Computational Fluid Dynamics
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Lecture – 27
FD approx. on a non-uniform mesh and need of analysis of obtained discretization

Before we end the discussion on finite difference methods, let us consider the case of non-uniform meshes.

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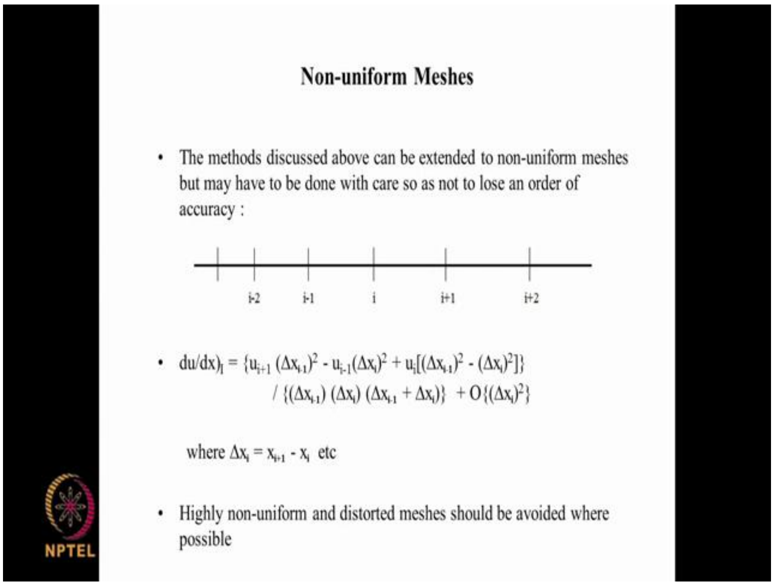

When we say non-uniform mesh, the spacing between adjacent and nodes is not the same as illustrated in this figure. We have this is a case of one-dimensional grid, where this is the example in the x direction and the nodes are located at these points. Here, this is the ith node, this is the i plus 1 node, i plus 2, i minus 1 and i minus 2 like this and you can see that the spacing between these 2 is different from between these 2 and between these 2 like this all right is shown to be gradually increasing and almost by a proportion like a geometric progression, it is not necessary that it should increase in in a specific way that we do use this kind of geometric progression of the grid spacing, for example, in regions of large gradients like in boundary layers and so on, but agreed with a spacing which is changing is a non uniform mesh.

So, for such a case we can define a delta x, but the delta x is not the same. So, we can associate the delta x of node i. We can define it as delta x i as x i plus 1 minus x i. So, delta x i here is this distance, you could also say that is the average of this from the midpoint of this to midpoint of this, but this is one definition and with this definition you can go back to the earlier terra series expansions and then still write for example, u of x, x u at i plus 1 as u at i plus dou u by dou y at i times delta x i plus dou square u by dou x square times delta x i square and so on like that.

So, when we are expanding u i plus 1 in terms of u i, then we get delta x i and if you want to derive a central difference formula, you also need u i minus 1. So, when we expand this in terms of terra series. Now, u i minus 1 is given as u i minus delta x i minus 1 which is this distance delta x i minus 1 is x i minus x i minus 1. So, that is dou u by dou x at i times delta x i minus 1 plus dou square u by dou x square at i times delta x i minus 1 whole square by factorial 2 and so on. So, you can do this kind of expansions. Now, if you have uniform grid spacing when you add this expansion and this expansion this term will cancel out for the first derivative, when we are deriving the central difference formula, but in this particular case because delta x i is different from delta x i minus 1 they do not cancel out.

So, in order to derive a proper formula we have to multiply 1 by the expansion of this by delta x i minus 1 and this by delta x i, and then do further algebraic to cancel out those second derivative terms in order to derive a second order approximation. So, there is more algebra more manipulation that is required, but essentially the same principle applies that we had used for the derivation of a the central difference formula higher order accuracy accurate formulas and slightly more complicated formulas arise, for example, d u by d x at i is now expressed in the simplest case, it is u i plus 1 minus 2 u i plus u i minus 1 by delta x square.

