Computational Fluid Dynamics
Prof. Sreenivas Jayanthi
Department of Computer Science and Engineering
Indian Institute of Technology, Madras

Lecture – 31
Tutorial on Stability Analysis

In today's lecture we are going to do a stability analysis, and today's lecture will be in the form of a tutorial we take a specific problem and then try to do it on the board. So, that you can follow closely what I am doing and then you can replicate it for other problems of your interest today’s problem is to do a stability analysis for the same one-dimensional wave equation which we have been considering earlier, but with an implicit scheme. So, we are going to do FTCA scheme for the 1-dimensional linear wave equation and what you mean by FTCS scheme is forward in time, central in space, but an implicit scheme and we will try to do the von Neumann stability analysis for this and investigate the stability conditions for this particular case.

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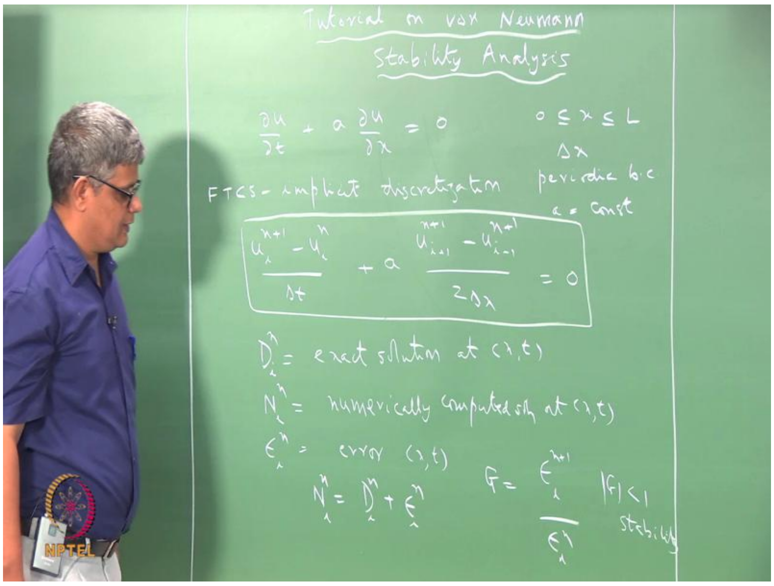

So, it is a tutorial on von Neumann stability analysis, the equation that we take is duo u by duo t plus a duo u by duo x equal to 0. This is on a domain 0 less than x less than capital l and with uniform grid space you know delta x, with periodic boundary conditions. So, that and a is constant. Therefore, we have a linear equation with periodic boundary conditions, uniform grid spacing and 1-dimensional form we can apply the von Neumann stability analysis for this.

And we are going to consider the forward in time, central in space, implicit scheme discretization. So, that we can write this as u i n plus 1 minus u i n by delta t, is the forward discretization forward differencing for the time derivative, plus a time central in space. So, this will be u i plus 1 minus u i minus 1 by 2 delta x. Since it is implicit the space derivative is evaluated at n plus 1 and this is equal to 0. So, this is our discretization scheme and we can also write this as we can take this delta t this side and get an expressive formula for this.

So, when we have a scheme like this 1 and we want to investigate the stability using the von stability analysis. We consider d to be the exact solution, and n to be the numerically completed solution. And x epsilon to be the error and each of these is a function of both x and t because we have this u here is a function of x and t therefore, we index we denote the space x by the space index i and the time by the time index superscript n here, and this will be the exact solution at time of x t and this is also at time of x and t and this is at time of x and t. And as the solution progresses d also changes d i n is changing and n i n is changing and epsilon i n is changing and we say that the numerical solution is the exact solution plus an error and therefore, we can say that n i n, is d i n like this. And what this means is that we have an exact solution plus an error which is coming, which is changing at a particular location with time because as n changes and if the scheme is such that this error can accumulate.

Then if we run enough number of this times test the accumulated error may become much bigger than this. So, as to give us an computed solution n, which does not look like d as long as epsilon is very small, n will be almost equal to d if epsilon is growing. Then after certain number of time steps epsilon will become large it will become. So, large that n will be more like error than like d, and that is what we would like to avoid. So, we would like to see under what conditions the error will grow and eventually take us to a situation where this error dominates the exact solution and therefore, the completed solution is no longer like the exact solution.

So, this is the stability idea and we are going to define an amplification factor, which is error at a particular special occasion at n plus 1 times divided by error at the same special occasion at n and if the amplitude of this, if mod g is less than 1 then, we can say that error will not grow, if it is greater than 1 we have possibility of the error increasing with time in magnitude and therefore, overwhelming the exact solution to give rise to a computed solution, which is no longer like the exact solution.

