Computational Fluid Dynamics
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Lecture – 34
N-S equation in compressible flow- Mac Cormack Scheme

We have entered Module 4, finished half the course already and we are into the solution of coupled equations. And we are specifically concerned with the question of how to solve together the set of continuity momentum energy and other equations. In the previous lecture we have written down the equations for compressible flow and incompressible flow for a two dimensional case, and we noted several aspects to those equations.

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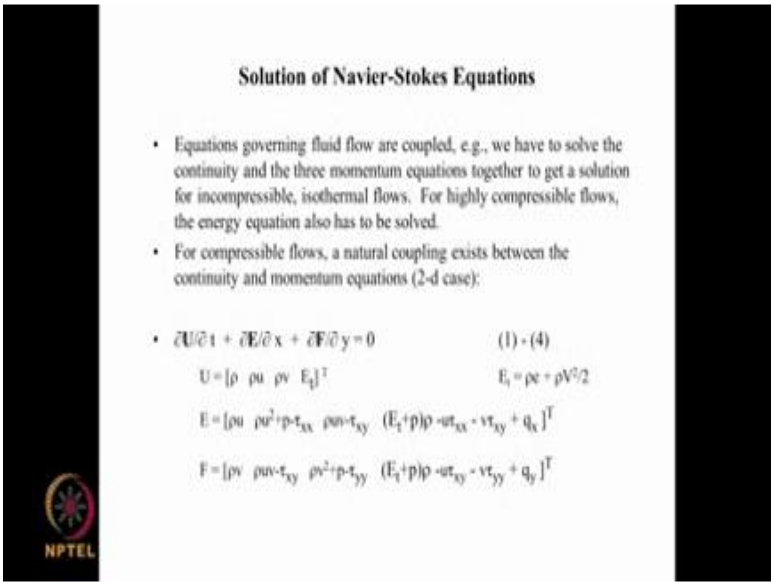

Including the fact that these are coupled in the sense that no equation can be solved by itself, and secondly, that they have non-linear terms, and thirdly that not all of them are in the standard scalar transport equation in the model scalar transport equation. The question of how to solve these things all these things together comes into picture. And we said that we will start with the compressible flow and then gone into the special difficulties that are associated with the incompressible flow.

Now looking at a compressible flow equations where we take advantage of a natural coupling that exist between the continuity and the momentum equations. So, we are looking at the equations written in a special way these are nothing, but the navier stokes equations and the energy equation that written in a special way for a 2 d case. And the special way is that you are writing this as it is like the continuity equation, like the one dimensional wave equation that we considered without any diffusion term. If you look at this equation here this is dou u by dou t plus dou e by dou x if you put that equal to 0 you get the one dimensional form. And if you put dou f by dou y equal to 0 you get the two dimensional form.

And what is special about this is that the three variables that are coming here u e and f are vector quantities. This is a set of four equations representing the continuity equation the x momentum conservation y momentum conservation and the energy conservation equation. So for each of those equations u term takes four different values, similarly e takes four different values and in that sense this is written in a vector form. For example, u here in the case of continuity it takes rho and in the case of x momentum it is rho u and in the case of y momentum its rho v and in the case of energy equation this is the total energy which is expressed as rho times e where small e is the internal energy plus rho v square by 2 which is the kinetic energy.

In that sense u takes on four different values for the four different equations. And similarly e and f, so if you want to reconstruct from this the x momentum the continuity equation you take the first value of this, first value of this, and first value of this. So, if you substitute that here we get dou rho by dou t plus dou by dou x of rho u plus dou by dou y of rho v equal to 0. And that obviously is the continuity equation.

Now, if you are looking at the x momentum equation you must take the second term here, and the second term here, and the second term here. And what is the second term here this is rho u. whereas, the second term of e here consists of 3 components, rho u square plus p minus tau x x. And we know that in the x momentum equation we have terms associated with dou by dou x of rho u square is what is coming here and then you have minus dou p by dou x since it is brought to the left hand side it is become dou p by dou x, and we also have dou by dou x of tau x x. And when we bring it to the left hand side it becomes minus dou by dou x of tau x x. All the three terms are going into dou by dou x things.

