Computational Fluid Dynamics
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Lecture – 30
Properties of Numerical Schemes: Stability analysis

In this lecture we look at the second important property, which is necessary for us to have the check for converges, which is the stability we would like to formulate in this lecture the problem of stability, that is the definition of stability and what is what do we mathematically mean by stability? And we can also we will also learn to see how we can make an analysis for stability of a given discretization scheme.

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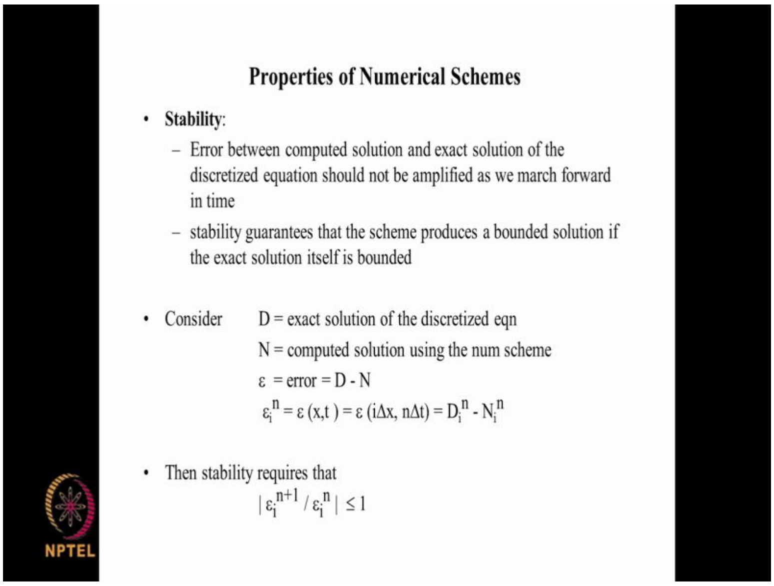

So, when we talk about stability we are talking about the error between the computed solution and the exact solution of the discretized equation. And we are talking about this difference between the two, between the exact solution of the discretized equation and the computed solution of the discretized equation, that this error should not be amplified as we much forward in time. Because we are accumulating errors from the previous times steps and we are also gathering errors from the approximations made at of the derivatives spatial derivatives.

So, stability guarantees that the scheme produces a bounded solution if the exact solution itself is bounded. So, in that sense it is a very important property, and we have seen that consistency al1 is not sufficient for us to ensure converges, and good solution, but if we also have stability then we could get possibly a converge solution. So, in this sense how do we express this problem this concept of stability.

Consider d to be the exact solution of the discretized equation, the capital d and similarly capital n to be the computed solution using a given numerical scheme. And both d and n will vary with position and time in a functional space of x and t, they going to vary with x and t. So, you could say that d i n with subscript i and superscript n is the exact solution of the discretized equation at the spatial location i, and at the temporary location given by the index n similarly we can say that n sub i sup small n is the computed solution, at the same spatial location i and the temporary location n and the error at the ith location at the nth time step; is therefore, given by the difference between the two d i n minus n i n has given here. And this error is going to change with time and it is going to change with the location, stability means that the error at a particular location does not get amplified with time.

So, that is the error at spatial location I, at n plus 1 divided by the error at n this ratio here is always the magnitude is less than 1. So, that it does not become either amplified in the positive values or negative values. Example if you have 1, 0.2 and if the ratio is 2 then next time it is going to be 3.6 and then after that it is going to be some eleven point something and after that it is going to be some thirty five point something. So, the value is going to increase tremendously and similarly if the ratio is actually minus 3 then, it will go from 1.2 to minus 3.6 plus 11.8 minus 35. So, it will keep on getting amplified isolating negative and positive. And if it is going to increase magnitude wise in that way it does not matter is accelerated between positive and negative or just accelerating. It will soon grow to such a high value that the true value of that functional solution at that location. Example 10.5 may be the value here and if the error goes to 35 and 110 like that that error itself becomes much larger than the functional value and so, you will use the real functional value variation if the error becomes very high.

