Computational Fluid Dynamics
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Lecture – 21
Concept of wellposedness of mathematical problems

So, the last lecture in the second module on governing equations deals with the boundary conditions and initial conditions. These are important because we have a partial difference equation and the specific mathematical problem is not complete without the initial and boundary conditions as appropriate. So, what kind of boundary condition and initial conditions can we bring here?

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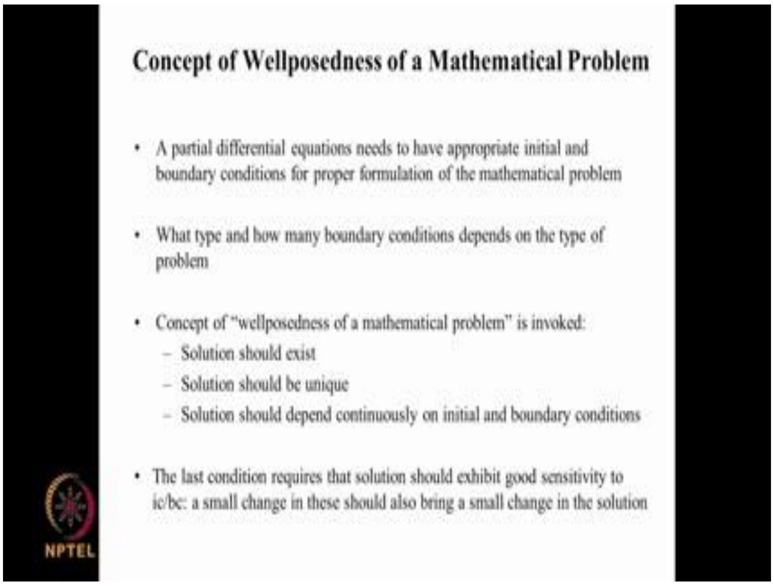

Here, we need to consider the concept of wellposedness of a mathematical problem to answer the question and the question is that what type of boundary conditions and how many boundary conditions are to be specified to complete the mathematical problem. This depends obviously, on the type of problem that we are considering here and we would like to have as many as necessary for a wellposed problem and only as much as is necessary for wellposedness in the problem.

What do we mean by wellposedness? The wellposedness, it has an intuitive sense that is there should be a solution to the problem to the mathematical problem that we are posing and the problem, the solution not only should exist, it should be a unique solution and thirdly it should be it should depend continuously on the initial and boundary conditions that we specify.

So, we are looking at the boundary conditions and initial conditions to make a difference to the solution that we get for the mathematical problem because it is a same equation whether the flow of that particular fluid is going over a car or inside the car or inside a house all those things it is a its a same set of equations. What makes the problem different from one problem different from other problem must therefore, come from the initial and boundary conditions that we specified and here we would like to specify them in a such a way that we have a solution and that the solution is unique. So, that if a number of solution are there for the same problem then it is a no consequence and if there is no solution also it is of no consequence, and crucially if the solution is very highly sensitive to the boundary conditions and initial conditions that we specify, then it is of no use or if it is not at all sensitive to the boundary conditions and initial conditions then again its of no good because then in what sense is a solution unique.

So, it is the goodness of wellposedness of the mathematical problem is seen in the type of solution that it produces and the type of boundary conditions and initial conditions that we need to be specify in order to get a good solution and so together this complete the specification of the mathematical problem and that is what we have in the slide here. A partial differential equations needs to have a appropriate initial and boundary conditions for proper formulation of the mathematical problem, and what type and how many boundary conditions depends on the type of the problem and concept of wellposedness is something that we have to keep in mind in a specifying the boundary conditions, and the concept is that it is essentially that there should be a solution and the solution should be unique and it should depend continuously on initial and boundary conditions.

So, the last condition requires that solution should exhibit good sensitivity to the initial and boundary conditions. A small change in these should also bring about a small change in the solution and only a small change, if it is a prescriptive solution prescriptive change to a small solution then we have to worry about how accurately we have to specify the boundary condition and initial conditions, and the question of how accurately we know and control these things in real cases also comes into picture. So, we will cut that particular sentence. So, a small change in the boundary conditions should produce a sensitive change in the solution that we get for this so that is the concept of mathematical problem.