But in this particular case its u i plus 1 times delta x i minus 1 whole square minus u i minus 1 times delta x whole square, you can see that we had to multiply the expansion of this with delta x i minus 1 and multiply this by delta x i and then do further manipulations that is how this is coming here, plus u i times delta x i minus 1 minus delta x i whole square divided by delta x i minus 1 times delta x i times delta x i plus 1 plus delta x i and this whole thing is formally second order accurate with respect to delta x i.

So, let us see whether its stands a simple test which is at if delta x i minus 1 is equal to delta x i then we should be recovering the earlier formula. So, in such a case this is delta x square delta x square and this becomes 0. So, you have u i plus 1 minus u i minus 1 times delta x square divided by delta x delta x. So, these 2 will cancel this out and then you will have 2 delta x. So, that gives us u i plus 1 minus u i minus 1 by 2 delta x and that is a second order accurate formula.

So, in that sense this tends to the simple formula for the case of equal delta x, but for unequal delta x you have to have more complicated formulas. So, it is pretty straight forward to derive a second order accurate formulas for the first derivate and second derivatives that we encounter in our typical fluid flow problems in terms of Navier-Stokes equations energy balance equation and so on, and so it is not very much more difficult to deal with non uniform meshes, but we would like to make a point that highly non uniform and distorted meshes should be avoided where possible because this expression here is formally second order accurate, but let us say that delta x i is much greater than delta x i minus 1, in which case this term will be very small. So, it is negligible and this term will be very small negligible, and it boils down to first order accurate formula.

So, although it is formally second order accurate, if you have large changes in delta x then you could it could effectively function as if it is a first order accurate formula and not a second order, which is why we would like to make sure that sudden changes in delta x are should be avoided and we make use of a geometric progression typically with an expansion ratio of the order of between 0.7 and 1.4, 1 by 0.7 is about 0.4, 1.4. So, it is of that kind of a range that we can use and we should also keep in mind the geometric progression builds up very fast. So, if we use a geometric progression with an expansion ratio of something like 1.4 for 50 grid points then the delta x variation can be very, very large between the smallest and the largest.

So, you need to really see whether you would like to have that kind of variation and accordingly adjust with geometric progression based on the number of grid points, you want to put and the domain length and what should be the grid size in the center and what should be the grid size close to the wall where there may be gradience and so we have to take care when we make the mesh.

So, non-uniform meshes are probably unavoidable, up to second order accurate expressions they do not pose real great difficulty, as you go to higher and higher order accurate then the algebraic becomes a tougher and tougher. So, and we would like to point out that sudden changes in grid spacing are should be avoided where possible. So, with this we have covered the finite difference approximations for derivatives and we also applied it to a couple of example problems, and we have also seen time discretization.

In this lecture, we will do a tutorial problem, but not on the board, but in the computer and this will also be an assignment for you, so that you can work out on your own and the idea of this exercise is to see for ourselves, why it is not a trivial thing to solve Navier-Stokes type of equations? We have seen that using finite difference approximations it is possible to reduce a governing equation into an algebraic equation. In the case of explicit schemes, it is possible to march forward from grid point to grid point and get a solution very, very easily, but is it as simple as it, that is what we are going to do in in this example.