So for g mod of g less than 1, we have stability, and mod of g, is greater than 1, we have instability and that is what we should be avoiding. So, under what conditions are this particular scheme stable is the situation that we would like to evaluate and for that for a linear initial value problem with periodic boundary conditions. We can make use of the von Nuemann analysis wherein we decompose the error that x of t into a sum of Fourier components and m varies from 0 to capital m and m is given by k m is given by m phi by l and m is 0, 1, 2 up to capital m where capital m is l by delta x.

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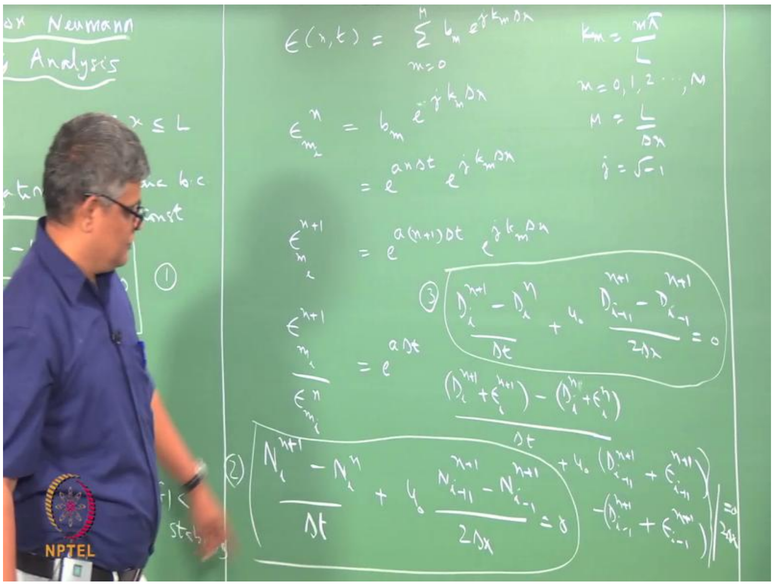

So, these are the Fourier components these are the m capital m plus 1 number of Fourier components which are present in this series and each of this is a sinusoidal component with an amplitude and with a sinusoidal variation given by this exponential this function with j equal to square root of minus 1, and because we are looking at a liner equation the contribution of each of this Fourier components can be added at any time to give rise to the total error at a particular special occasion, at that particular temper location and because this super quotient principle is possible to look at the evolution of the error we can take any particular wave component and we try to investigate the stability condition. So, we look at the specific component mth wave component at location i m times of location n, and we write this as b m minus j k m at delta x, where k m is the corresponding wave number for this and because we are going to look at whether its going to increase with time. And we are going to take the ratio here, we can also write this as exponential of a n delta t exponential of j k m delta x here.

So, that this is at time t and the error of the same component mth wave component at the same location at the next time step n plus 1 is therefore, given by exponential of a times. Now we have already used a here lets change, this it is easier as u naught in a way it is a linearised conviction term. So, we can retain this n plus 1 times delta t j k m delta x here and the ratio between the 2 is essentially for the mth wave component n plus 1 divided by the same component at the same location that at the mth time is now given by exponential of a delta t and therefore, if a is positive then this is going to be a positive quantity and if it is greater than 1 in magnitude then it is going to give raise to.

So, we make we take this particular wave component and then we substitute that in our formula here starting with this and we can call this as equation 1 and the numerical value that we get here is obtained from this. So, we can say that equation 1 is satisfied. So, as to give rise to n i n plus 1 minus n i n divided by delta t plus u naught times n i plus 1 n plus 1 minus n i minus 1 n plus 1 divide by 2 delta x equal to 0. So, this is our second equation and we also know that d i n is an exact solution and we can also say that d i in plus 1 minus d i n divide by delta t plus u naught times d i plus 1 n plus 1 minus d i minus 1 n plus 1 by delta x is equal to 0 and that is equation number 3 here. We also have this expression and we can substitute this here and then we can get a fourth expression which is d i n plus 1 plus epsilon i n plus 1 is this quantity minus d i n plus d i n minus plus epsilon i n this whole thing divide by delta t plus u naught times this here which is d i plus 1 n plus 1 plus epsilon i plus 1 n plus 1 minus d i minus 1 n plus 1 plus epsilon i minus 1 n plus 1 this whole thing. So, this minus this whole thing divided by 2 delta x equal to 0.

Now, we can club all these together and we will be getting this and this whole thing is equal to 0 and once we take out the d components.

