Computational Fluid Dynamics
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Lecture – 25
Solution of Poisson equation in rectangular duct – Tutorial

This lecture is in the form of tutorial, where we are going to make use of a knowledge of being able to write difference approximations for any derivatives both central differences, 1 sided differences, higher order derivatives, higher order accurate schemes, all that thing will put to solve a Poisson equation on a rectangular geometry its very much similar to what we have done in the very first week of this particular course, but we going make it more spicy by looking at different boundary conditions. That we won't have been able incorporate at that at that time because we did not know this finite difference approximations 1 sided difference approximations and so on.

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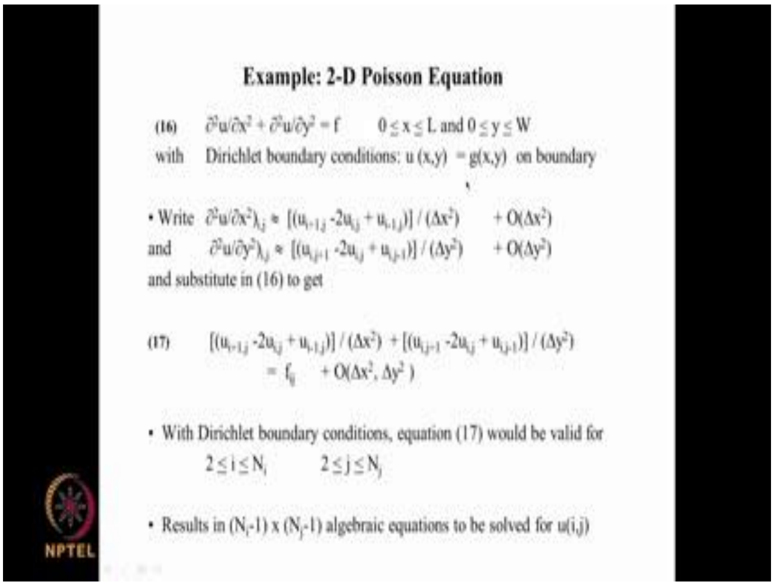

So, the mathematical problem that we are looking at is given by equation 16 here dou square u by dou x square plus dou square u by dou y square equal to f, within the domain between 0 and l in the x direction, and between 0 and capital w in the y direction. So, this is a rectangular for example, duct with the length of l and width of w, and this u here. So, this equation is very similar to the equation that we had for the w velocity component, and f here was the minus 1 by rho 1 by mu times dou p by dou z and the u that we have here was actually the w. So, in that sense this equation is very much similar to the equation that we had in our very first lecture for flow through a rectangular duct. But we are going to now do it with the knowledge of finite difference approximations, and how do many different ways we can do it. So, we consider in the first case the Dirichlet boundary conditions. So, by that we mean where the value of the variable is specified on the boundary. So, which was also what we had in the first example where we imposed the no slip boundary condition. So, we said that w was equal to 0 in all boundaries.

And here g is the boundary g of x y is describes a boundary and on all points of x y lying on the boundary value of u is given by g. So, g can vary or it can remain constant it does not matter. So, for a given boundary point define by x i y j the value of g is given is known for this 2 d problem.

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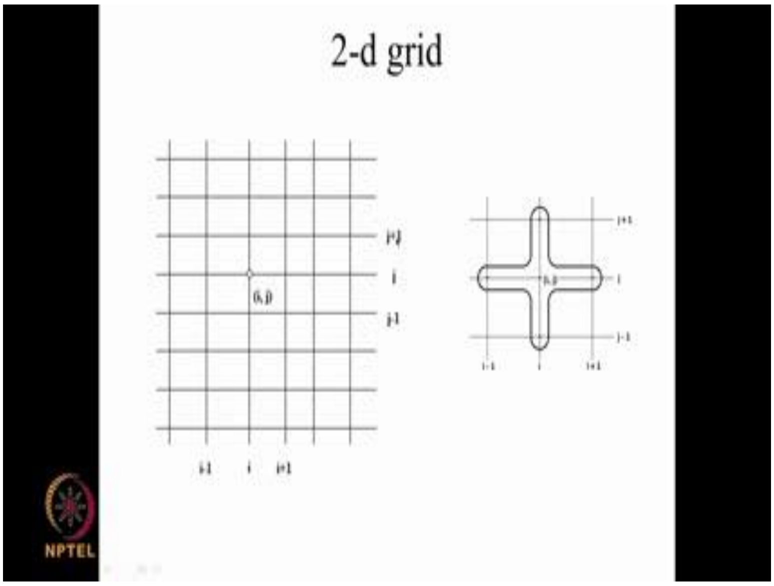