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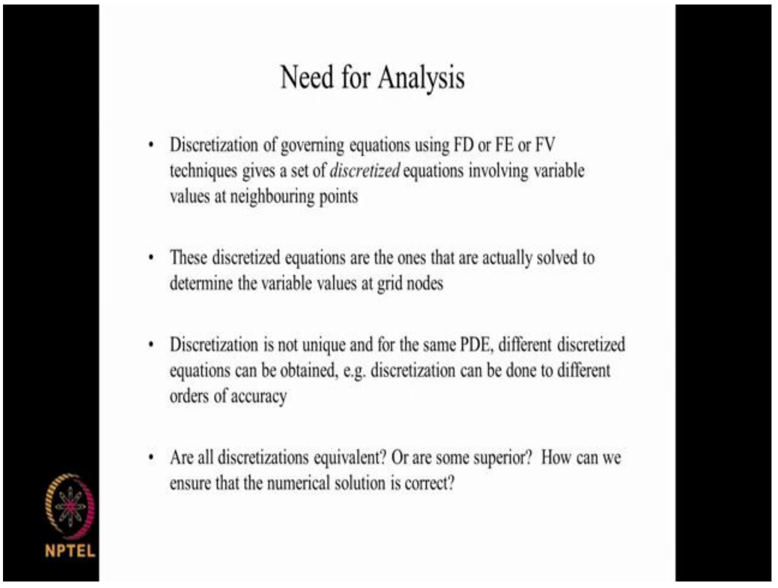

So, the objective of this test is that we know that for a given partial differential equation, a number of discretizations are possible. Now, the question is that are all these discretizations equal equivalent, obviously, they are not in terms because there are some with higher order of accuracy and some with lower order of accuracy, but we also know that order of accuracy is something which is only vague, it is not something definite we cannot say that this will have so much error because what we mean by order of accuracy is the magnitude of the leading term of the truncated series and we have also seen that most of the first order, second order schemes will neglect lots and lots of terms and only the first few terms are taken into account.

So, there is no guarantee that the series starts converging within the first few terms. So, for large grid space definitely the order of accuracy is not a good reflection of the accuracy of the solution. So, what is small is something that depends on the solution that depends on the problem and where we are looking at within the problem and so on. So, we would like to keep those notions in the mind, but we cannot say that this particular simulation has this much of error because this is first order accurate and second order accurate that first order, second order only gives us an indication of how rapidly the error would decrease by grid refinement in the limiting case of very small delta x, but it is not really applicable for course grids.

So, with this particular thing we would say that we have seen a number of discretizations possible for the same equation, and we would like to see whether all of them would give us nearly the same solution or some schemes, some sets of combinations of approximations will give much better solution than some others.

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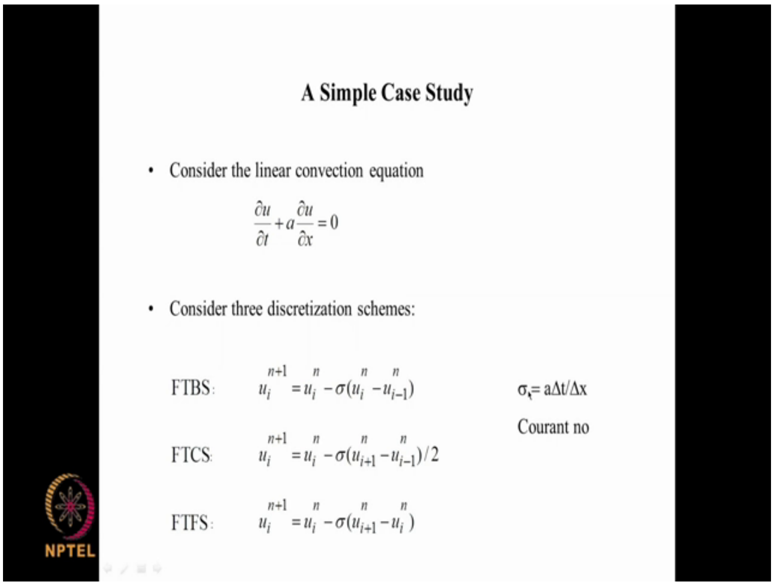

So, for this we take a very simple example, a case study and the case study is that of a linear convection equation which is given by dou u by dou t plus a dou u by dou x equal to 0. Now, this is 1 part of a simple advection diffusion equation or a scalar transport equation and so this can be considered as the left hand side of one-dimensional example momentum equation with a which is linearized in the sense that in this equation, u is a variable and a is considered as a given constant and this is a partial differential equation first order partial differential equation and we will see some initial conditions and boundary conditions for this and once we have an equation like this then we can readily put in number of discretizations. So, we are considering three discretizations which will enable us to make a solution and then these discretizations are somewhat similar, but they are also different and we would like to see whether all the three will give us the same solution.