So, we do not want this ratio of the error at n plus 1 to the error at n to be greater than 1 in magnitude. Because if greater than 1 it starts accumulating and because we tend to take very many values to reach a particular time step time location because we would like delta t to be small. So, that we can maintain good accuracy, that means, that in order to reach a particular time denote we have to take lots of steps. So, in taking the lots of steps if you have an amplification factor even if it slightly greater than 1 it can accumulate very fast. So, we would like the error to be such that error at n plus 1 divided by error at n is always less than or equal to 1 in terms of magnitude. So, this we can say is the formulation of the stability. Now the question is how do we know the error and if you know the error.

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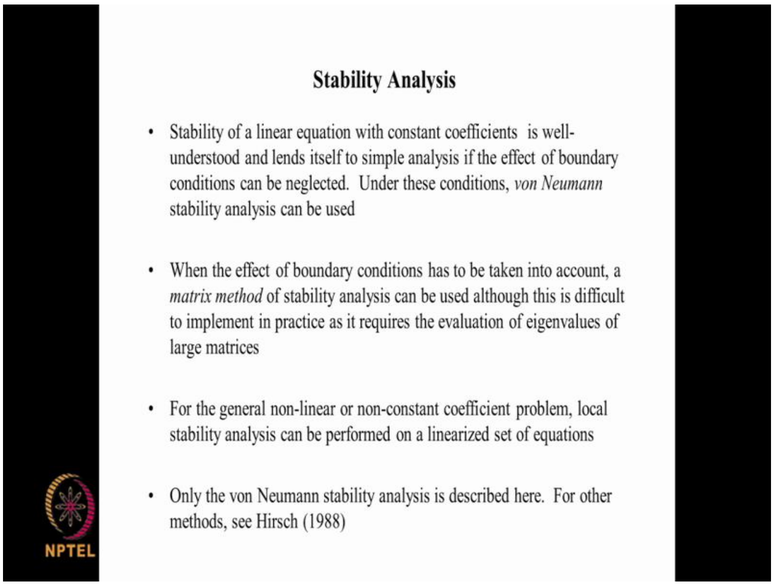

Why do not we just add it or subtract it to get the true solution.

So, in all of stability analysis we do not explicitly compute the error. We only look at the whether a given scheme has this property of amplifying the error or not. Whether or not the scheme has property of amplification, as for as the stability analysis concerned stability of a linear equation with constants coefficients is well understood, and lends itself to simple analysis if the effect of boundary conditions can be neglected ok.

So, if the boundary effect of boundary conditions can be neglected and if you can treat the boundary conditions effective as periodic boundary conditions then we can have a powerful analytical tool which known as the von Neumann analysis which can be performed on a uniform grid. And the non-even form grids then we have more difficulty, when the effect of boundary condition has to be taken a matrix method of stability can be used all though it is difficult implement in practice as it requires the evaluation of eigenvalues of large matrices. And similarly when you have non-even form spacing again you can make use of the matrices method.

For general non-linear and non-constant coefficient problems local stability analysis can be performed on linearized set of equations, but what we have as stability analysis is the linear stability analysis. So, and we are going to discuss the simplest of a stability analysis which even though it is simplest it still gives a feel for the possibility of instability gross instability which manifests very quickly. So, this is known as the Neumann method, von Neumann stability analysis and this analysis is strictly applicable for linear equations with uniform spacing and with periodic boundary conditions. So, under those conditions it is possible for us to evaluate the amplification factor all though we do not individually evaluate error at n plus 1 and error at n.

