Computational Fluid Dynamics

Prof. Sreenivas Jayanti
Department of Computer Science and Engineering
Indian Institute of Technology, Madras

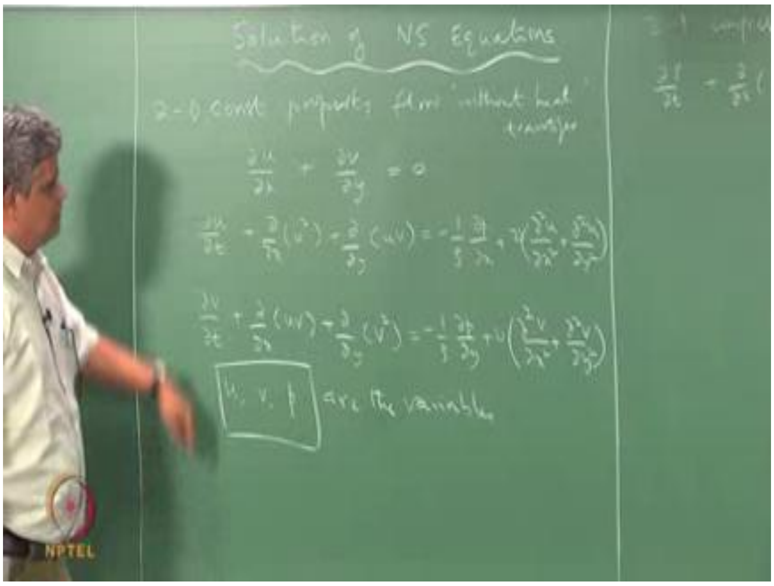
Lecture – 33
Introduction to the solution of coupled N-S equations

We are going to start a new module today. And this module is about how to solve the equations that govern the flow. What we have been, what we have done so far is to find out what the equations are that govern fluid flow. We derived the Navier -Stokes equations which are a statement of the continuity equation, and a momentum balance equation. We also derived the energy balance equations, species balance equations and we also looked at the boundary conditions and initial conditions for this.

So, we now know what kind of equations needs to be solved. And we have also in the third module looked at a one typically equation the scalar transport equation, and we looked at the issues that arise when we want to solve this. We looked at the issues of related to accuracy we looked at issues related to consistency, stability, and convergence, and the dispersion errors, and diffusion errors that can come out of the numerical solution. And based on all this things, we are at a position that we can make now we can go into the next stage of solving not just one equation, but all the equations so as to get a solution for a fluid mechanics problems.

So, in this thing, we would like to first write down equations and then we would look at what are the issues involved in it, and then we will go into the actual techniques that are needed here. So, in this introductory lecture of module-4 which is dealing with the solution of Navier-Stokes equations. We are going to outline our strategy and our methodology; we are just going to talk about the solutions. And so in that sense, we are not going to straight away discuss the solution techniques, because this we need to understand a bit more before we would like to go into that.

(Refer Slide Time: 02:42)



So, what we are looking at is Navier stokes equations and these equations for the simplest case of constant property flow. We can write this as the continuity equation as dou u by dou x plus dou v dou y equal to 0. And let us also consider a 2-D flow – two-dimensional flow, so that we have fewer numbers of equations to write. And we also have the x-momentum equation as you dou u by dou t plus dou by dou x of u square plus dou by dou y of u v equal to minus 1 by rho dou p by dou x plus nu times dou square u by dou x square and dou square u by dou y square. For constant property, single-phase flow where there are no buoyancy effects, and also those things we can neglect the effect gravity that can be subsumed in the pressure if necessary.