So, for this we have the first discretization is what we call as forward in time and backward in space, FTBS scheme. So, time derivative is discretized as a forward in time. So, that is u i n plus 1 minus u i n divided by delta t and the space derivative dou u by dou x is put in the form of backward difference. So, that is u i minus u i minus 1 divided by delta x equal to 0. So, you have u i n plus 1 minus u i n divided by delta t plus a u i minus u i minus 1 by delta x equal to 0 and that can be rearranged to give a straight forward expression like this u i n plus 1 equal to u i n, which is coming from the time derivative minus we take this to the other side. So, we get minus here and the sigma is what is known as a courant number after a famous mathematician who brought out the courant number criterion stability and this is a times delta t when we take this to the other side it becomes delta t in the numerator divided by delta x which is coming from this, minus sigma times u i n minus u i minus 1 n.

And we see immediately that this is an explicit method because on the left hand side we have the variable that we want to compute on the right hand side, all the variables are appearing at the previous time level this is an explicit method and we can also see the u i and u i minus 1, which makes it backward in space and u i n plus 1 and u i n will make it forward in time. So, in that sense this is a very simple prescription and from a given initial condition, we can march forward in space and time with this. So, this is one possibility and we know that we are making this as forward difference. So, it is first order accurate and here also this is backward difference involving only 2 points. So, this is also first order accurate. So, we have this approximation here is first order accurate in time and space. So, that is what we have in FTBS.

In FTCS, we have the same forward in time because forward in time with explicit method will enable us to get a quick solution, and at this stage we do not we want to test with simple expressions. So, you have the same u i n plus 1 minus u i n, but the space derivative is expressed in terms of central differences. So, you have this as u i plus 1 minus u i minus 1 by delta 2 delta x and we will also make it explicit. So, we get u i plus 1 minus u i minus 1 because there is a 2 delta x have here is a 2 delta x is consumed because there and the delta x is consumed by this courant number sigma here.

And other possibility that we consider we can consider many, many possibilities, but let us consider another thing in which instead of using a backward differencing, we use forward differencing even for space. So, this will be given as u i plus 1 minus u i divided by delta x. So, that will give us u i n plus 1 equal to u u i n minus sigma u i plus 1 minus u i n. So, these two are same except that we are taking the difference between u i and u i n minus 1 in backward spacing, and here its u i plus 1 minus u i forward differencing here and both FTBS and FTFS are first order accurate in time and first order accurate in space, whereas FTCS is first order accurate in time and second order accurate in space.

If you look at the way the solution evolves that is how for a given set of initial conditions, how the u i n plus 1 changes with time, how the special variation changes with time is governed entirely by obviously, by the initial condition and the courant number here and what is this courant number it is the velocity a linear convection a we will come to that and delta t by delta x. So, the delta t is the time step that we choose and delta x is the time step is a space increment that we would have chosen here.

So, what we do is that we can do in any kind of programming platform or even in excel or even by hand, you can do this and. So, we have to fix a value of delta x and we have to fix a value of delta t and we have to fix the value of a here. So, for the sake of getting a numerical solution, we put a to be 1 meter per second and delta x to be 0.01 and so we take a to be 1 meter per second and delta x is 0.1.

