Computational Fluid Dynamics
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Module – 03
Lecture – 28
Need for the analysis of discretized equation

We are starting part b of the module 3 today, and in this part b we are going to look at the analysis of discretized equations. And the idea being that as shown in the last case study, not every find a difference approximation for every derivative gives us a solution which is satisfactory. So, we would like to see under what conditions we will get a proper solution, and how to solve these equations properly. This analysis is essentially in two parts.

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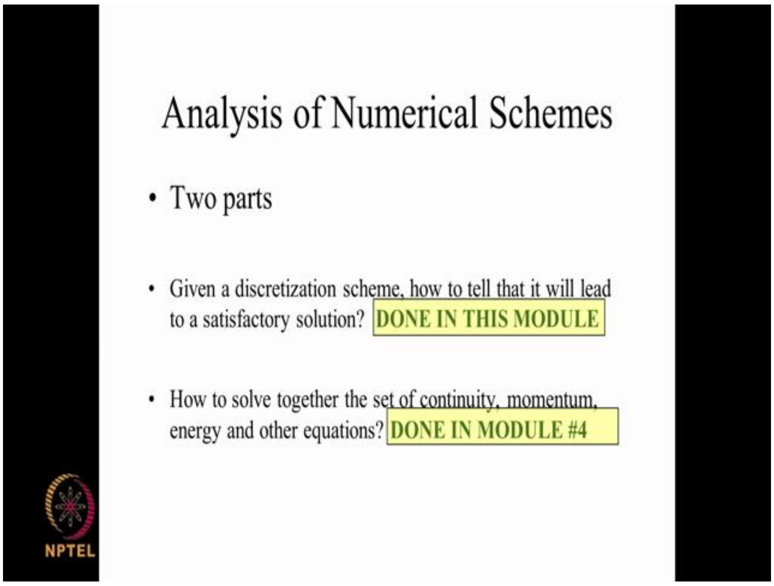

In the first part we have we try to answer the question given a discretization scheme how to tell that this scheme consisting of the specific approximations that we make further derivatives that appear in the governing equation. That particular scheme will lead us to a satisfactory solution. So, this is the question that we address in this module, but there is also the analysis of the numerical schemes answering a different question that is how to solve together the set of continuity momentum energy equation, and other equations and that is going to be done in the next module. So, that is module 4.

So, as far as this part b of the module is concerned, we are going to look at one equation and it is like the scalar transport equation and in fact, the elements of those equations to come up with the methodology for analyzing a given discretization scheme and telling a priori whether it is going to give us a resample satisfactory solution. So, that is what we are going to do in part b of this basic concepts of CFD in part a what we have seen is that we can we have seen the deriving derivation of the finite difference approximations, for any derivative to any order of accuracy that is decide and as part of this derivation.

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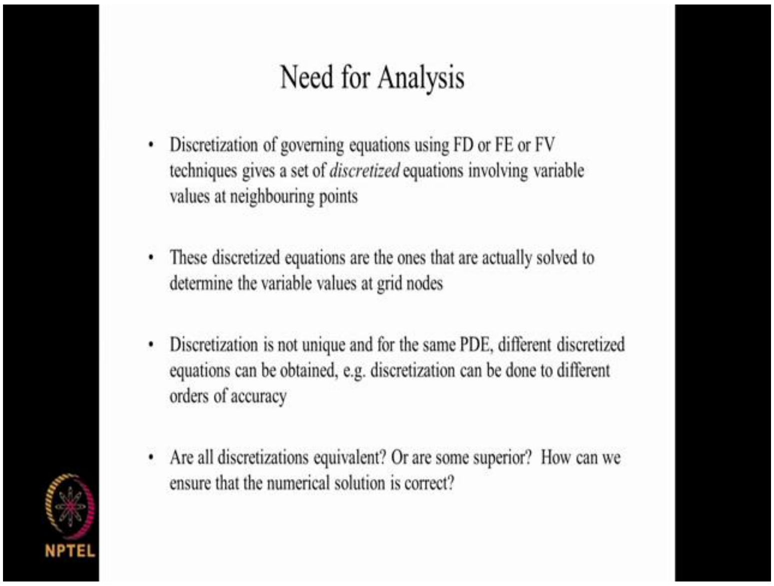

We have seen that for a given derivative there are number of approximations are possible and in when you have number of such derivatives that appear they can be number of possible schemes by which the overall partial differential equation, can be converted to an algebraic equation at a particular space and time location. This can be applied for all the points that which you want to have a solution and we can have a set of equations which will then be use to find the solution the questions that we wanted to answer were that are all the discretizations equivalent, and other some discretizations that are superior and how can we tell that we will have a numerical solution which is correct.