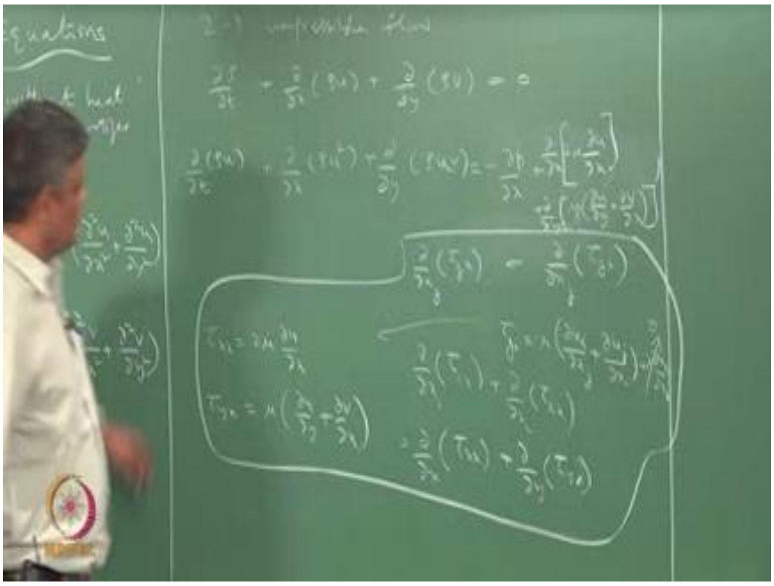
So, we have an x-momentum equation like this. And we can also write the corresponding y-momentum equation balance equation, which can be written as dou by dou x of u v plus dou by dou y of v square equal to minus 1 by rho dou p by dou y plus nu times dou square v by dou x sqaure plus dou square v by dou y square. If it is a 3-D, then we will have one more term here dou w by dou y, and then we will have one more term here dou by dou z of u w, and then one more term here dou square u by dou z square here.

Similarly, more terms will appear here and we will have one more equation for the representing the consideration of linear momentum in the z-direction. So, we have these three equations which describe two-dimensional constant property flow without any heat transfer effects without heat transfer, mass transfer and chemical reactions and all that. So, in that sense it can be considered as the simplest possible.

And in this case, we can see that we have three equations, and we have three variables, (Refer Time: 05:32) are the variables. So, mathematically you can say that you have got a problem in which the equations are given and we have as many equations as the number of unknowns, and we should be able solve this.

Now when we consider when we relax the assumptions somewhat, and we look at a compressible flow case, which can be of interest, for example, to aerospace type applications or even chemical engineering applications, mechanical engineering applications in which their significant pressure changes like the blow down process in which you have big pressure vessel, and either in an accident scenario or in an actual controlled scenario, you open a valve and because of the pressure difference the gas of the liquid is coming out, and you have significant pressure changes, so you have compressible effects that come into picture.

(Refer Slide Time: 06:40)



So in the 2-D compressible flow case, we can similarly write down the governing equations as now density changes and we can put this as del rho by del t plus dou by dou x of rho u plus dou by dou y of rho v equal to 0. This is still 2-D, by 2-D, we mean that there is no variation in the z-direction. So, if let us say that this board here represents the z-direction, and so z is increasing in this direction, what we mean by a 2-D flow in which nothing is happening in the z direction is that if you take, for example, velocity profile here, when you have a certain variation may be like this and you take the temperature profile it may be varying like this.

And if you go to some other z and here you plot the velocity profile, it is exactly like this and the temperature profile is like this. You go to some other location in the z-direction, again you plot the temperature profiles and then velocity profiles, they are exactly like that. At different z's there is no variation any of the velocity components any of the temperature components, any species component then you say there is nothing happening, no changes are happening in the z-direction.

So, you can say that it is not three-dimensional flow, it is may be two-dimensional and variations in the z-direction can be neglected. So, you do not have a dou by dou z of rho w here, because dou by dou z representing variation z -direction is 0. So, in such a case, you say it is a 2-D flow. And if you have for example, flow between two parallel plates like this, and the width of the plate is very long compare to the gap between the plates, then you can say that in the direction of the width there is not that much change and the flow is going only this direction. So, along the height between the two parallel plates, there is variation and may be in the flow direction there is variation, but not along the width. So, if you put the gap here as the y-direction, and the flow direction as x, in the z-direction there is now change so that is the two-dimensional flow.

So, two-dimensional flow or three-dimensional flow or even one-dimensional flow is related to the flow geometry. And you can imagine certain geometric cases, where you can have two-dimensional flow; and for those kinds of two-dimensional flow cases, we are writing down the corresponding equation assuming compressible flow. So, therefore, density changes, and we have to write more involved continuity equation. And the x-momentum conservation equation is also like involves density variations rho u square plus dou by dou y of rho u v equal to minus dou p by dou x plus dou by dou x of nu dou u by dou y plus dou v by dou x plus dou by dou y of mu.