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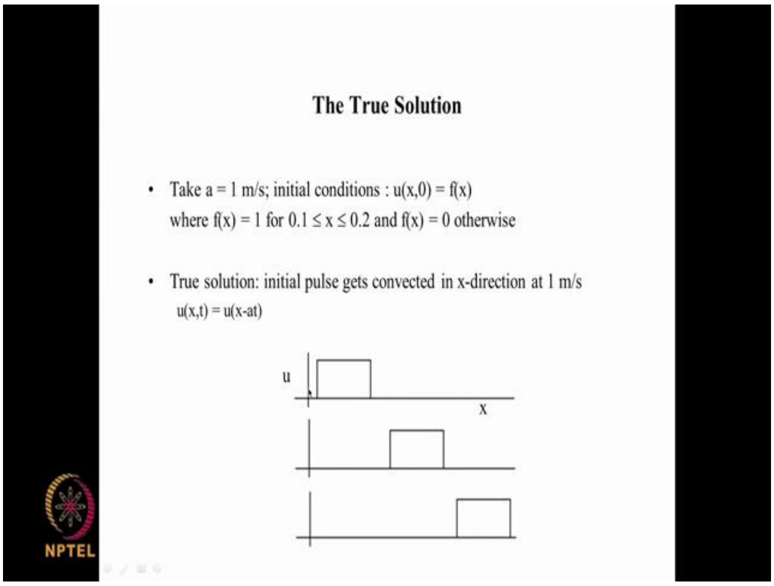

And u of x 0 so that is the initial condition is given as a spatial distribution and the spatial distribution is a simple rectangular pulse where its equal to 1 over the interval of 0.1 to 0.2 and 0, everywhere else the true solution to this is this pulse would get transported in the positive x direction at a speed of 1 meter per second. So, you can actually verify that by substituting solution like this that is u of x t equal to u of x minus a t is a solution for this par this partial differential equation so that means, that if you have an initial pulse like this this is traveling in the x direction at a speed of 1 meter per second.

So, after say 1 second it would have traversed 1 meter here and after 1.5 meter seconds, it would have traversed the center of this would have moved by 1.5 meters per second, otherwise the shape would be the same if you had initially a triangular pulse like this it would remain triangular it would remain triangular here. So, what would be looking from our solution from the three different cases is that is the shape being maintained and is the pulse moving at the speed that is expected which is 1 meter per second. So, that is what we are looking for and we would like to see it to what extent the three methods that we have put up which do not seem to have any problem within themselves those seem to be quite good and straight forward things to what extent they would simulate reproduce this expected behavior.

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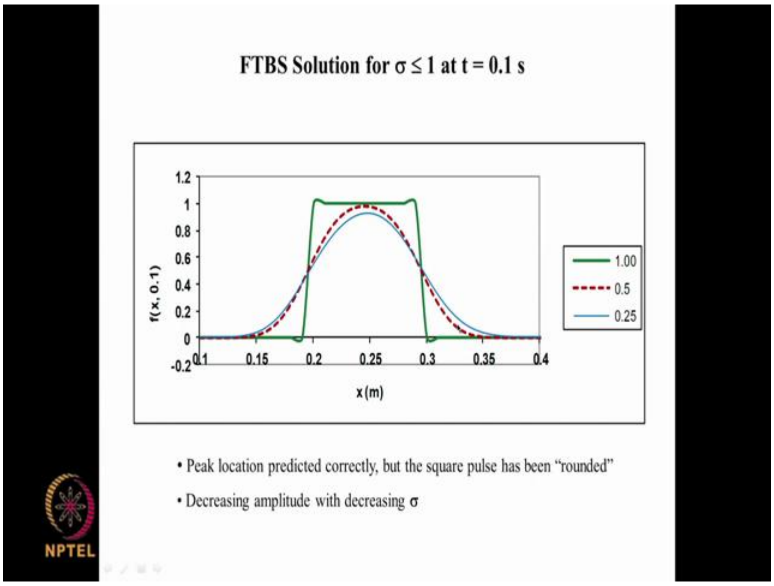