Similarly, the f term here is dou by dou y of rho u v and that is there and the other term tau x y or y x symmetric. So, this is dou by dou y of tau y x and that has been brought to the left hand side to be made into dou by minus dou by dou y of tau x y. If we now want to reconstruct the x momentum equation we take this value for u and all the three together for each and these to together for f, so that gives us dou by dou t of rho u which we know plus dou by dou x of rho u square plus dou p by dou x minus dou x by dou t of x tau x x, and plus dou by dou y of rho u v plus dou by dou y of minus tau y x other things. So if you were to take the tau s on to the right hand side and then the p on to the right hand side then what we will have for the x momentum equation is, dou by dou t of rho u plus dou by dou x of rho u square plus dou by dou y of rho u v equal to minus dou p by dou x plus dou by dou x of tau x x plus dou by dou y of tau x y. It is same as the x momentum equation that we had earlier.

Similarly, in the y momentum equation we have rho v is the term that is attributed to u and the e term has rho u v minus tau x y and the f term here has again p times like this rho v square plus p minus tau y y. So the pressure appears in dou by dou y term in the y momentum equation and it appears as dou by dou x term in the x momentum equation as is expected.

So, now let us look at the last equation that is energy equation. And for the energy equation we take the last term for u here again the last term for e and the last set of terms for f. Now let us just spend some time on this so that we are comfortable with these equations. For this we put it is as dou by dou t of e t and where e t is rho e plus rho rho v square by two and this is what we had as the dou by dou t term. And here we have e t times rho plus p times; i think there is a u missing here it should be rho u here and rho v here. So this is dou by dou x of rho e t times u plus dou by dou y of rho e t times v, and similarly dou by dou x of rho p u plus dou by dou y of rho p v. And then you have the work done by the stresses tau x x times u and tau y x times v, then tau y times v, and then tau y x times u here, and then the heat fluxes.

So, all the terms come together in the form of the energy balance equation. For simplicity we have written these as tau's, but these tau's are given by the Newtonian fluid approximation that is tau i j is equal to mu times dou u i by dou x j plus dou u j by dou x i plus lambda times dou u k by dou x k. It is a same definition that is there for the tau here. Similarly q x is equal to minus k dou t by dou x and q y is minus k dou t by dou y. These equations although these are written in an unfamiliar form are just regrouping of the earlier equations that we have seen in a specific format which enables us to make use of the methods that we know before. That we have studied before, so we are going to just rewrite it like this. And say that these this equation here in terms of the vectors u e and f represents actually 4 equations and the 4 equations in the case of 2 d are the conservation of continuity equation, conservation of x momentum equation, conservation of y momentum equation, and conservation of total energy consisting of internal energy and kinetic energy.

So, this is the formal equation that we would like to solve. These are the coupled equations that we are trying to solve.

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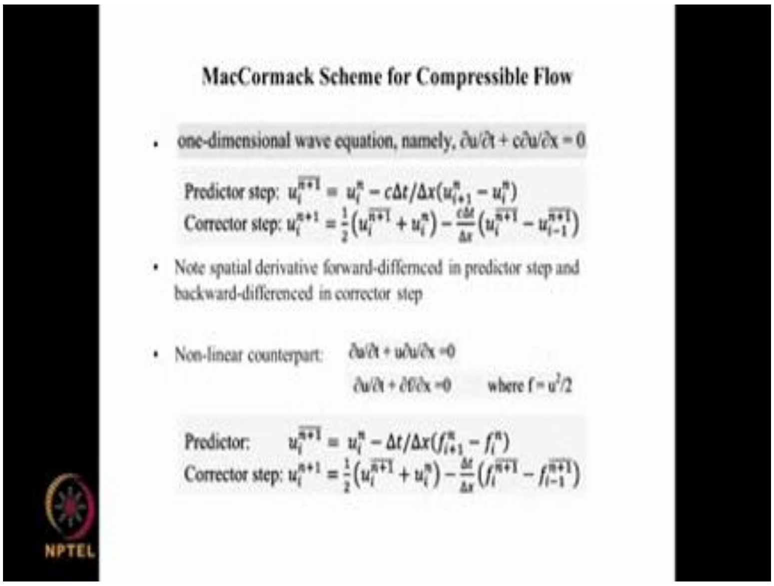