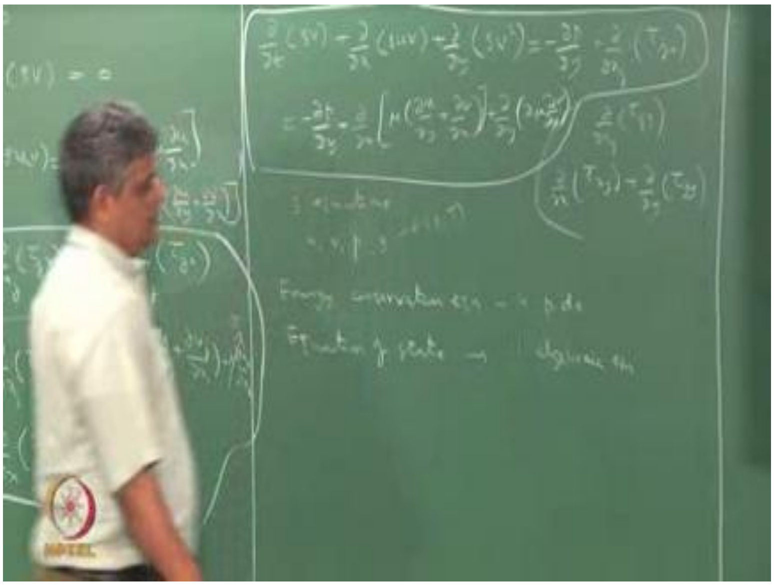
Here this is let us write minus dou p by dou x plus terms representing the stresses - viscous stresses, and so here the term is actually dou by dou x j of tau j i. Where tau j i or tau j i is mu dou u i by dou x j plus dou u j by dou x i plus lambda times dou u k by dou x k. So, this is the stress viscous stress in the case of a Newtonian fluid, and this is what is the corresponding expression in the case of the viscous stress in the x or y-momentum equation here.

And for this particular case, we can safely neglect this term because usually the second coefficient of viscosity is not important, so we will just neglect that particular thing and anyway dou u k by dou x k may not be such a big issue. So, we will simplify it a bit and we consider only the viscous stress arising out of the dynamic viscosity nu. So, this is the expression here, and we would like write the corresponding expression in the x-momentum equation. So, for the x-momentum equation, this term here becomes dou by dou x j of dou of tau j x because this is i th momentum equation, and this represents stress acting in i-direction on the j th face, so this is x here.

So, we will have two terms. So, we will have dou by dou x of dou by dou x 1 of tau 1 x plus dou by dou x 2 of tau 2 x. So, one here is the same as dou by dou x of tau x x plus dou by dou y of tau y x. So, now we can make use of this expression here to evaluate this tau x x tau y x. And from this, we can write tau x x as we can see that x x means that these two become the same so that is 2 mu dou u by dou x and tau y x is mu times dou u by dou y plus dou v by dou x.

So, these are the terms which go into the corresponding expression, and these terms go into this expression. So, with this, let us now rewrite this expression, so that we have a clear idea coming directly from the definition. And so this will be dou by dou x of two mu dou by dou x plus dou by dou y of mu dou u by dou y plus dou v dou x. So, this is the x-momentum equation, and we can similarly write the y-momentum equation.

(Refer Slide Time: 15:10)