So, this is the solution that we are getting from FTBS at a time of point 1 second and we have three different solutions here, for three different courant numbers. So, the when we say three different courant numbers here courant number is a delta t by delta x a is fixed as 1 meter per second delta x is fixed as 0.01 meters. So, delta t is given by the courant number. So, if you fix courant number to be 1 delta t will be equal to 0.01. If you fix it to be 0.05 then the delta t will be 0.05 and for a courant number of 0.25 the delta t will be even smaller 0.0025 and what we have seen from our error analysis and accuracy and all that, if we decrease delta t then we should be getting more accurate solution because for a first order if you decrease delta t by a factor of 2 then error should decrease by factor of 2 and so on, that is the kind of thing.

So, we would expect more and more accurate solution as we go from 1 to 0.5 to 0.25 because delta t is changing, delta t is decreasing and the solution that is actually shown here, there are these small curves here which have more software glitches because trying to fit some smooth length for this, otherwise the solution that we see in green has a pulse of maximum of 1 here and it spreads between 0.2 to 0.3 at the end of 0.1 seconds. You can see that as part of the initial guess, we gave the pulse to be 0.1 to 0.2 and in 0.1 seconds, it would have moved by 0.1 meters. So, it would have gone from 0.1 to 0.2 to 0.2 to 0.3 and that is exactly what we are getting.

We are getting it going from 0.2 to 0.3 with the same value here. So, we are getting very nearly the exact solution and whatever small changes here are because of the software is trying to make these things. So, if you plot the individual values you will see the exact solution being produced by this green line, but if we make it more accurate as per our definition of accuracy by reducing the delta t by a factor of 2 here from here to here then we see that the shape is no longer rectangular. It is more like a diffused kind of a thing, this what we would have in practical systems, this pulse wont remain like a square pulse it will be diffusing in this direction and if you make it even smaller time step it should be more accurate, but we find that its actually less accurate as compared to the better solutions here.

We see that the peak value has decreased and it has spread more. So, in that sense it seems to be (Refer Time: 24:25) to the arguments that we made earlier that more the smaller the delta t the more accurate the solution and here we have a solution which is not working like that. Now, what will happen if we go out to a courant number which is greater than 1.

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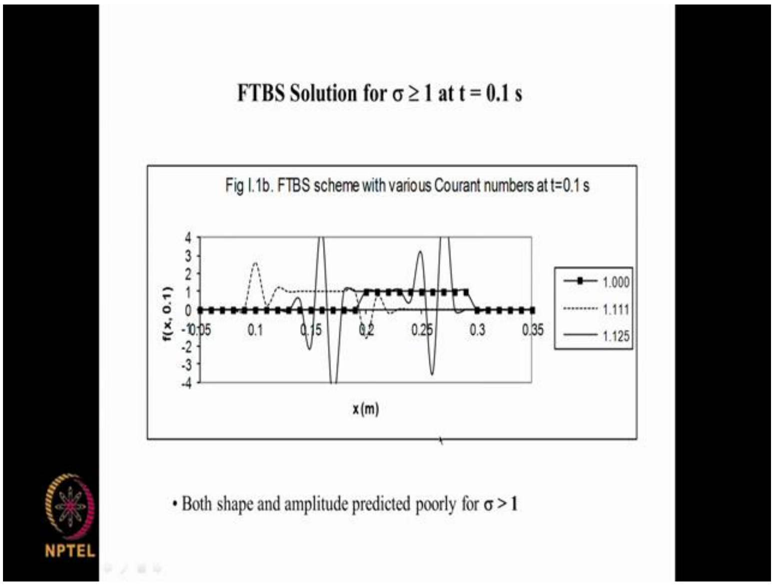

So, we found that best solution is at 1 and if we increase it by a factor of 11 percent. So, delta t is increased by about 10 percent, 11 percent here and this is what we are getting. So, you can see those pulses of 1 going from 0.2 to 0.3. So, we have exact solution here and then it is 0 like this, but slight increase in the delta t has given us a variation like this by the dotted line and this is not like the square pulse that we expect, and it is not confined to between 0.2 and 0.3 which is what we expect, which is what we are getting with sigma of 0.1 and if you are increased by another small amount from 1.111 to 1.125, this is the value it is going to deeply negative and deeply positive and this is not like the rectangular pulse that we are expecting.