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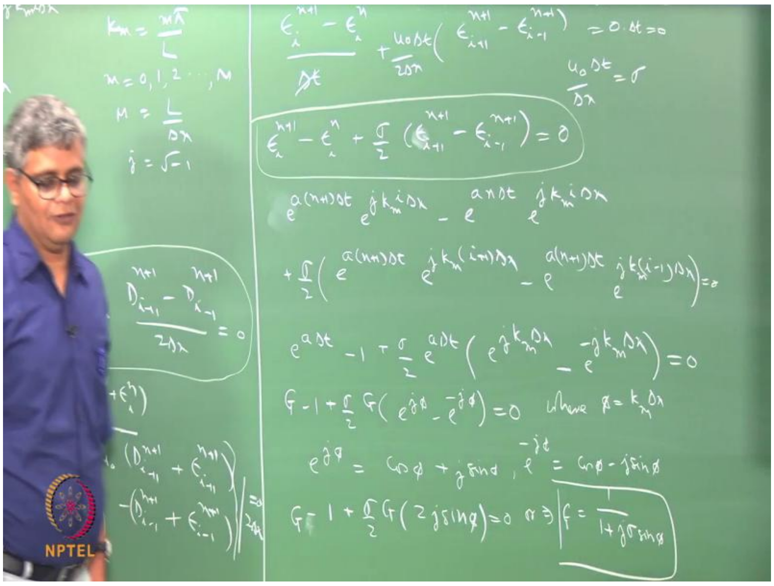

We will have an error evaluation equation as epsilon i n plus 1 epsilon i n plus 1 minus epsilon i n divide by delta t plus u naught by 2 delta x times epsilon i plus 1 n plus 1 minus epsilon i minus 1 n plus 1 equal to 0. So, we can now take this delta t here. So, that we have times delta t which is still equal to 0. And this can be written u naught delta t by delta x is (Refer Time: 15:59) number sigma. So, we can write this as epsilon i at n plus 1 minus epsilon i n plus sigma by 2 times epsilon i plus 1 n plus 1 minus epsilon i minus 1 n plus 1 equal to 0. So, this is the error equation here and then this error equation is decomposed into. So, many Fourier components and we look at the mth component of this here and therefore, we substitute this form into here and we can write this as the mth component will be will be exponential of a n plus 1 delta t exponential of j k m i delta x i think here i forgot to put i because this is at x this should be i times delta x. So, this is this is this term minus exponential of a n delta t exponential of j k m i delta x plus sigma by 2 times exponential of a n plus 1 times delta t exponential of j k m times i plus 1 delta x minus exponential of a n plus 1 delta t exponential of j k m i minus 1 delta x. So, this whole thing is equal to 0.

So, this each of this term this whole equation is now divided by this term. So, that we have if we divide by this we get a delta t because, this and this will cancel out and n part will cancel out with this. And we get exponential of a delta t minus 1 plus sigma by 2 here and here, a n delta t again will cancel out here leaving with this exponential of a delta t and similarly here exponential of a delta t will be there. So, we can write this as exponential of a delta t times here in this k m i delta x cancels out, we will have e j k m delta x minus here again this is already considered and i delta x will cancel out and we will have minus j k m delta x with the minus sign and this whole thing is equal to 0. So, we have now this equation is just a simplification of this in which we have substituted the mth wave component in the error propagation equation.

Here and the divided by this quantity in order to get g because this is precisely what our application factor is because this is epsilon i n plus 1 by plus epsilon i n and that gives us this. So, this is our g here. So, we can write this as g minus 1 plus sigma by 2 this is again g times e j 5 minus e minus j 5 equal to 0 where phi is equal to k m delta x and as we have mentioned earlier as k m changes as the wave number changes this phi will take values from 0 to phi.

So, in that sense we can virtually take this phi to be a continuous function from 0 to phi even though it has discreet values. So, as delta x tends to smaller and smaller values capital m becomes larger and larger so; that means, if the number of steps taken to go from 0 to phi will increase and it becomes in the limiting case of delta x to 0 it becomes almost a continuous function of phi all the way from 0 to phi and we also have the relation like e j phi is cosine phi plus j sine phi and e minus j phi is cosine phi minus j sine phi here and. So, if you take this to this minus this you will be getting minus 2 j sine phi g minus 1 plus sigma by 2 times g times 2 j sine phi equal to 0 or g equal to 1 by. So, this 2 and this 2 will cancel out and. So, we get g times 1 plus sigma j sine phi minus 1 equal to 0 or g equal to 1 by 1 plus j sigma sine phi this is our amplification factor. So, now, we have to investigate the conditions of under which g is going to be greater than 1.