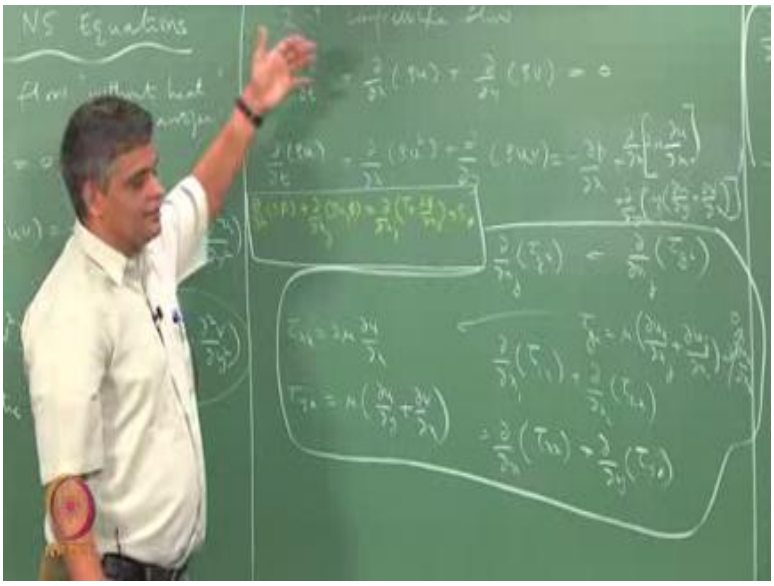
We can write the y-momentum equation as dou y dou t of dou v plus dou by dou x of rho u v plus dou by dou y of dou v square equal to minus dou p by dou y, and here we have plus again dou by dou x j of tau j i. So, this is dou by dou x j of tau j y in the y th momentum direction. So, this will be dou by dou x of tau x y plus dou by dou y of tau y. And we can make use of this expression here and then substitute this. And therefore, from this, we can get the proper expression this whole thing equal to minus dou p by dou y plus dou by dou x of mu dou u by dou y plus dou v by dou x plus dou by dou y of 2 mu dou v by dou y. So, this is our y-momentum equation and this is x-momentum equation, and this is the continuity equation.

Do we have anything more we need to have because its compressible flow, if you know look at the equations that we have we have three equations, and the variables are u, v, p and rho - density is also coming to picture here. And density is usually a function of pressure and temperature or it is given by an equation of state and it is a function of temperature also.

So, we need to also solve the energy equation; and we also need to have the equation of state. So, in the case of 2-D compressible flow, we have four partial differential equations, and typically one algebraic equation representing the equation of state. In the case of incompressible flow with constant properties, we have three equations, and for the there are three unknowns here. So, these are the kind of equations that we are solving for the simplest case.

So, let us now take a moment to consider what we are trying do here and what these equations are. Before we go ahead and try to solve these things, and apply our knowledge finite differences and knowledge of consistency analysis and the stability analysis and all those things, before we do that we would like to see what this equations alike. Now, when you look at this equation, it looks like pretty straightforward here, but this is an equation which involves two variables. In our scalar transport equation that we had we had only one variable we had an equation like this.

(Refer Slide Time: 19:33)



Our model equation, which we are going to write in yellow, is dou by dou t of rho phi plus dou by dou x j of rho u j phi equal to dou by dou x j of gamma phi dou phi by dou x j plus source term. So, this is our equation. And in this equation, we are suppose to know the source term, we are suppose to be given the diffusivity, we are suppose to be given the density and the velocity, and the only variable in this equation is phi. But here when we look at this, this simple equation looks very simple, but it has two variables u and v; so in that sense, it is not as the like the model equation.

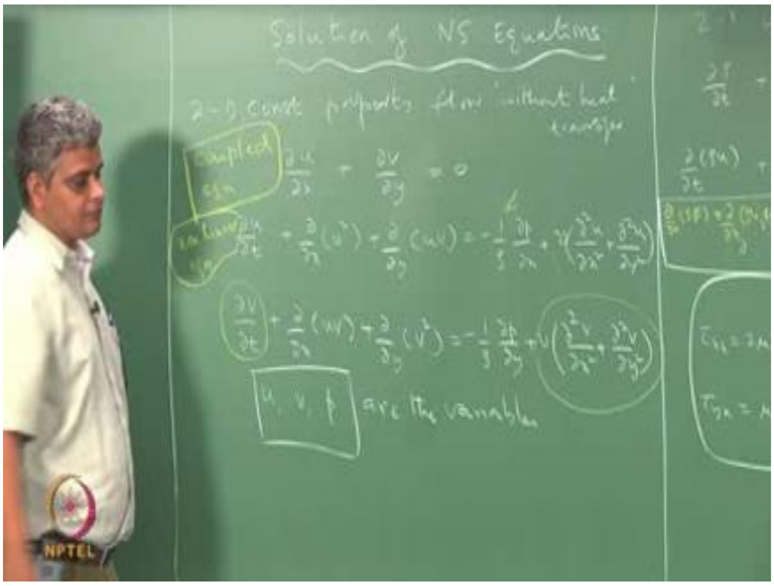
When you look at this equation here, this looks like you have similar kind of terms dou by dou t term, advection terms and diffusion terms like that. And you can consider this to be like the model equation with a time dependent term, and the two terms arising out of the advection term, source term which is a pressure gradient and the diffusion term so this looks like this, but it is not quite the same here.