Now, one method that we start with is the MacCormack explicit method. And let us first see what this method is for a simple case. For example, for the case of one dimensional wave equation you can write it as dou u by dou t plus c times dou u by dou x equal to 0, we are quite familiar with this. For this particular equation MacCormack method solves it in two step process; predictors step and as a corrector step. We have seen that ftbs, ftcs those kind of a approximations for this, but this is a more advanced method which will actually help us deal with the non-linearity when we extent it to the real case. In the simplest form you have a provisional value of u i n plus 1 given by u n plus 1 over bar here, and this is evaluated as the old value minus c delta t by delta x times u i plus 1 minus u i minus u i n. And if you look at this step carefully this is nothing but ftfs, because if you bring this here and then divide by delta t so this dou u by dou t is being evaluated as u i n plus 1 minus u i n divide by delta t.

And then dou u by dou x here is being evaluated as u i plus 1 minus u i by delta x. This is the f t f x explicit form which we know is unstable. So, now this is corrected the value obtained by this is corrected in the next step as u i n plus 1 equal to half of u i n plus 1 bar plus u i n minus c delta t by delta x times u i n plus 1 bar minus u i minus 1 n plus one bar. So, what does is it mean here? If you look at a this it is not quite like the previous one this term here is the space derivative, so dou u by dou x and you have the delta t that is coming from there and here its written as u i minus u i minus 1. So, its backward spacing, but it is intermediate values the predicted values that are used here.

So, if you look at this and this the difference is these are the old step values and these are the provisional step values or the predicted step values and here it is used as forward differencing and here they have used a backward differencing. So, taken together we will get a central scheme in the case where you have no non-linearity. In the case of non-linearity then there can be some small differences. So, the predictor step and corrector step formulation allows us to put an overall second order central scheme, but on the whole despite using forward differencing here because of the correction that is introduced here we have an overall scheme which is conditionally stable. And what we would like to note here is that the spatial derivative is forward differenced; i plus 1 minus i in the predictor step and backward differenced in the case of corrector step. And that is something that we we would need to note and put accordingly.

And the other thing is that we evaluate all the values of i, in this one d case all the i values at n plus 1 bar and then use those values here in order to evaluate the corrector step. This is old value, this is a new value at i and these are the new provisional values of u i at n plus u i and u i minus 1. Now if you have it as a non-linear equation here c is constant, and what we talk typically have in our navier strokes equation is, a non-linearity like dou u by dou t plus u dou u by dou x equal to 0. Same as dou square u by dou by dou x of u square term is where that it is coming, so that can be written as u dou u in a sense with some manipulation.

This is a non-linear equation because you here is also being solved here and this u and dou u by dou x will make it non-linear. And this we put not in this particular form, but in the form of fluxes. Just as if you go to the second which lecture of the first module we expressed when we are doing the finite volume method, we expressed dou square w i dou x square plus dou square w by dou y square as del dot some f so the same kind of flux kind of terminology is being brought forward here. It is written as dou f by dou x where f is equal to u square by 2. Once you substitute this then this comes out, so these are mathematically equivalent, but this form is like the form that we are dealing with here.

So, in that sense we have this particular form they are anticipating having to use that form for the set of coupled equations. Now if you have an equation like this, the MacCormack scheme is applied in the following way you had c here now it is c equal to one and so you can write this as u i n plus 1 bar equal to u i n minus delta t by delta x the value of flux at i plus 1 minus f at the flux at i at the n th step. Now this is forward differencing. When you come to the corrector step u i n plus 1 which is the final value that we are looking at is half of the provisional value plus the old value minus delta t by delta x and f i minus f i minus 1 making it backward differencing based on the flux values evaluated at n plus 1 bar.

So, this and this one are similar except that you have putting it as a flux here in this particular thing and that allows you to bring in the u into this and then make the evaluation here. So that is how we can proceed. Now this is all one dimensional form. Now, how do we do the multi dimensional form and how do we do the coupled equations. So, we extend this to this vector equation, this equation involving u e f as vectors in the straightest forward way here.