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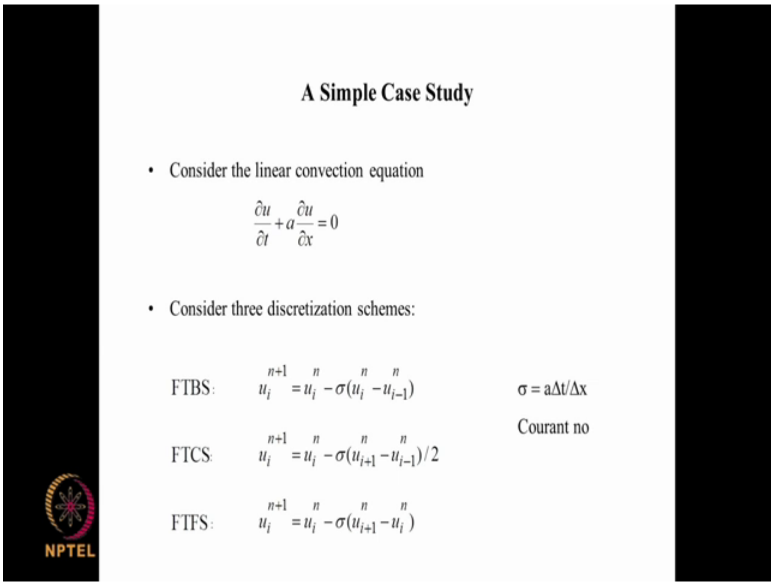

And we try to answer this questions by taking a simple case of linear convection equation or the linear wave equation dou u by dou t plus a dou u by dou x equal to 0 and we consider three explicit different schemes the forward in time backward in space forward in time central in space and forward in time forward in space and we expressed the discretized equation in the as a function of the parameter courant number sigma which is given by a delta t by delta x and since these are all explicit methods we can match forward in time and space.

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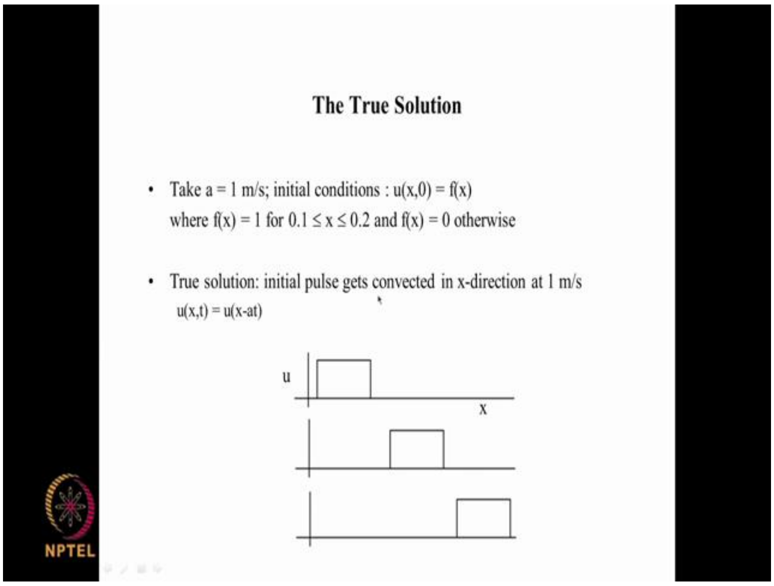

And what we are expecting was that for a given initial condition where f x equal to one over an interval from x equal to point one to point two and 0 everywhere else, we are expecting that this rectangular pulse to be carried forward in the x direction as time progresses at a speed of one meter per second.