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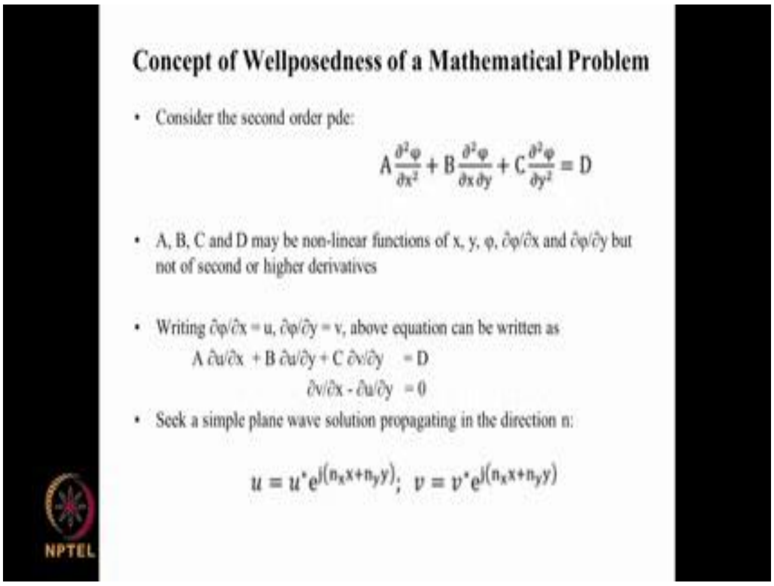

How it is embedded in the equations is something that we need to examine and what it actually means to the specific equations that we are solving is also very important for us, and only then we can ask we can think of the type of boundary conditions and what the concept of wellposedness is means to us from a CFD point of view.

So, here we know that our governing equations are second order partial differential equations and considering here a generalized second order partial differential equation involving two independent variables x and y. So, this is a dou square phi by dou x square plus b dou square by phi dou x dou y plus c dou square phi by dou y square equal to d and A, B, C, D are may be non-linear functions of xy phi dou phi by dou x and dou phi by dou y, but not of second or higher derivatives. So, it is in a way we are considering quasi-linear second order partial differential equation, which is the type of equations that we have as you seen in the last tutorial all are governing equations are second order or less, and they are of this particular form in the general form of that we have written here.

So, this second order partial differential equation can be converted into to first order equations by writing dou psi by dou x dou x equal to u and dou phi by dou y equal to v. We can convert into two coupled equations as shown here and in order to look for the nature of the problem, we will talk about the hyperbolic and elliptic and those all kind of things. So, in order to get a true sense of what is actually involved and in order to relate to the earliest statements that we made about the directional advection and the a directional diffusion concept that is embedded in the governing equations, we want to bring that into play here and see how it actually means to our boundary conditions.

So, here what we are looking at this is at; we are looking a general second order partial differential equation. We have converted into to first order equations and we are looking for a plane like solution, plane wave type of solution of this particular form. You have u and v, u is given in terms of u star times a travelling wave travelling in the direction n with direction cosines of nx and ny and similarly v is also travelling in with an amplitude and a direction propagation like this. So, if these were to be solutions of this when we substitute this into this, we should be able to get a condition for the existence of non trivial solutions.

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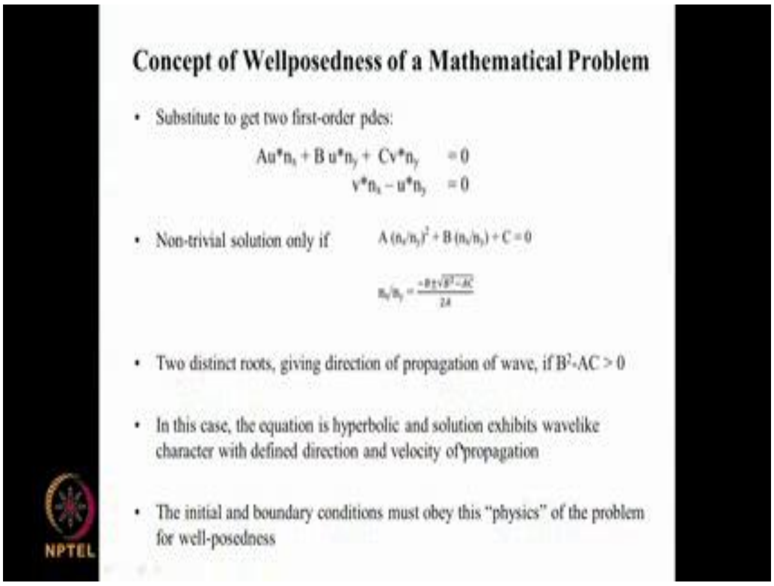

So, we substitute that u and v into that and then we do some manipulation and then we get a condition from non triviality of the wave type of solution, and that is possible only if nx and ny the directional cosine directional cosines of the plane wave are such that they satisfy this equation and the direction cosines given by this equation minus b plus or minus square root of b square minus ac by 2 a.

