs.

In the same way we have put this particular equation. Now idea is we want to have a second order accurate implicit method. We already have a second order accurate explicit method that is Mac-Cormack scheme. So, we would like to improve on it. So, we cannot sacrifices the second order accuracy that is implied in the Mac-Cormack thing. So, we would like to do better than that by going for an implicit method. The beam warming method is certain way of deriving a second order accurate implicit method and we first expand u and i n plus 1 around u i n. So, we write this as u i n plus 1 equal to u i n plus delta t by delta t times dou u by dou t at i n plus delta t square by two factorial dou square u by dou t square i n plus terms of the order delta t cube. So, that pretty straight forward.

And here the derivation is such that you are expanding u i n about i n plus 1. So, this is something unusual, but I would like you to note that in this case the derivatives are evaluated it i n. This is expansion about point i comma n or x comma t and this is expansion about i comma n plus 1. So, x comma t plus delta t, so that is the expansion of u i n as u i n, the u i n plus 1. So, you have minus delta t dou u by dou t, but its evaluation 10 plus 1 minus it should be plus delta t square by factorial 2-Dou square u by dou t square i n plus 1 and so on plus terms of the order of delta t cube.

So, nothing different here, nothing new here except the fact that this is expansion about i n plus 1 and this is expansion about i n. Now, you subtract this from this and what will you get? You get u i n plus 1 minus u i n and you have equal to u i n they do not cancel out. In fact, this goes that side and then that becomes 2 u i n. Once you do the sums you will see that u i n plus 1 is equal to u i n plus half of delta t times dou u by dou t i n which is coming from here plus dou u by dou t at i n plus 1 which is coming from here because you are subtracting this minus becomes 0. So, you have plus here. So, it is a mistake here.

So, when you put here and then you delete; you subtract this from this they do not cancel out because this is dou square u by dou t square at i n and this is dou square u by dou t square at i n plus 1. If this is also at n then you can cancel out the two because this is at i n plus 1 they do not cancel out, so you have half of delta t square by factorial 2-Dou square u by dou t square at i n minus dou square u by dou t square at i n plus 1. So, we can write u i n equal to u i n half of delta t times dou u by dou t at i n plus dou u by dou t at i n plus 1 and this can be written as dou by dou t of dou u by dou t times delta t. So, that becomes a delta t cube term, we can neglect this and say that we are neglecting the order of delta t cubed.

This expression here is third order accurate as of now here. Now we know that dou u by dou t is equal to plus dou f by dou x is equal to 0. So, we can say dou u by dou t equal to minus dou f by dou x. So, we can substitute that here. So, we have dou u by dou t at i comma n and that can be written as minus dou f by dou x at i comma n and here you have dou u by dou t at i comma n plus 1. So, you can write this as minus dou f by dou x at i comma n plus 1. So, we are making use of this equation here to convert these express in terms of time derivatives at i n plus 1 and all that in terms of the fluxes that are coming here. This is a final expression now when you bring the delta t here then this becomes a second order accurate expression here.

Now, the fluxes this f is actually u square u square by 2, it is a non-linear thing. So, there is we need to make it linear, we need to linearize it. So, we write f n plus 1 that is coming here or f n that is coming here as f n plus 1 equal to f n the space in the x does not matter here it is a valid for all the things. So, we can write this as because n plus 1 and n here we can write it as dou f by dou t times delta t plus terms of the order of the delta t square. So, this is f n and f is a function of u for example, we said f is u square by 2. So, we can write this dou f by dou t as dou f by dou u times dou u by dou t and plus terms of order of the delta t square that is a coming here.

And we represent this dou f by dou u as a and we can write this dou f by dou u by dou t as u n plus 1 minus u n divided by delta t. So, we are not bring anything here except bringing the nomenclature of a being equal to dou f by dou u and this being evaluated u n plus 1 minus u n divided by delta t and why we are doing that because in this case we are getting dou f dou by x like this way we are getting a derivative. This will become dou by dou x of u n plus 1 minus dou f by dou x of u n like that.

Finally, f n plus 1 is being evaluated as dou f f n plus a times u n plus 1 minus u n and this approximation is also second order accurate. So, when you look at this terms coming from this approximation of second order accurate in time and these things are also being approximated in a second order accurate kind of a thing here. So, it is not over yet.