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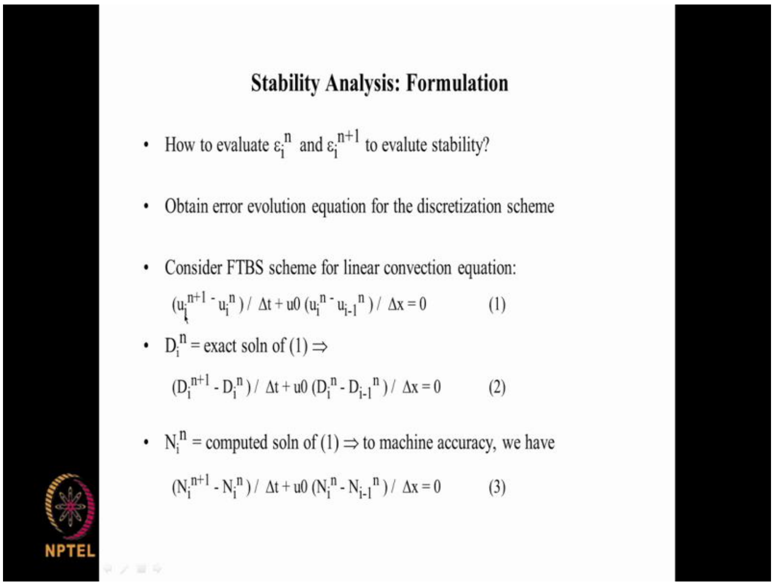

So, how do we do this evaluation of the stability? So, we evaluate we obtain an error evolution equation in the following way; for example, if we take the FTBS scheme for the linear convection equation it is a linearized convection equation that is why we put it as u naught is the wave speed, and confusing though, but this u is a variable u, So, u have dou u by dou x dou u by dou t plus u naught times dou u by dou x equal to 0. In a way if it is left hand side of the momentum equation with the 1-dimensional form of the left hand side of the momentum equation neglecting the influence of pressure and neglecting with customs and it is also linearize around point u naught.

So, for this if we apply forward in time here and backward in space here, then we get an FTBS scheme like this. Now, we say that d i n, capital d i n is the exact solution of this equation so; that means, that if you substitute d i n into this equation appropriately then this equation would be satisfied therefore, at point i n plus 1, this is d i n plus 1 and here it is d i n divided by delta t plus u naught times d i n minus d i minus 1 n divided by delta x equal to 0. This is because capital d i n is the exact solution of the discretized equation by definition.

And n i n is the computed solution of this equation to machine accuracy and therefore, we can say that n i n plus 1 minus n i n divided by delta t plus u naught times n i n i n minus n i minus 1 n divided by delta x equal to 0. What is the difference? Whereas d does not have any error, n has error. And what this 1 is saying is that there is some error at n i at n plus 1 and there is also error at n and these are cleverly because n i n plus 1 is actually derived from n i n it is carrying the error from n i n and it is also modulating the error by the influence of errors from the neighboring things to get a new value of error relating numerical solution n i n plus, 1 such that this overall equation is satisfied.

So, the difference between equation two and equation three is d is the exact solution and this is exact solution plus an error, but the error is not random error, it is the built up error built up from previous time at n time step, and also neighboring space points. So, it is a built up error and the error builds up in such a way that you get a numerical solution which also seems to satisfy the discretized equation, because this is solved to machine accuracy and machine accuracy can be sixteen decimal places or and so on. So, for all practical purposes the value of n i n plus 1 that you compute from the previous value satisfies the governing equation.

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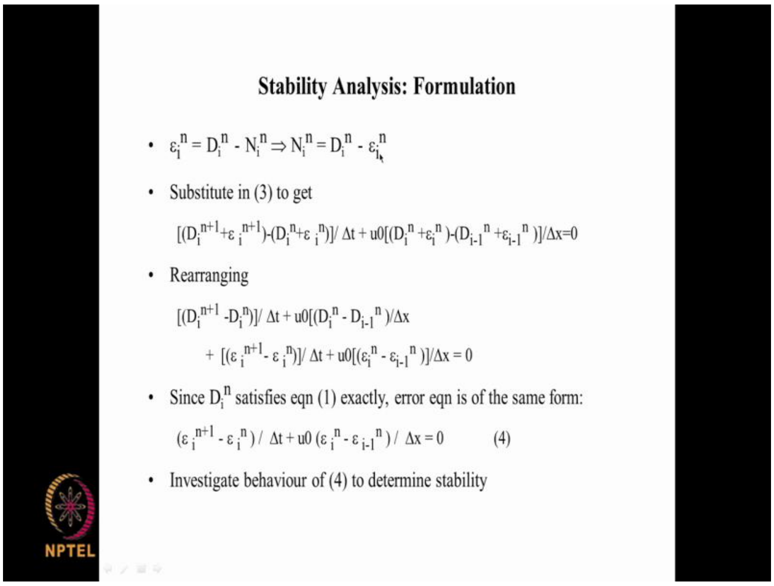