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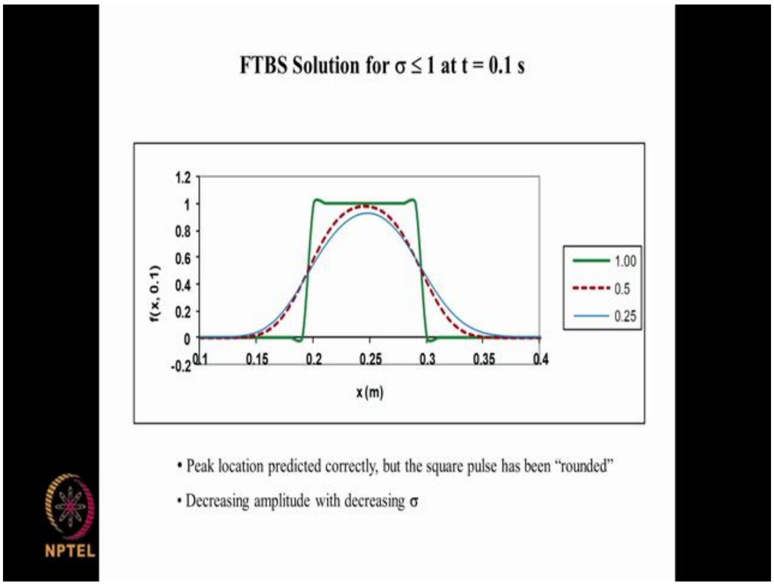

And then we compared the solution at different time at a point at a fixed time of 0.1 second using different schemes like the FTBS, FTFS and FTCS and using different values of courant number. So, that is for a given delta x we simply varied the delta t. And we were expecting certain trend to be reflected in the solution for example, if you take a specific scheme like FTBS and then if we were to reduce the courant number that is reduce the delta t we were expecting more accurate solution.

We are not really getting that, and if we were to increase delta t courant number we are expecting again probably because of larger accuracy larger discretization error as delta t increases, we are expecting slightly worse result than what would be getting with certain value of courant number.

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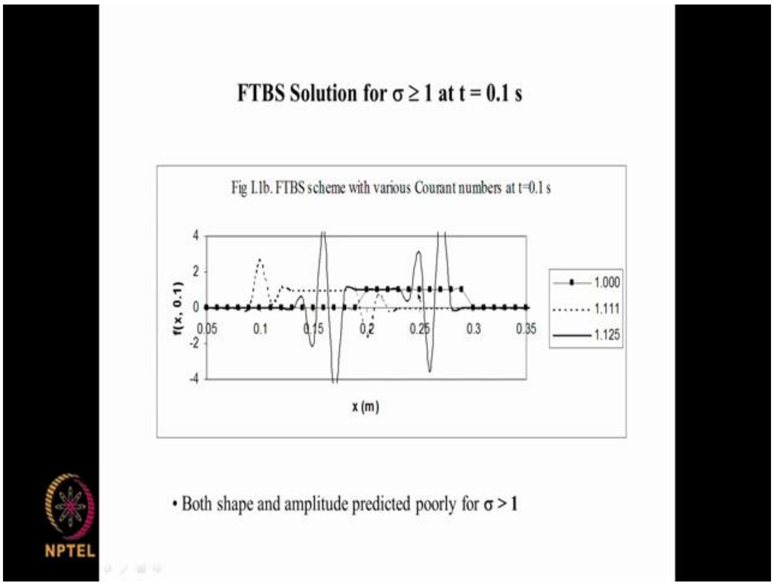

And we are finding that when courant goes beyond the certain limit, it seems to be giving us very odd unacceptable results unstable results.