Because, in this case, in a way, we are looking at some kind of linearized thing and velocity here is known and phi is a variable here. Here when you look at dou by dou x of u square one u corresponds to this u here and the other u corresponds to phi. So, there is phi, but the u is not known. Similarly, when you come to this, this u here corresponds to the phi the variable that this equation is being solved for; and this v here is not known. And it is not known from this equation, it is not known from this equation here.

And when you come to this term here, the source term is introducing a new variable p, which is not part of this it is not given it is one of the variables here, so it is one of the variables which are represented by these equations and that is not known. So, this is not known when we are trying to solve for u from this equation, just as we discretized this and solve phi from this equation, at that case in the model equation everything is known except phi.

And here we would like to solve this equation first u here, but unfortunately the coefficients in the advection terms are not known, this source term is not known, and here we can say that the viscosity is given. So, everything is known about the diffusion term here this is like the corresponding model equation term, but other terms are not known.

(Refer Slide Time: 22:53)



Similarly, when you come to the y-momentum equation, this is like the model equation with phi equal to v. So, this term is like the model equation and again this term is like the model equation that again here we do not know p, so the source term is not known. And when you come to this the coefficient here there it is given and here it is not known and the coefficient here is not known, whereas there it is known. So, we now have a scalar transport equation with unknown coefficients and unknown source terms, but diffusion term and the temporal term are known. So in that sense, the solution is of this is not like the solution of this, and there is a difference.

So, when you are trying to solve this let us say for u then we do not know v. If you are trying to solve this for v, then you do not know u. And when you are trying solve this for u p is not known v is not known and so in that sense, but when you take all of them together then you have three equations and three unknowns. So, these are coupled equations. You cannot solve any one equation in isolation like you are doing for the model scalar transport equation; and not only that these are non-linear equations. Why do we say non-linear? Here this is supposed to be u phi, but this u square so it is a same phi here that is appearing as with coefficient here so this makes this non-linear.

This term here makes it coupled; because you are solving for u and v is not known at it can be obtained only from a different equation. So, in that sense, when we look at the constant property flow in the simplest case, we have coupled equations and non-linear equations. Now you come here in this particular case of compressible flow, again you have the same situation that this equation here is introducing is involving three variables rho, which is not known, u which is not known and v which is not known. So, you can solve this only for one variable and the other two are coupled to the other equations that are present. And again here you have u here and rho, non-linear term here, coupled term here, unknown term here, whereas these things are this part is known, but this is coupled.

Similarly, the y-momentum equation again you have unknown term here, unknown terms here, non-linear term here, unknowns' term and coupled term here, and a known term here. And in addition to this, we have the energy conservation equation which we are not written down, but that is another equation which has with introduces a new variable either temperature or enthalpy. And it also will be involving the advection terms in which the velocity terms are not known; and so in that sense that equation represents another equation which is like this scalar transport equation model equation, but which cannot be solved in isolation.

So, in the case of constant property flow, we have non-linear coupled equations which are three in number here; and here we have non-linear coupled equations with four partial differential equations, all of which are coupled none of which can be solved independently. And we also have the equation of state. So, we have coupled non-linear equations. In the case of 2-D constant property flow with heat transfer, in addition to this, we will have the heat transfer equation; that heat transfer equation we will see can be is decoupled from the momentum equations.

So, in that sense, one you solve the momentum equation, you get u, v at every point; and then the corresponding heat transfer equation will be of this form with known coefficients, and that equation can be solved at that time it will be like the model equation. But there is a problem the model equation can be solved only after we solve the Navier-Stokes equation and get u, v, p. So, in that sense in the case of constant property flow with heat transfer incompressible flow with heat transfer the energy balance equation is decoupled from the other equations, but it can be solved only after solving the continuity and momentum equations.