So, we make use of a 2 d grid defining it like i in this direction, x in this direction and y in this direction and we have I, i plus 1 i minus 1 and that kind of notations for grid lines in the x direction. Grid lines perpendicular to the x direction. So, this is all i minus 1 this is all i i plus 1 like this and grid lines perpendicular to the y direction are denoted by the symbol by the index j here. So, you have j here, and the immediate upward neighbor is j plus 1 immediate downward neighbor is j minus 1. So, the i immediate right neighbor is i plus 1 and immediate left neighbor is i minus 1. So, the intersection of this y line and j line will identify the node i j. So, this node here has a left neighbor which is given by i minus 1 j, right neighbor which is given by i plus 1 j a bottom neighbor which is given by i and j minus 1 and a top neighbor given by i and j plus 1.

So, with this kind of notation we can go back to the 2 derivatives that are appearing in in this and we can write a central difference approximation for dou square u by dou x square. At the grid node i j this is a second derivative with respect to x here. So, we can write the corresponding expression as u i plus 1 j minus 2. 2 u i j plus u i minus 1 j divided by delta x square and this is second-order accurate in delta x square in in x. So, this is an approximation for dou dou square u by dou x square at the same j. So, j remains the same here and when you look at this approximation this is bringing in the value of u at the center at the node i j and also to the immediate left neighbor and to the immediate right neighbor. So, when you go back to thing here we have i here i j here i minus 1 j and i plus 1 j. So, these 3 points here are being used to represent dou square u by dou x square at i comma j.

Similarly, dou square u by dou y square is a second term which is appearing in this. The variations with respect to y here, we can again write a central difference formula like this u i j plus 1 minus 2. 2 u i j plus i j minus 1 by delta y square and this will be invoking i j i j minus 1 and i j plus 1. So, the immediate bottom neighbor and immediate top neighbor. When you know substitute both this approximations in this and then, put this f here as f i j i comma j here, then you an overall expression which is like this, so, this is equation 17 is a discretize form of equation 16 at node i j. And in this equation here you have u i comma j u i plus 1 comma j u i minus 1, j u i j plus 1 u i j minus 1 are appearing and together so; that means, that the discretization of the governing equation at node i j at node i j here, results in an algebraic equation involving the left neighbor the right neighbor, the bottom neighbor, and the top neighbor and itself giving rise to a computational molecule like this.

So, this actually encloses all the points that are being in involved in the algebraic equation for this particular node i, and this presences the difficulty that we cannot evaluate this until we know all the neighboring points and we never know the neighboring points independently. So, that is why we write down this equation for all the points and then we solve them simultaneously. So, this is a computational molecule for this Poisson equation here and we notice that the computational molecule depends only on the derivatives. So, the value of this f i j is given by a function it is an algebraic function may be constant. Whatever it is it is not bringing u into picture if f is a function of u then there may be part here and will come to that in more advanced other parts of this particular course.

So, now this is a computational molecule this has to be applied at every grid point at this point, this point, this point, this point and all the points and how many points where it has to be applied depends on the boundary conditions that we have.

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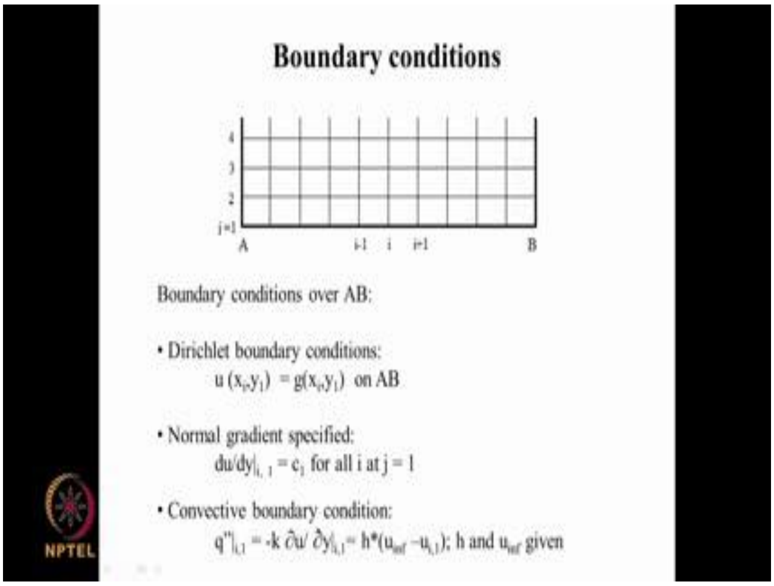