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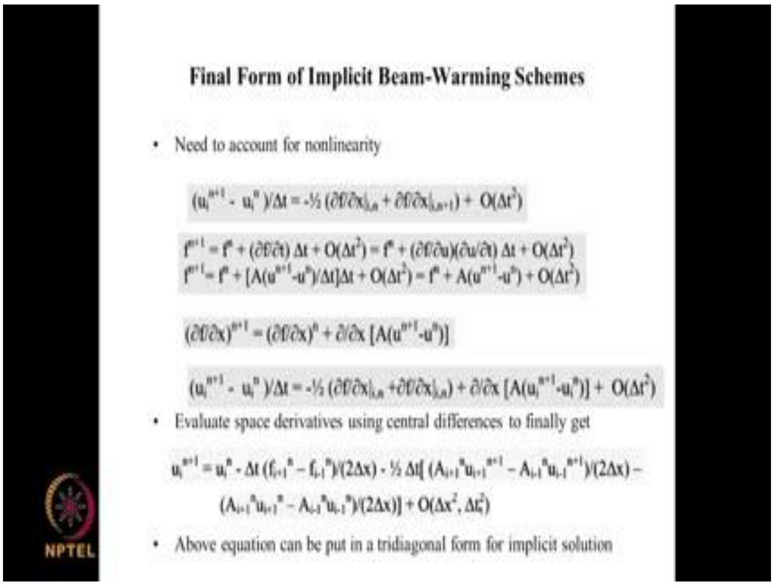

In order to account for this think it is just the same thing forgot to delete that. So, we have f n like this and therefore, now we can write dou f by dou x at n plus 1 as dou f by dou x at n plus dou by dou x times a times u n plus 1 minus u n and why is that. So, we differentiate this with respect to x, we get dou f by dou x at n plus 1 equal to dou f by dou x at n plus dou by dou x of this whole thing and that is what we have here. We can make use of this expression here to write this as u i n plus 1 minus u i n divided by delta t equal to minus half dou f by dou x at i n plus dou f by dou x at i comma n which is coming from here plus dou by dou x of a times u i plus i n plus 1 minus u i n by delta t plus delta t square.

Now, we have this dou f dou x here at i n and these kind things and you have dou by dou x here these things are evaluated using central differences, so that we can have second order of accuracy in space. So, once we do that we can write this as u i n plus 1 equal to u i n times delta t here and f i plus 1 minus f i minus 1 by 2-Delta x is coming from here and you have this plus this 2 and this 2 will cancel out to give you this. This time is being evaluated as again you have the minus delta t that is coming from here and you can write this as a i plus 1 n times u i plus 1 n plus 1 minus a i minus 1 u i minus 1 n plus 1 by 2-Delta x.

Now what are we doing here? We are evaluating dou by dou x of this quantity here and this a here is a function of u it is dou f by dou u can be a function of u here and we are evaluating this a at nth time step, so that this can be evaluated and we are evaluating this whole thing using central differences. So, we are writing this as a i plus 1 here, u i plus 1 and we are writing for this one a u i n a i minus 1 and u i minus 1 here.

But out of this these two terms both are functions of u, a is being evaluated at n and u is being evaluated n plus 1. So, I think this is n here. This is written as this plus a i plus 1, n u i plus 1 n and a i minus 1 n and u i minus 1 n. So, you can see that this quantity here is being evaluated as half of u is being evaluated n n plus 1 and u is being evaluated at n. So, you are essentially getting n plus 1 and n average here. We also have the a's always being evaluated at n and specially u's are being evaluated in such a way that there is central differences i plus 1 and i minus 1 and i plus 1 and i minus 1. So, together this gives as an approximation for the u i n plus 1 evaluation.

If you look at this expression here what do we need in order to get u i n plus 1. We know u i n f i plus 1 n, so this is at nth step. So, this can be evaluated explicitly and here a i plus 1 n, so explicit evaluation. Here you have u i plus 1 n plus 1. So, this we do not know and you have u i minus 1 n plus 1 this also not known because this at n plus 1 time step and here you have i plus 1 n n, so this is known here, this is known, this is known. So, in the whole expression here in order to get u i n plus 1 you also need to know u i plus 1 n plus 1 and u i minus 1 n plus 1. Together you have the equation here involves u i i n plus 1 as a function of u i plus 1 n plus 1 and u i minus 1 n plus 1.