Now, we have defined error at i n as d i n minus n i n and therefore, you can say that n i n equal to d i n minus epsilon i n and therefore, we can substitute this into equation 3 here, where i we have n i n you substitute d i n minus e i epsilon i n and where i we have n i n plus 1 you substitute as d i n plus 1 minus error at i n plus 1 like that and if you do that you get this equation, and if you re rearrange this bring all the ds together and then all the epsilons together you get this equation. And in this the first part is exactly equal to 0 because d i n is the exact solution of the governing equation and we have also seen that here. So, d i n plus 1 minus d i n by delta t plus u naught times d i n minus d i minus 1 n by delta x is exactly equal to 0.

So, we substitute that here and we get an equation for error. So, this as we have already said error is building up and error at n plus 1 is now related to the error at the previous time step at the same location and the neighboring spatial locations, as per this particular formula. So, this equation here gives us the error evolution equation how error builds up if this equation has a solution such that the error does not build up, then we are safe, but if this has a solution of the form that error does build up then we have problem here.

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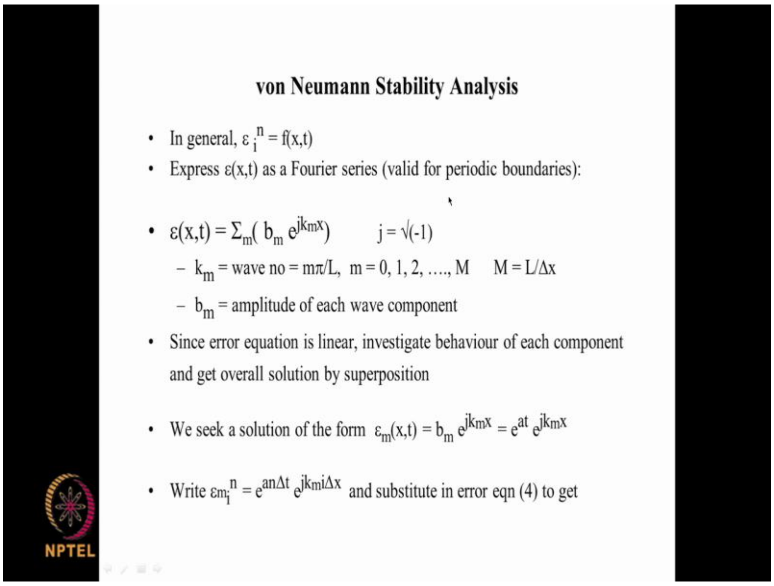

How do we know whether this has the building up property or not? So, what we say is that let us assume that this has the buildup property and if that is the case what is the exponent of that. So, what we are looking at is because we are dealing with the linear equation, all we have we are started with a linearized wave equation 1, dimensional wave equation. So, it is a linear equation and in a case of linear equation we can always superpose different solutions.

So, we make use of that possibility to decompose the spatial distribution of error that is epsilon x, as as a fourier series and because we are dealing with the assuming that we have periodic boundary conditions, it is possible to express the error at any time t as a finite number of wave modes of this particular form here. So, we can substitute we can express the error at any time t, as some over m of b m times exponential of j k m x where j is square root of minus 1. It is a imaginary number here and where k m is the wave number which is given by which takes the values of m pie divided by capital l where goes from 0, 1 two up to capital m where capital l is the domain in the x length and capital m is l divided by delta x where delta x is the grid spacing and b m is amplitude of each wave component.