So, now this is our rectangular domain this reaches up to width of w and length of l here if you take this particular boundary here; a, b then you can have three different types of boundary conditions which we are already (Refer Time: 08:55) to in our earlier classes Dirichlet boundary condition is a boundary condition. Where the value of the variable is specified on the boundary now here this boundary point corresponds to j equal to 1, so u of x i. So, that is at for all these nodes which are on the boundary here, i is changing, but j equal to 1. So, x i y 1 and the corresponding function here g x i y y 1 we substitute value of x and y 1 and you get g here and that determines the value of the node on the boundary in a Dirichlet boundary condition we can also have a normal gradient specified thing. So, now, for this boundary here this bottom boundary normal variant is dou u by dou y. So, you can say dou u by dou y at i comma 1, at this point, this point, this point is specified and that is given by some c 1 which maybe a function of i for whatever it is and this is applicable for all i for j equal to 1. So, this is normal gradient specified boundary conditions. So, this can also be a boundary condition now in in this case in this case the values of u at the boundary points are known, but in this case we do not know the value of u now you can also have for example, in case of heat transfer you can have a what is known as convective boundary condition. So, you can say that the heat flux at the wall is given as minus k dou u by dou y its usually dou t by dou y, but here a variable is used. So, you can say minus k dou u by dou y and its given by convective heat transfer coefficient, and the difference between the bulk value the value at infinity minus the value at the wall. So, the value at the wall is i u i 1 at this point at this point. This will be u i minus 1 one. So, this is a value at the wall and this is the value at the infinity or the bulk value and this is the heat transfer coefficient.

So, in this kind of boundary condition the heat transfer coefficient is given, and the value of the variable at infinity is a given thing, but we do not know the value of this. So, in the case of Dirichlet boundary conditions the values at the boundaries are known and it is only at the interior points get the values. In the case of normal gradient specified or convective boundary conditions or may be even derivative boundary condition, and all that the values at the boundaries also need to be evaluated. So, in this particular case for this equation with Dirichlet boundary conditions you have we need to evaluate only at the interior points. So, only for i equal to 1 value are known and j equal to 1 values are known and similarly i equal to n i which is the top most which is the right most value and top most value is known. So, only for in between the values, we need to find out. So, you have n minus 1 times n j minus 1, n i minus 1 times n j minus 1 number of algebraic equations are to be solved in order to get the value of u at all the interior points. So, this we have already done in our very first week of lectures and so, we are quite familiar with this, now if you want to do the normal gradient specified thing.

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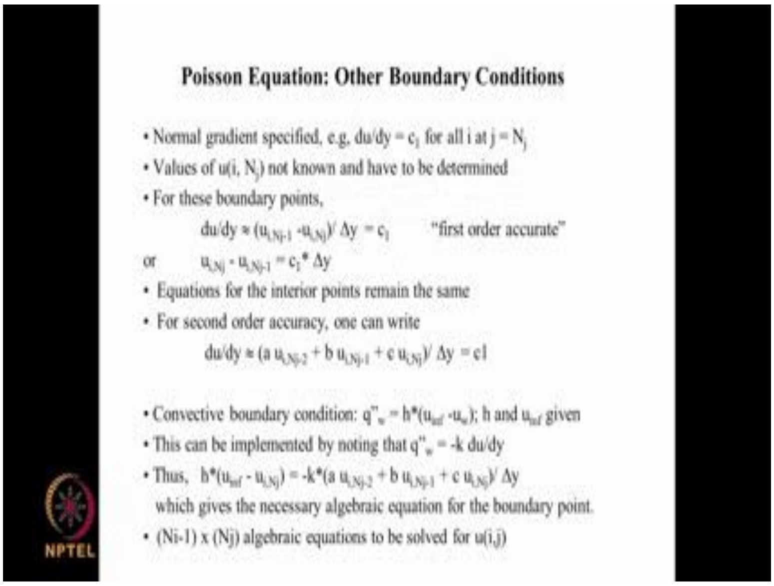