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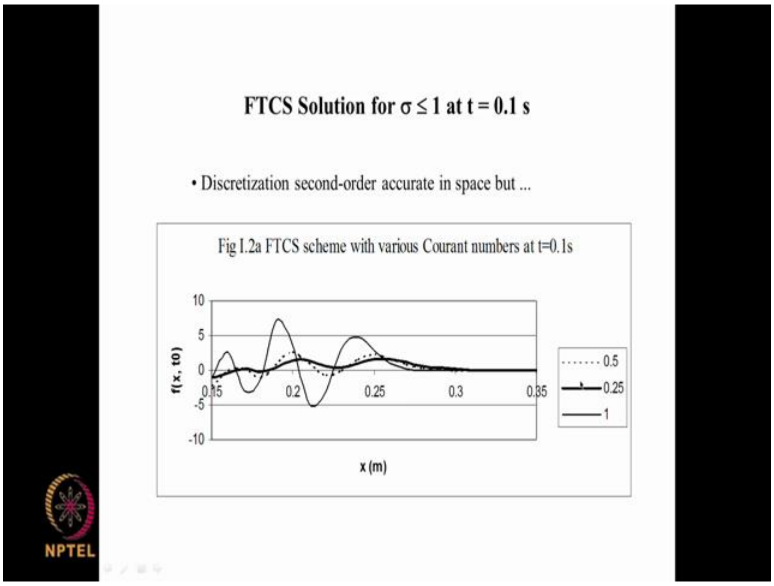

And we find that FTCS scheme, which is supposed to be more accurate at least for the space derivative than the backward in space scheme was actually giving very wrong results even for the courant numbers in which the FTBS scheme give somewhat, good results acceptable results and for the FTFS scheme.

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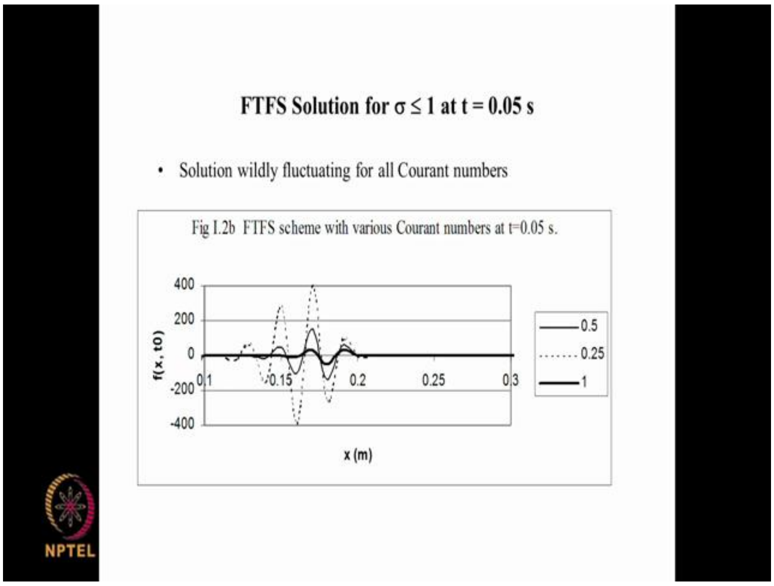

We found that although it is formally of the same order of accuracy in terms of accuracy of the same order as the FTBS scheme, it is giving no good solution for any value of courant number this is the kind of specificity of the solution sensitivity of the solution. To the scheme that we use and to the value of delta t and delta x that we employed, that is surprising and it is also in a way intimidate in because we do not know which ones to choose we how do we know that we have chosen the right kind of values and how do we how can we trust the solution when it is exhibiting this sort of erratic behavior. So, that is essentially.