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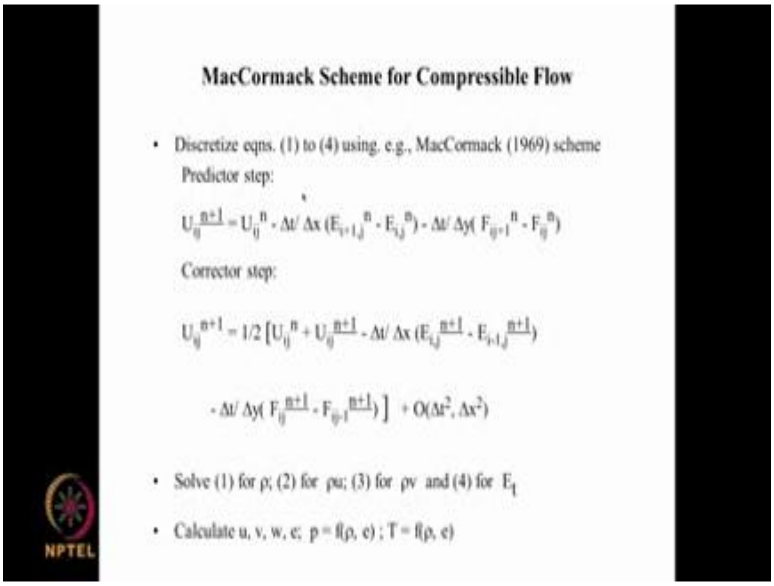

Because it is a two dimensional thing we have to now put it as u i j n plus 1 bar.

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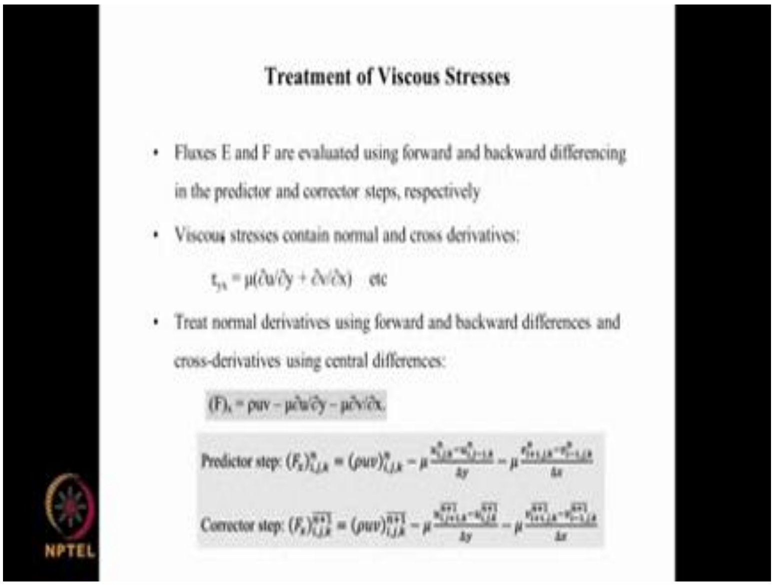

N plus 1 bar is u i j at n minus delta t by delta x and e i plus 1 j n minus e i minus 1 i j n divided by delta x. And f i j plus 1 minus f i j divided by delta y.

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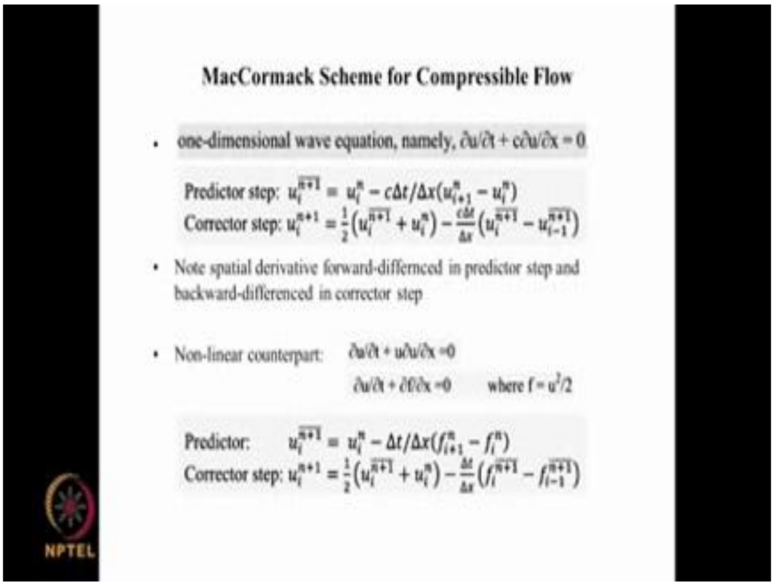