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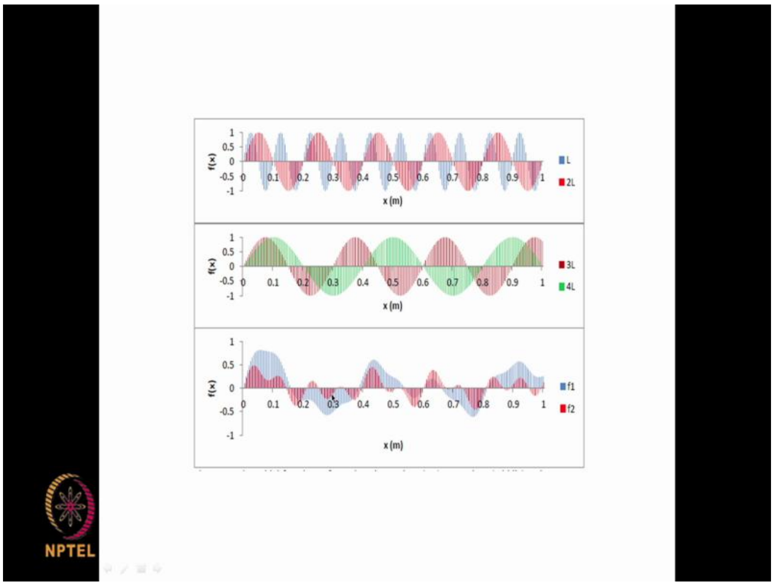

So, let us just try to visualize this for example, we are looking at an x domain here which is going from 0 to 1, and here we have a function f of x here, we see different color things you have this blue color, is a sinusoidal wave like this and it has a certain wave length which is given by roughly 0.1 here and you have a red color thing here which has exactly twice the wave length and it is varying like this. And you have a dark red color here which has even longer wave length which is, 3 times the base wave length here and then you have a green one here, which has four times the basic wave length. So, you have here four sinusoidal functions. Sinusoidal functions of wave length of 0.1, which is the blue color and light red color is wave length of 0.2 and dark red color which is the wave length of 0.3 and green color which is a wave length of 0.4.

So, you can superpose all these things you can for example, multiply the first wave by a constant and second wave by another constant third wave by another constant and forth wave by another constant and make it up as f and if you do that then you can get a variation of f1 which is given by this blue thing here, and you can see that it is somewhat like this, but not exactly like this and by using different coefficients instead of 0.1, 0.2 like that if use a different set of coefficients the same 4 sinusoidal components of wave length of 0.1, 0.2, 0.3, 0.4. If multiplied by some other constant then they can represent this red function here.

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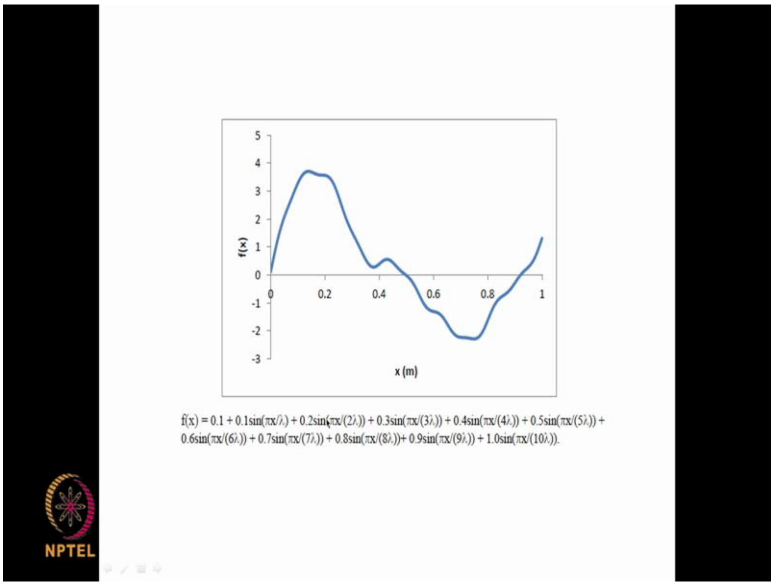

So, and to show this example here this spatial distribution here this spatial variation of f x is made up of ten wave components here. Where you have basic wave length lambda here plus 2 times lambda, 3 times lambda, 4 times lambda, 5 times lambda, 6 times lambda, 7 times lambda, 8 times, 9 times,10 times and each of them has a coefficient like 0.1, 0.2, 0.3, 0.4 like this and with i also put a dc component a constant value here. If I change this to 0, then this whole thing will come down the shape does would not change if I change this 0.1 to something else then this form changes, if I change this 1 to something else this form changes.