So, what we see here is that we have a solution methods FTBS solution scheme. It looks like it is wholly unreliable, it is giving correct results only for 1 value of sigma, only for 1 value of delta t for a given delta x. For other values it is either giving a diffused solution like what we getting here or an absurd solution. So, a small increase, greater than this is giving an absurd solution here, and actually decrease in the delta t, so as to get better accuracy is not giving good solution as what we got with this value, this is surprising.

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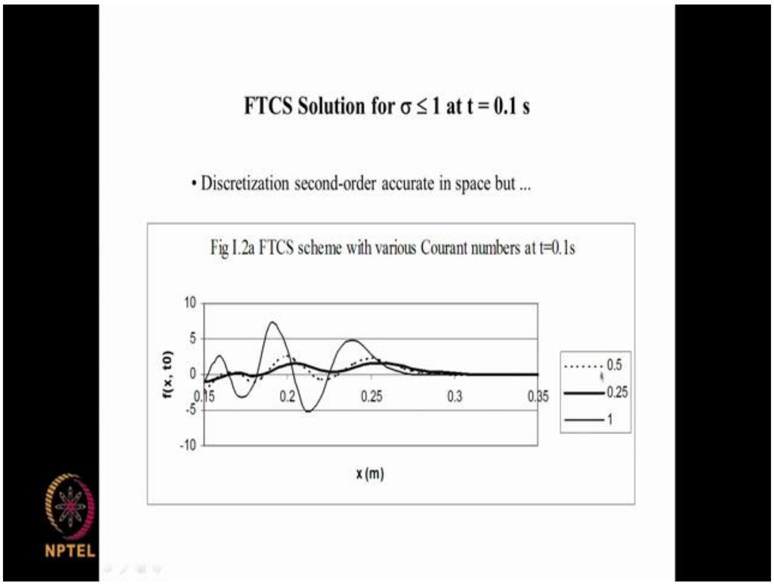

Now, you look at the other FTCS, we expect this to be more accurate than the FTBS scheme because this is second order accurate and space here and here we are looking at again at the same time of 0.1 seconds and again with three different values of courant number, courant number of 1.5 and 0.25 here and we should be ideally getting a pulse between 0.2 and 0.3 with an amplitude of 1. So, we should be getting something like this and what we are getting is totally different things, it is for example, the peak value here is with 1 courant number, we are getting a maximum value somewhere here of the order of 7 and a minimum value somewhere here and it is not like the square pulse that we should be expecting here and you for even smaller kind of things, we are not getting anything like the rectangular pulse.

So, what should have been giving a better solution because of its inherent second order accuracy is not actually giving us a good solution and all because we changed it from second order to first order to second order in space.

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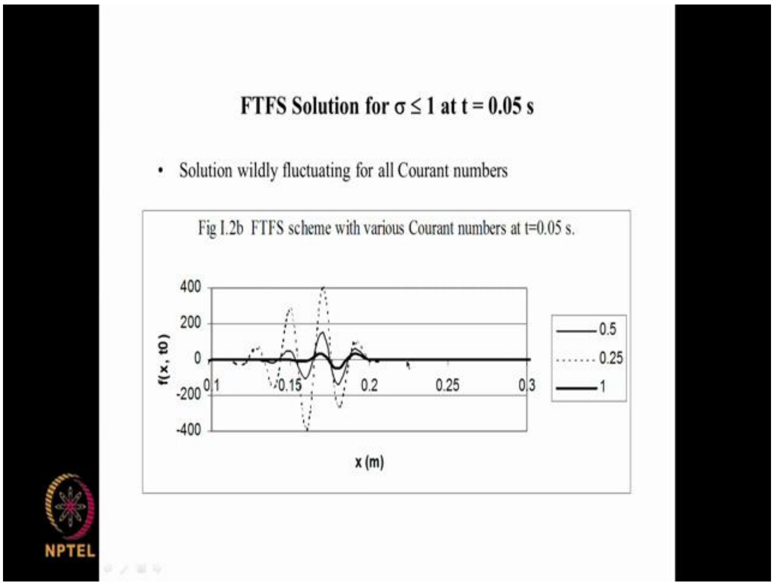