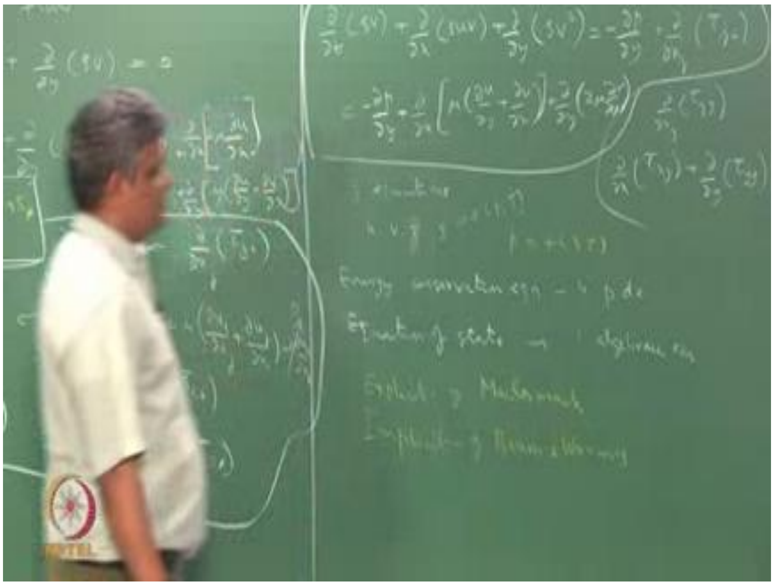
So, there is some kind of decoupling and it can solve separately, but only after you have solved the coupled equation. So, this is a feature of the equations that we are trying to solve and the equations the features are we have coupled equations; no single equations can be solved in isolation. And we have non-linear component to this and so that is giving a problem. And we have in the case of compressible flow, larger number of equations and larger kind of couplings here; and so the solution is not going to be just like you solve the models transport equations several times we have to think of ways of dealing with the non-linearity and the coupling.

And so what we are going to do as strategy is that despite the complexity additional complexity that is there in compressible flow; in the sense that here the equation involves only two variables, and here it involves three variables, and here we have three equations and here we have five equations. And all these things, there is a natural extension that is possible from the concepts that we have learnt by solving the model equation here, and then we will try to create a template, we will examine the templates that that have been proposed for the solution of these coupled equations for compressible flow calculation.

And in compressible flow, each equation including the energy equation is like the model equation. In the first case, it is like the model equation with phi equal to 1 and diffusion term is 0, and source term is 0. In the x-momentum equation, it is like the model equation with phi equal to u here and then you have the rho u term coming here, and the source term is given by this. And in the case of the y-momentum equation, phi is equal to v and then it is somewhat similar here, whereas if you consider the case of incompressible flow this equation is not like this because there is no dou by dou t term. So, in that sense this equation and the corresponding continuity equation are different this particular equation is not like the standard model equation.

So, we will look at how we can extend the template that we have created which is that we deal with this using forward in time, and this can be backward in space preferably with up wending with so that we do not have these oscillations and (Refer Time: 31:28) oscillations. And then we have central differencing approximation for this. So, those kind of template for the solution of the model equation, we will try apply here; and then we see we can develop a method by which we can solve all these coupled equations, but it is not a direct extension, it is some modified extension of this.

(Refer Slide Time: 32:09)



So, we will start with the case of compressible flow, and we will look at one method which is, we look at one explicit method of Mac-Cormack for the solution of this coupled equations. And we will also look at one implicit method of beam and warming, and so these are couple of different approaches for the solution of coupled equations that appear in compressible flow.

And we will see that these kind of methods in theory could be extended to incompressible flow calculations, but there are certain restrictions in this that make it not suitable for this kind of problems. Where you have obviously, you have an equation for u which is given one could extract, one could interpret this as an equation for u and one could interpret this as an equation for v, but one cannot interpret this as an equation for p, because p is not a variable in this. So, in that sense the three variables that are here are not appearing in all the three equations.

Whereas here if you consider rho, u, v and t as the four primary variables, and p are given as a function of rho and temperature from the equation of state, then you have four equations, and you have one equation for each of this.

This is an equation for rho; this is an equation for u; and this in equation for v; and the energy conservation equation is a equation for T, so that kind of interpretation is possible. And it is not possible in this case, we have to do something more. So, the kinds of methods that have been developed for compressible flow are significantly different from the kind of methods that are used for incompressible flow. So, in the first part of this module, we look at compressible flow cases; and in the second part of the module, we look at methods for incompressible flows. So that is what we are going to do in the next several classes.