So, when you look at this it is exactly the same. Here dou u by dou x is u i plus 1 minus u i divided by delta x and the same thing is being put here, e i plus 1 minus e i divided by delta x. Similarly this is dou by dou f by dou y, so this is written as f i j plus 1 minus f i j divided by delta y. We are not making any modification here for the fact that u e and f are vectors it is being used directly here it is not necessary to make any change in this particular case.

Now, the corrector step is put as u i j n plus 1 this is the value that we are actually seeking and this is half of u i j n plus u i j n plus 1 bar, it is a same as what we had earlier there should be a bracket here. And minus delta t by delta x times e i j n plus 1 bar minus e i minus 1 j n plus 1 bar. So this is again the same as what we had here u i minus u i minus 1 based on the n plus 1 values or f i n plus 1 bar minus f i minus 1 n plus 1 bar. So, that is what we have here.

So, the e and f are the fluxes and the fluxes are being evaluated in backward differencing in the corrector step and forward differencing in the in the predictor step. So that is the way that we can write down the scheme here. And the overall scheme is second order accurate in time and second order accurate in space. So, it is second order accurate and not first order accurate, so you can say that it is more accurate and more likely to resolve the gradients as compared to there is less diffusion in this particular scale, so there is some merit in this.

And how we actually do this, given that u e f are vectors is that in each case for each equation we know the value what are the terms that are appearing here u e and f. We first solve equation one here for the continuity equation. That gives us dou rho by dou t plus dou by dou x of rho u plus dou by dou dou y of rho v equal to 0. So, that equation is solved for you. Now after we get u we solve using this and this again, but taking the second values for rho u, and then the third equation for rho v, and the fourth equation for e t. Since, we have rho from the first equation we can substitute this here and we can get u and we can substitute in this and we can get v, and e t is rho e plus half rho v square. We know rho and we know the total e and we also know u and v. So, from that you can get e, and once you get the internal energy and once you get rho you can get pressure from the equation of state as a function of rho and e. And we can also get temperature as a function of the density and internal energy of that particular medium.

So, this is one method by which we can solve all these equations we can solve all the equations together in a compact form, but at each time first solving one equation the continuity equation and that equation itself we solve as a predictor and corrector for all the i and j points. And then we go to the second the x momentum equation terms here. We substitute these into this and then we substitute them into the corresponding form here and we solve all of them to finally get rho u i j at n plus 1 for all the i and j. And then we solved the same equation for rho v to get rho v at n plus 1 for all i j, and then we also get e t for n plus 1 for all i j and from that we can deduce u v and from this we can deduce internal energy and from the calculated density and internal energy we get pressure and also temperature.

We can recover u v w p t in this particular way making use of the continuity equation to get density, the x momentum equation to get rho u and since we know density v we can get u from that. And the y momentum equation to get rho v and from that we get v. And from the energy equation we get total energy and since we know the kinetic energy components and density we can get the internal energy. And we get these by solving the appropriate partial differential equations using the MacCormack scheme and from known rho and e at every point at every, i j we can get the corresponding pressure at that particular i j. Similarly temperature at that i j

So, in this way we can get all the equations, but there is slightly more elaboration that is needed here because we have not said anything about tau and q s and all that so let us just look at that treatment of viscous stresses. Here we have fluxes e and f are evaluated in the MacCormack method using forward and backward differencing in the predictor and the corrector steps respectively.