Now, if you look at the third solution FTFS solution and this is at even shorter time 0.05 seconds. So, it should have moved only by 0.05, it should have been centered between 0.15 and 0.25 here. So, it should have been a square pulse and what we see here are amplitudes is order of 400, it should have been 1 and again it does not matter what kind of courant numbers we have here, in all the cases we are getting absurd solutions.

So, what is this telling us? It is telling us that there is definitely more to the solution than just substituting the finite difference approximations for the derivatives and putting together a scheme.

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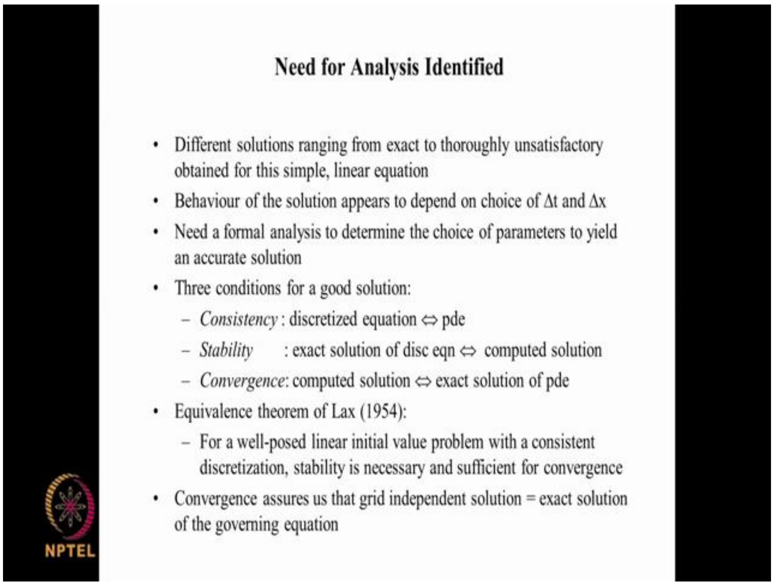

So, we see that different solutions ranging from exact to thoroughly unsatisfactory have been obtained for this simple linear equation and what we are dealing with in actual Navier-Stokes equations are non-linear equations and coupled equations. So, and behavior of the solution seems to depend on the choice of delta t and delta x. So, this is bad because if it is something do with the equations, we can blame the modeler or the mathematician, but these delta t and delta x are the choices that we have to make in order to get a numerical solution. So, now, the blame is on us, we have to find the correct values of delta x and delta t in order to get a solution and we see that sometimes it is coming very rarely, but most of the time it is not coming even for the simplest case using approximations that have nothing apparently no problem with them.

So, this is a situation that we have with the simple linear convection equation kind of thing, and this brings out the difficulty in getting a numerical solution if we take a very naive approach, and so what we are going to do in the next module is we will do a proper analysis of the discretization that we obtained from these kind of defensing schemes and then we see what are the conditions in which we can get a proper solution.

So, we look at concepts like consistency, stability, convergence, boundedness all these concepts that have been brought into play here in order to come out with a discretization scheme which we can confidently say that will give us an ok type of solution, if not the exact solution at least we should be getting an approximate solution which should look, which should have some features of the of the real solution. So, that is what we are going to do in the second part of this third module.