So, the same 10 components if added differently with the different proportionality coefficients here can give is to different functional forms and what we are saying is that the error in this space at each of these grid points 0.1 point naught 1 point naught 2 point naught 3. If you put them here they may be like this this error can be represented as the sum as the contribution of different wave components which are there and the contribution of each wave component is given by the coefficient 0.1, 0.2, 0.3 like that and the sum of all these things will give rise to the spatial distribution of error at time equal to t and time equal to t plus 1 at t plus delta t you will have a different error distribution. Because error has evolved even that can be represented as a combination of the same sinusoidal functions, but with different coefficients 0.1 instead of 0.1, 0.2 you may have 0.25, 0.3 which we have seen here example the sum of these four with one set of coefficients will give rise to this b function blue function here, and with another set of coefficient will give rise to this red function here.

So, that is what we are saying here the error at time t at a particular time is no function of space this spatial function of epsilon x is represented as a coefficient times a sinusoidal component, coming from this exponential of j k m where k m is the wave number which is inverse of the wave length which is given by this thing here, and the number of wave lengths wave components which coming to this decomposition Fourier decomposition is given by capital m where capital m is the total domain length divided by the delta x which is a thing here which is the spacing and for a domain the bounded by periodic boundary conditions, we can expand the error function the spatial distribution of error in terms of this finite number of wave components.

Now, since the error equation is linear we can study the error behavior of each wave component and see whether this particular wave amplitude will grow in time or will decline in time. So, we can say that this is this is amplitude like the 0.1, 0.2 that we that we put earlier and this is a sign function involving the lambda here. So, this amplitude if it is growing in time for this particular wave then, it is going to contribute to the growing error if the amplitude decreases with time then, we can say that it is decline it is going to not build up it is going to disappear soon.

So, what we are looking at we are looking at a solution for a particular wave component m small m as b m times exponential of j k m x or since we are looking at whether it is going to increase or decrease this time we express this as exponential of a times t where t is increasing. When you go from t to t plus delta t if a is positive then b m will increase if a is negative then, b m will decrease. So, that is what we are looking at and this is 1 particular wave component, this can be now because we have wave component wave contribution coming from different is I, i plus 1 i minus 1 and different ts n plus 1 n minus 1 and n like that.

We can write a discreet function of this as the error resulting from the nth wave component at ith space location. And nth time step as exponential of a times. Where a is the constant to be determined n where n here delta t and exponential of j k m i delta x. So, wherever you have x you substitute it as i delta x wherever you have t you substitute as n delta t here, and we substitute this in the error evolution equation here. So, let us see what we have.

If you substitute if you substitute these into this equation here and divide by the error at epsilon i n, that we are normalizing it then we get we get epsilon i n plus 1 divided by epsilon i. And what is epsilon i n plus 1 divided by epsilon i as per this? This is equal to exponential of a n plus 1 delta t exponential of j k m i delta x divided by exponential of a n delta t times exponential of j k m i delta x. So, this cancels out that gives us exponential of a n plus 1 delta t divided by exponential of a n delta t that is nothing, but exponential of a delta t. So, with that thing we can once we substitute we get an expression like this.