For example; dou u by dou y equal to c 1 for all i at j equal to n j or j equal to 1 in this particular case, the value of u at i and j is not known and this needs to these values that is all the i values on this j equal to n j need to be determined. So, we can write in approximation of d u by d y at i n j as u i n j minus 1 minus u i n j by delta y equal to c i now this is first order accurate approximation for dou u by dou y at i n j. So, this is because it is a derivative with respect to y n j minus 1 and n j is changing, but the, i index is not changing and we can see that this is n j minus 1 and n j. So, there should be a minus sign here there should be minus sign here, So, this is backward differencing approximation first order accurate because its involving only 2 immediate neighbors and that is 1 possibility. So, since the values of this dou u by dou y is given here, you can now write this equal to c 1 and and therefore, from this you can write an algebraic equation u i n j minus u i n j minus 1 equal to c 1 times delta y. So, here i have corrected the minus sign. So, this is an algebraic equation for this node u i n j.

So, in the case of boundary point where the boundary value is not known, we do not use the governing equation we do not use Poisson equation, but we make use of the boundary condition in order to get an algebraic equation for this. Now we can instead of using first order accurate thing we can also make use of a second order accurate 1 sided differencing formula here and because we are looking at n j which is a top most boundary. We do backward differencing involving n j n j minus 1 n j n j minus 2 and we can easily derive the coefficients a, b, c for this, using the method that we have already written in which case we write this dou u by dou y at i n j as a u i n j minus 2 plus b u i n j minus 1 plus c u i n j divided by delta y equal to the same first derivative which is given here, and this will give us at different algebraic equation for the same point u i n j n j. So, here we have 1 and minus and here you have a, b, c like that. So, you have a different discretized equation for this particular point here this particular node here and this again is obtained from the boundary condition here.

So, now when you have a convective boundary condition for example, q wall equal to h times u infinity minus u wall, with h and u infinity given we know that the heat flux from the convective side from external side must be equal that heat flux from interior which is minus k dou u by dou y, and dou u by dou y. So, this is effectively condition is minus k dou u by dou y equal to h times u infinity minus u wall and. So, we can write this as h times u infinity minus u i n j equal to minus k times dou u by dou y at n j, if we make use of this second order expression you have this expression like this. Now we can rearrange this in terms of known quantities. So, we can bring in u i n j times minus h here and you have this thing here. So, we can rearrange this to get an algebraic equation involving u i and j u i n j minus 1 u i n j minus 2, and in this we have a, b, c, k delta y and also h will coming to picture this gives us a different algebraic equation again for the boundary point.

So, in this way we make use of we incorporate the boundary conditions, in to our set of equations that need to be solved if it is a Dirichlet boundary conditions then the values is known values is incorporated, when we when we derive for example, if this is the boundary point. So, in a Dirichlet condition this value is known. So, this is spread into the equation for this. In the case of normal boundary conditions normal gradient specified or convective boundary conditions we are directly evaluating the the value of the variable at these boundary nodes, by writing an algebraic equation derived from the boundary conditions not from the partial differential equation. So, in this particular case we have n i minus 1 times n j algebraic equation to be solved for u i j. If you have 1 normal gradient boundary conditions specified here. So, if you. So, in that way we can bring in any kind of boundary conditions into the governing equations, and we this boundary conditions for interior points is given is converted in to an algebraic equation using this and at the boundaries depending on the Dirichlet boundary conditions or the normal boundary conditions or convective boundary conditions, you make use of either this equation or this equation or this equation to find the boundary equations. So, together this constitute a set of algebraic equations which can be put in the form a x equal to b and then we can use variety of methods for solving this, in this particular way, in this example, where we can claim that we know how to discretize a given partial differential equation. Using finite difference approximations, having understood this we will go into another feature of these finite difference approximations, which is specific for time dependent problem. So, we will in the next lecture we look at the how to bring in time difference time derivatives and what special difficulties or features that are brought in because of time derivatives as a post to space derivatives. With that we will be able to look at all derivates that appear in the governing equations, that is we have time derivatives like dou u by dou t and we have first-order derivatives like u dou u by dou x plus v dou u by dou y those type of things we know how to derive finite difference approximations for the first derivatives. And we also have derived finite difference approximation for second-derivatives like mu dou square u by dou x square dou square u by dou x square and all that.

So, we will be armed with the a set of designer finite difference formulas, that we can design as per our demand or as the problem demands in terms of one-sided or central or this accuracy or first-order accuracy, second order-accuracy and we can convert any given partial differential equation into an algebraic equation at a given node. And we do that for all the nodes and then convert into a x equal to b form. So, on the next lecture we look at what to do for finite difference for derivatives involving time derivatives.

Thank you.