In 1-D case this gives as a tridiagonal matrix involving u i u i plus 1 and u i minus 1 and we can write this as a tridiagonal matrix and then we need to solve this. So, to that extent this is an implicit formulation. So, when we look at the beam warming method for the solution of this equation here we have derived a second order accurate in time expression of delta t square through manipulation. We finally, brought it as a dou f by dou x and all this things here and we have accounted for non-linearity here and these special derivatives here and here are evaluated using central differences throughout, so as make it second order accurate in space.

Second order accurate in time second order accurate in space and in case it because u i n plus 1 requires u i plus 1 n plus 1 and u i minus 1 n plus 1 and other terms which are involving the values at nth time step. Finally, this is linearized because this u i n plus 1 has this A here, this A involves u, so this is where the non-linearity is coming. But we have avoided the non-linearity by evaluating this coefficient A at nth time step. So, this is known as the (Refer Time: 26:41) substitution.

The non-linear coefficient is evaluated using the old value and the actual variable value is evaluated using at the current time step. So, part of this non-linearity is eliminated by making this coefficient based on the previous time step value. So, this particular overall scheme here is implicit, it is second order accurate in both time and space, and its linear and in this case of one dimensional calculation it gives us a tridiagonal matrix equation and people have efficient method for the solution of tri diagonal matrix method. So, that is advantage of a beam warming method.

So, in the next lecture we will see how this is actually can be used for the solution of our coupled equations these equations.

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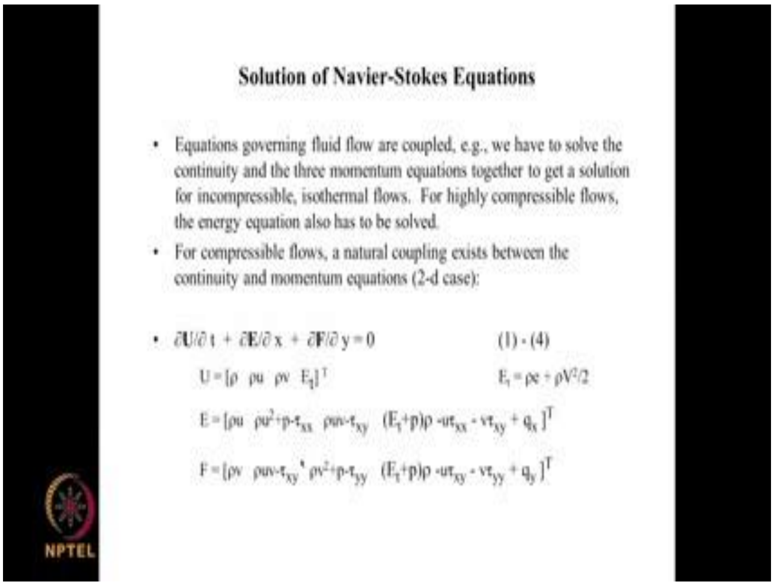

Because these are the once that need to be solved in the real case it is not just the simple one equation, but we can see how the method has come about. So, we can see how we have put together, how the method of beam warming method has been put together to solve this equation by incorporating special features, the features that we would like to have, and what are those features here? When we finally get the final prescription we have u i n plus 1 being evaluated using a method which is second order accurate in time, second order accurate in space, so that means, that we have good accuracy.

Something that is linearized it is f is a non-linear term; it is linearized by having the coefficient here evaluated at nth time the previous time step value. So, through the (Refer Time: 28:46) substitution process this has been linearized. Finally, we have a desirable a form of the implicit scheme, and what is that desirable form? It is a tridiagonal form, tridiagonal form as we will see in module 5, in next module, is a desirable form of a matrix equation for which efficient methods are known.

So that means, that we can solve the equations efficiently very quickly without requiring too much of memory and without requiring too much of computational time, so it incorporates all this feature and what we have seen now is the case for one dimensional flow case, and also for a single equation dou u by dou t plus dou f by dou x equal to 0. We will see how this whole thing is packaged together to solve all the equations that is the Navier stokes equations plus the Energy equation for compressible flow in the next lecture.