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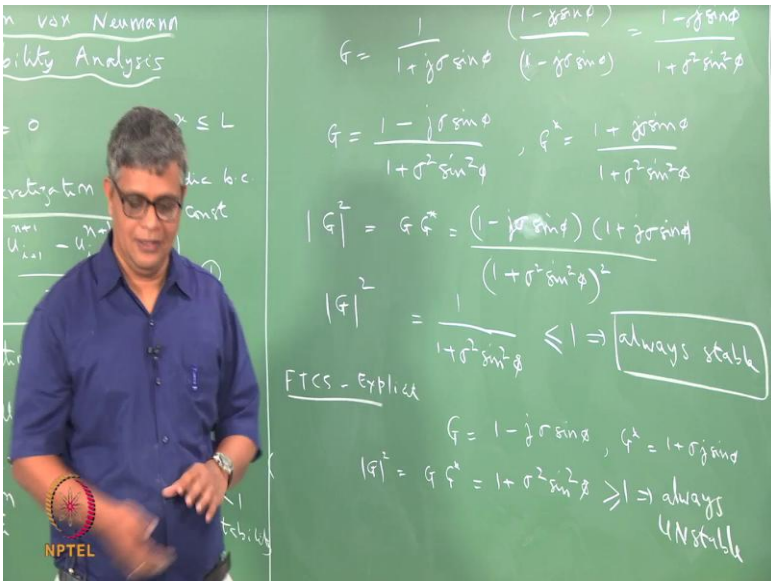

So, we have g equal to 1 by 1 plus j sigma sine phi, and we can multiply and divide by 1 minus j sigma sine phi. So, as to get 1 minus sigma j sine phi and we get 1 minus sigma square j square sine phi and j square is minus one. So, we get 1 plus sigma square sine square phi this is a real quantity and therefore, this is our g here. So, therefore, we can write this g as 1 minus j sigma sine phi by 1 plus sigma square sine square phi. And the complex conjugate g star will be this 1 plus j sigma sine phi divide by 1 plus sigma square sine square phi and we can write, the square of the modulus of g as g times g star and this is equal to 1 minus j sigma sine phi times 1 plus j sigma sine phi divide by 1 plus sigma square sine square phi whole square because, it is the same thing which is common and this again gives us 1 plus sigma square sine square phi. So therefore, it is 1 plus sigma square sine square phi.

So, this is square of the magnitude the numerator give us 1 plus sigma square sine square phi. So, we cancel with 1 of these and then we have this. And what we see is that this is always less than 1, in the special case where phi is equal to 0 of 180, then this equal to 1. Because sigma is u naught delta by delta t by delta x and we have sigma square here then, this is always less than or equal to 1 and therefore, it is always stable this is in contrast to the FTCS in the case of FTCS explicit where we put this as u i n plus n and u i minus 1 n here then we have seen that g in this case is equal to 1 minus j sigma sine phi and therefore, j star is 1 plus sigma j sine phi and therefore, the square of the modulus is g times g star and that is equal to 1 plus sigma square sine square phi and this is always greater than or equal to 1 except for the special case of phi equal to 0 or 180 this is greater than 1 so; that means, that this is always unstable.

So, the same FTCS explicit version is always unstable, but the implicit version is always stable and this is something that we get usually when we go from stability from a stability point of view from explicit method to implicit method. There are explicit methods which are unconditionally stable just like this FTCS implicit. And there are explicit methods which are unconditionally unstable like this 1 here, but usually when an explicit method is conditionally stable or unstable the corresponding implicit method may prove is very likely to be to prove unconditionally stable. And therefore, when we go for an implicit method then we gain in terms of stability, and that is something that we would like to demonstrate.

And I hope from this tutorial we have learnt the mechanics of how to do the von Neumann stability, analysis for a given discretization. So, we start with finding the error evaluation equation either this or this we saw this from the discretization. And then we investigate the stability of the mth wave component, by substituting the corresponding expression for epsilon i n s as epsilon as a n delta t, times epsilon j k m i delta x here and then we find the amplification factor and from the amplification factor, we find the magnitude and we see under what conditions the magnitude is greater than 1.

And if the magnitude is greater than 1 for certain conditions then it is conditionally stable or conditionally unstable. If for all conditions if the magnitude is greater than 1 then, it is always unstable and if there are if for all conditions of sigma and phi and all that if it is stable if it is less than 1 it is always stable. In the next class we are going to look at some more schemes and then we are going to look at the stability analysis, and we will move on to the 1 dimensional scalar transport equations and look at the stability.