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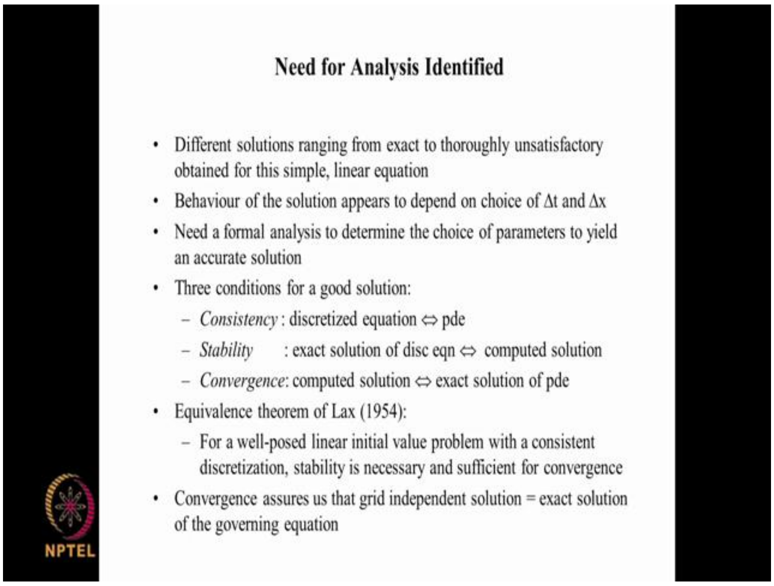

Why we need to have an analysis of a given discretization scheme, as to see that it would actually give us a proper solution.

So, here when we look at the analysis we notice that are solution CFD solution is done in stages, that is we first convert the partial differential equation to a discretized equation and then we solve the discretized equation we do not solve the partial differential equation. So, the solution is obtained in two steps; one is the converting the pde into a discretized algebraic equation and then solving the set of algebraic equations in order to get a solution. So, if you want convergence.

So, where we define convergence as the computed solution must be the same as the exact solution in the limiting case. Why in the limiting case? Because we know that we are making approximations of the derivatives. These approximations involve truncation errors of the order of delta x or delta x square or delta x cube like that and similarly for delta t. So, in the limiting case if the delta t and delta x go to 0, that is for very fine grids we are expecting we can expect convergence, between the computed solution and the exact solution of the partial differential equation.

So, in order to get convergence we would have to have consistency which is that the discretized equation that we are solving that we are approximating the partial differential equation. So, that must be mirror reflection or it must be the same as the partial differential equation in the limiting case that must be one condition. So, that we can say that we are solving the correct equation and not some approximate form which leads to a different kind of approximation than, what is there in the partial differential equation and the. Secondly, that the discretized equation that we are solving must reach the exact solution of the discretized equation and not some approximation or a solution which is not intended to be there which is the which is known as a stability condition.

So, that is that in order to get convergence of the convergence between the CFD solution and the exact solution of the partial differential equation. We need to ensure that we have consistency which is the discretized equation, which we are solving is the same as the partial differential equation which we are suppose to solve in the limiting case, of delta t and delta x tending to 0. Secondly, the discretized equation that we are solving using certain a time stepping using certain delta x stepping in the process of doing this discretized solution which is a computed solution which has finite arithmetic in the process of doing this, we do not expect the errors from errors to build up through the successive mathematical operations that we undertake in order to get to the solution.

So, there is a possibility of build up when you do repeat mathematical operations like addition, subtraction, multiplication and division. And if you do many of those and if at every step because you have finite set of arithmetic so; that means, that when you add two plus two it is not exactly equal to 4; it may be 3.999 something there is a small difference that may be there. So, the difference in every mathematical operation that you carry out we would like them not to add up and to build up.

So, as to compromise the computed solution, that is a stability aspect. So, if you have consistency and if you have stability and if you have a linear equation there is an equivalence theorem of lax proposed by a lax which is that for a well posed linear initial value problem with a consistent discretization stability is the necessary and sufficient condition for convergence. So, this is a very useful theorem for us to have. So, that convergence can be achieved for a certain mathematical problem, which is that it is a linear initial value problem with well post boundary conditions.

So, well post mathematical problem if you have well post mathematical problem and if you have a discretization which is consistent in the sense that the truncation error between the involved in the discretization goes to 0. As the delta t and delta x and other terms go to 0 other parameters goes to 0 and if we also have stability that is in the sense that errors from whatever source that we are that are present either from the boundary conditions or from a or from approximations made in different derivatives of the equation or in the round of errors all these errors do not build up. So, as to give rise to a bad solution, so, if that stability of the competition is guaranteed then we can expect to get a convergence of the computed solution towards the exact solution in the limiting case of delta t and delta x tending to 0.