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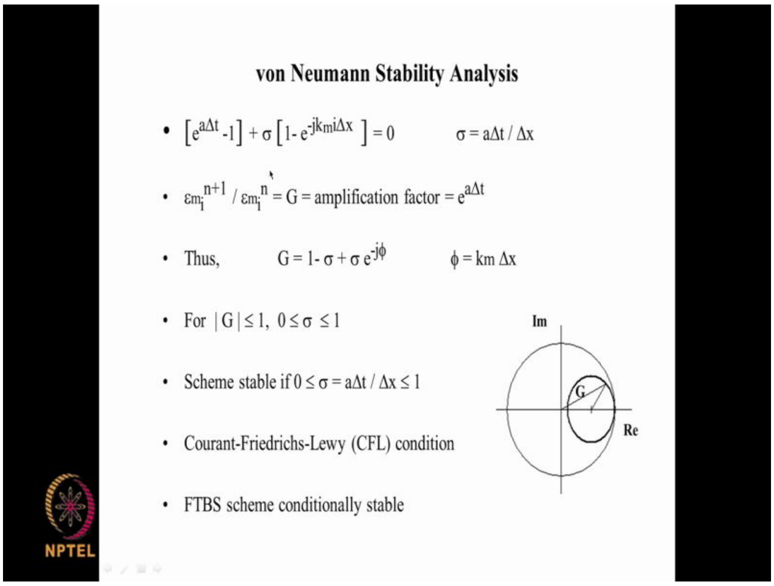

This is coming from the time derivative terms and this is coming from the space derivative dou u by dou x and this is coming from dou u by dou t and this sigma is a number a delta t by delta x.

So, this is the after substitution of this assumed wave component variation with time and space in this particular wave and substitution of this into this and simplification will give us an equation like this and we know that epsilon m i n plus 1 divided by epsilon m i n is the amplification factor for this wave component. And that is given by exponential of a delta t and therefore, we can write this as g minus 1 plus sigma times 1 minus this whole thing equal to zero or j g equal to 1 minus sigma plus sigma times exponential of minus j phi. Phi here is just nothing, but k m delta x and depending on the value of k m where k m as we have seen here as per the decomposition for different wave numbers this k m, can go from 0, 1, 2, 3 up to capital m here. So, each of them will correspond to an angle here in this imaginary versus real plot of this amplification factor in general, this amplification factor can be imaginary. So, when you put it in this particular form then you will get a variation like this.

So, for a given value of sigma and for phi going from 0 up to this you can get a function for g a value for g, and if that value here is such that the magnitude of this amplification factor is less than 1. Then you have stability if it is less greater than 1, you have instability now we can see that for the modulus of g to be less than 1 sigma has to be less than 1 from this. So, this function will this thing here is the unit circle and this is a circle which is made by this function g here and this circle which is centered at 1 minus sigma value with this as the radius ok.

So, if sigma as long as sigma is less than or equal to 1 then g will be less than or equal to 1 if sigma is greater than 1 then this value will be the center of this will be lying outside the imaginary circles. So, this will have at least for some values of g there will be g will be greater than 1. So, and; that means, that there are some wave components which will become unstable and they will contribute to a growing wave. So, if there is a decline wave and a growing wave the decline wave will go to zero very quickly errors coming from that, but the growing wave will amplify the errors and that will dominate the solution and therefore, they whether or not a dominates the solution whether or not the amplitude of a particular wave component goes up or not is given by this amplification factor and which we see depends on sigma which is a delta t by delta x.

So, choosing the appropriate values of delta t and delta x such that your sigma is less than 1 will ensure that you will get a stable solution and that is what the stability analysis and this particular condition is known as CFL condition or Courant Friedrichs Lewy condition for the 1 dimensional wave equation and this is the famous CFL condition first successful analysis of condition for stable scheme is essentially captured by this. So, from this we can say that FTBS scheme is conditionally stable. Conditionally stable in the sense that only when sigma is less than 1, given that sigma is a delta t by delta x and we are assuming the wave speed to be positive quantity. So, this delta t and delta sigma is always positive.

So, if sigma is between 0 and 1 then this is stable if it is greater than,1 this may become unstable and that is what we have seen for the FTBS scheme we saw that for sigma of 1 0.11 it has becoming unstable, but for sigma of 1 or 0.5 and 0.2 five it is not exploding it is giving some shape. So, in that sense FTBS scheme is conditionally stable as demonstrated by this von Neumann stability analysis. In the next lecture we will explore other schemes using the same von Neumann stability analysis both to get an understanding of what this stability is how to apply this stability analysis scheme and also what it implies.