Now viscous stresses are somewhat strange here because they contain both normal derivatives and cross derivatives. So, what do you mean by normal derivative and cross derivative? If you tau y x this is mu dou u by dou x plus dou v by dou y plus dou v by dou x. Now, if you look at this equation here we have dou by dou x of e and if you look at this one here this e has a component tau x x and similarly it has tau x y here. When you consider the y momentum equation there is a component like dou by dou x of tau x y. That means, that you have dou by dou x of of this whole thing.

Now, here we have dou by dou x of u of so you will have dou square u by dou x dou y here and dou square v by dou x square here. So, there is one which is coming as dou x square normal derivative and there is one which is coming as dou x dou y it is a cross derivative. And these kinds of things will also appear in other equations. The result of the viscous stresses is that there are some components which appear with cross derivatives so that is dou x dou y here, and some which appear as normal derivative dou square x. Similarly, we can have dou square y and dou x dou y terms when we consider other things.

So, how do we treat these terms? The prescription is that treat normal derivatives using forward and backward differences and cross derivatives using central differences. If we consider the term f, this is the term and if you are looking at the x component of it so that is this particular thing here in the x equation is, sorry. Rho u v minus tau x y and that is what we have here. Rho u v minus tau y x where tau y x is mu dou u by dou y plus dou v by dou x, so we substitute this here. And so the x component of f term is rho u v minus mu dou u by dou y minus dou v by dou x. So how do we write this in the predictor step?

So, in the predictor step we have f x i j k in three dimensions at n is evaluated. For example, this term is evaluated as rho u v at i j k plus mu times this one here is dou u by dou y, but this is dou by dou x. So that means, that this f term is actually dou by dou y so this becomes dou square y by dou x dou square u by dou y square, that is a normally derivative. In the case of normal derivative in the predictor step we use i j k minus i j minus 1 k, so this is backward differencing and the same term in the corrector step is written as i j plus 1 minus i j, so this is forward differencing. When you look at how we are evaluating this term in the predictor step and in the corrector step; we see that in one case we have make make use of backward differencing, the other case we are make use of forward differencing. We can use here forward differencing in which case we have to use here backward differencing.

Now, if you look at this term here, this is also some dou v by dou x this is also another derivative. And how do we evaluate this? We evaluate this as, this is dou v by dou x and it is evaluated as v i plus 1 minus v i minus 1 by 2 delta x, there should be 2 here. So that is being evaluated as a central differencing. And why is it being evaluated as central difference, because you are looking at f term and f term always appears with dou by dou y. So that means, that this particular term here is dou by dou y of this here, so that becomes dou square v by dou x dou y. So that makes it a cross derivative and cross derivatives must be evaluated using central differences and therefore we write this as dou square x dou y that particular thing is evaluated as v i plus 1 minus v i minus 1 by 2 delta x at n. And in the corrector step again it is v i plus 1 minus v i minus 1 by 2 delta x.

The same mistake, but the same idea here, so it is i plus 1 i minus 1 making it central here and central here. Whereas, this term which in the f term is dou square u by dou y square and therefore it is a normal term is evaluated as j minus j minus 1, so it is a backward differencing and here its forward differencing j plus 1 minus j.

With this detail it is now possible to go back to every term that is there in this and then put the appropriate way of differentiation here evaluating each of those derivatives and other things that are coming here, and then we can evaluate the whole thing. Now, one final thing we would like to notice that if you look at this evaluation here, this is u i j n plus one bar and it is expressed in terms of u i j n e i plus 1 n e i n and f i j plus 1 and f i j n. So, everything on the right hand side is a time step of n so this is known, and therefore this is explicit evaluation of u i j n plus 1 bar.

Now if you come to corrector step this is u i j n plus 1 value is expressed in terms u i j n this is known. U i j n plus 1 is already evaluated so this is known here. And e i j n plus 1 that is known because you have already evaluated all the u v p w and rho temperature all those things are evaluated so this is known. And again e i minus 1 j n plus 1 this is known, this is known, this is known. Again this is explicit. So this is an explicit predictor character corrector type of approach for the solution of the coupled equations and this is the MacCormack scheme which has been used for compressible flow calculations, is proposed in 1969 and it was a successful method.

So, in the next lecture we will see an alternative method and we will look at what is a difficulty with this, and we look at an alternative method for that; which is an improvement on this.