So, that is if we keep on finding a better and better solution a newer and newer solution for delta t and delta x becoming small and smaller. So, that is you take a certain delta a grid computation domain of say length l. Then you divide into 10 parts and you get a solution you will get a time dependent solution and you use a delta t of say one second and then you match forward for 10 seconds and you get a solution at the end of ten seconds over this domain between 0 and l now you do the same thing that is the solution at the end of 10 seconds. Now with 20 grids in the same length of between 0 and l and you use 20 times step to get there, that your delta t is smaller and delta x is smaller. Now you get a different solution and then instead of now using 20 cells may be you use 50 cells and may be you use 100 time steps to reach there you get a solution. So, as you keep on decreasing the grid spacing in both space delta x and time delta t, you will be getting a solution with which has which supposedly has lesser and lesser truncation error now if the scheme is stable. Then we expect to get a solution which is better and better in such a way that for very fine delta x and very fine very small delta t we can expect our computed solution to match with the exact solutions of partial differential equation.

So, that is what we mean by convergence that when we have very small delta x and very small delta t if convergence is guaranteed then we know that we are getting a solution which would match with the exact solution. So, convergence assures us that a grid independent solution, that is a solution which does not effectively change with a further decrease in delta x, because we have probably reached the limit of variation there in terms of the gradients that are present and in terms of the accuracy of the computer and all that. So, in such a case we can expect the exact solution to match with the converge solution, but it is not necessary that the solution that we are testing is equally sensitive to all changes in delta x and delta t.

So, for example, if we solve for the temperature variation and deduce from that a heat transfer co-efficient were heat transfer co-efficient is obtained from the gradient at the boundaries. So, the gradient at the boundaries may be either less sensitive or more sensitive than the temperature at a particular location within the domain. So, we have to be clear about this that we are solving for example, in a heat transfer problem we are solving for temperature and from that we are deducing the heat transfer coefficient.

The heat transfer coefficient is determined by the gradients at the boundaries, and temperature is determine by the corresponding temperature equation energy balance equation and it has many more terms and it has a variation which is different at different points within the domain. For example, if you are monitoring the maximum temperature in the domain it may be less sensitive than the gradient at a particular location, at a particular location at the wall and if you are looking at the average gradient so that you can get the average heat transfer coefficient. The average heat transfer coefficient is going to be less sensitive than the gradient at a particular location.

So, in that sense if you have a monitoring point at which you look at how the solution is evolving, then them the solution at the monitoring point may be more sensitive or less sensitive than the final solution that we would like to have. For example, in a in a case of flow through a duct then we specify a pressure gradient and we try to get the average velocity or you specify an average velocity we expect to get the mean maximum velocity and the pressure gradient. So, if you are looking at these bulk quantities like what is an overall pressure gradient for a given flow rate or what is a overall flow rate for a given pressure gradient or what is a overall heat transfer coefficient for a given heat flux or flow rate or for given temperature boundary conditions.

So, we are talking about overall values. So, these overall values do not exhibit the same sensitivity as a solution as part of the solution for example, temperature at a particular point or the velocity at a particular point or the maximum velocity or the maximum gradient. So, these things are going to be different. So, we have to keep this in mind when we talk about convergence, because when we talk about convergence and gradient dependence.

We are looking at some parameter may be the average heat transfer coefficient or gradient at a particular location or the maximum value of the velocity and we are seeing whether this this parameter this value here is going to change as i decrease my delta x and decrease my delta t. When we find that if you further and further decrease the delta x this value is not changing, then we say that we have convergence we have grid independent solution. Now weather the grid independent solution is convergent with the exact solution as partial differential equation is something that depends on the consistency and stability and it also depends on whether the solution that you are comparing is the same as the problem that you are solving.

For example, if you say that you are looking at the maximum velocity, the velocity at a particular point. That point is obtained from the partial differential equation, but if you say that you are looking at a gradient at a particular wall location. So, then that is not necessarily given by the partial differential equation it is given partly by the partial differential equation and partly by the boundary condition.

So, the errors in the way that you use the boundary conditions and errors in the way that you are discretisizing partial differential equation, they both come in to picture and it is slightly different. So, these complications are there when you are doing a practical problem, but let us just say from a purely conceptual point of view that if you can demonstrate that the equation that you are solving is an exact match of the partial differential equation. And if you can show that your numerical solution of the discretized equation is stable to small perturbations, small errors that may be coming from different values of different terms at different points of solution. If we you show that stable and it is errors do not build up perturbations do not build up and give rise to instability then you can hope to get a convergence converge solution in the limiting case at delta t and delta x tending to 0.

So, this is the way that we can ensure that we have discretized scheme involving set an approximations for the all the derivatives that that are present in an equation that discretization scheme is going to give us a recently accurate solution. So, we are looking at we are saying that in order to for example, in order to get a proper solution with the FTBS scheme or FTCS scheme or FTFS scheme we need to have two conditions. Firstly, that the FTBS scheme that we have here this particular approximation this particular way of getting u i n plus 1 is such that this scheme approaches the exact solution in the limiting case because at this stage we only have a proposal for a scheme as to how you can get u i at n plus 1. We have not actually implemented it. So, we are claiming that at this stage that these two differ only by the truncation error. So, if the error between the equation that you are solving and the equation that you are suppose to solve goes to 0 as delta t and delta x tend to 0. Then you can claim consistency condition that we have a consistence discretization.

So, if the discretization of this equation is consistent. So, for example, if FTBS equation is a discretization scheme is consistence then; that means, that the difference between this equation and this equation which is essentially arising from the truncation errors, from the approximations that you have made in in taking the first 2 terms in the taylor series expansion of this this gradient and similarly the truncation error that you made in this. So, the the sum of all these truncation errors the cumulative effect of all this truncation errors, is such that it tends to 0 as delta t and delta x tend to 0.

So, once you can make that claim you can say that you have a consistent discretization. So, the first question that we would like to answer is the FTBS scheme consistent similarly is the FTCS scheme consistent and if the FTFS scheme consistent, if one of these schemes if a specific scheme is consistent then the next question that will have to answer is that is the solution stable. So, that is we are starting with some n equal to 0 values some initial condition here, and then we are getting u i at n plus one at a particular point and then we are skipping to the next space point and the next. So, we match from point to point and then we go up to all the points at a particular time step level and then we move on so; that means, that we have a solution at u i at n plus one for example, at the ninth point and at a tenth time step which depends on the previous time step values and the previous time step values depend on the previous time step values and so on.

So; that means, that in the process of coming to the tenth spatial point, ninth spatial point at the tenth spatial time step you have build up you have accumulated the error that you made at the previous time step and at the neighboring locations, and then at the previous time step of those things at the eighth time step of the ninth grid point and then the tenth grid point and eighth point grid point like that. So, all those errors that you have made at previous time steps have they died down or are they accumulate are they accumulate in to such an extent that these perturbations will build up and make a solution unstable. So, that is the stability part that we would like to demonstrate. So, if we can show if we can demonstrate that the FTBS scheme is both consistent and stable then we can expect that the solution that we get by following through with this FTBS scheme is going to be a convergent solution.

Similarly, with the FTCS scheme and FTFS scheme or with any other we have approximating the derivatives because these are new three different ways they can be a second order of approximation the time derivative, they can be a third order accurate approximation in the space derivative they can be implicit they can be explicit they can be many, many combinations for any combination that we make, for any numerical scheme that we propose for example, this or this or this to find the solution we need to satisfy the consistency condition and the stability condition.

If we do that then, we can hope to get a convergent solution. So, in the next lecture we are going to look at how to verify the consistency of a given discretization scheme and we can we will also see how to verify the stability of a given discretization scheme. If we can come up with some generic methods for doing this analysis then we will be able to do this analysis for a given discretization scheme. See whether it is satisfies both the conditions, if it is satisfies both the conditions then, we can go ahead and get solution if it does not satisfy either of these conditions then we have no guarantee that the computed solution is going to be good, will say we will say that we will not use this scheme will give some other method to do this. So, that is an idea of this analysis, which is what we going to do in the next couple of lectures.