So, very familiar we are just deriving it in a proper way so obviously, there are two distinct roots are possible if b square minus ac is greater than 0 and that means, that when you have two distinct real roots here so that means, that you have two traveling wave type of solution is possible and that means, that in such a case the equation that we started out with will exhibit a wave like solution, and any equation which exhibits wave like solution with a well defined wave like character. So, that is with a well defined directional of well propagation and a velocity of propagation. So, that kind of equation is said to be a hyperbolic equation. So, you have a hyperbolic problem if you have a wave like solution is possible and what is important for us is that, the initial and boundary conditions that we specify should obey this physics of the mathematical problem.

So, what is physics that there is a wave like solution with a well defined sense of direction and magnitude of propagation speed of propagation? So, if we do not obey this principle at any time during our specification of problem or in the evaluation of the solution then we can get into the problems. So, this is something that we will see later on in terms of the solution, but this immediately brings us to the concept of domains of dependence and influence of the solution here.

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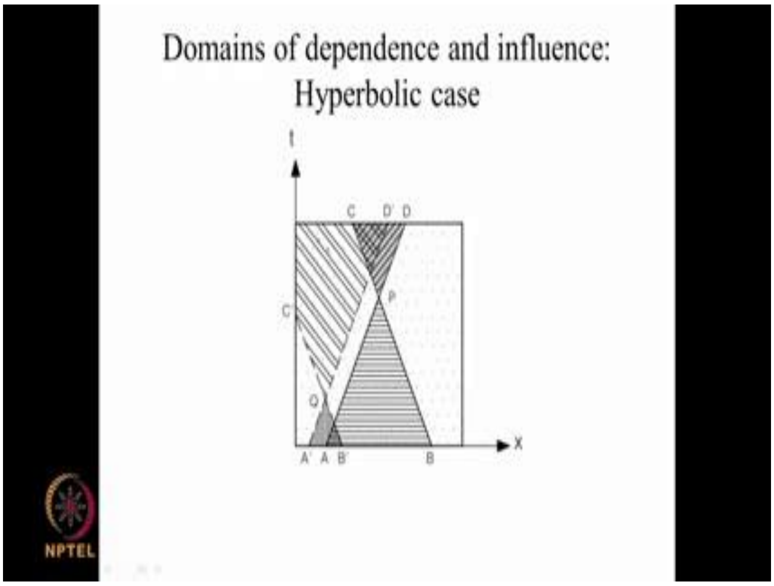

So, we are looking at the same at 2 dimensional case with x and t as a variables and this is our solution space over a certain domain in the x-direction over a period of time the along the y-axis here, and so at this rectangle here defines the possible x and t values that we can exhibit that the system can exhibit and for a finite physical domain extending from 0 to x here, as time progresses this domain expands in the positive direction, and here in this domain in the x and t consider a point p here and this at a given x and at a given t.

So, this p here that is a solution at distance of x and time of t is influenced by all the previous solutions, all the solutions within this triangle given by this horizontal lines here so that means, that the value of this p here depends on the value of p within this zone and any solution outside this zone will not influence this and how are the zones how is this area identified by the characteristics lines that we have evaluated based on this slopes here. So, you draw the lines and then extend it back all the way until you reach time equal to 0.

This is left handing characteristics and right handing characteristics and then you bring them here and then you have the values of the boundary. So, between a and b will influence the value of p here and that means, that p should be evaluated only with these this set a boundary conditions this set of initial conditions. So, when you have t equal to 0 whatever values of phi that you specify here become the initial condition and outer the initial conditions those initial values that lie beyond b or before a here, will not influence the solution at point p, but they may influence the solution at point q here, this q here is influenced by the values here and any change in the value of b will not influence q. Any change in the value initial condition at b prime will influence the value of p and. So, in that sense, if you have a solution scheme which expresses q in terms of the solution at b then it is wrong because it should not depend as per this nature of the travelling wave type of solution.

So, the initial condition at p equal to 0 at point b should not influence q and any the initial condition at any point b prime between this a and b must influence point p and so this triangle PAB is the domain of the dependence for point p and the domain dependence changes depending on where this point p is in this solution space, for q it is different and not only that if you extend this line further, the value of the solution at point p influences the values for the downstream, but not all points here, only within the triangle PCD. So, the solution at point p here as a domain or dependence PAB and a solution domain of dependence which is PCD here.

So, when we write down a finite difference approximations and all that, when we say that this value is given by the neighboring points then it should be those neighboring points should lie within this domain of influence and domain of dependence, if not then we can potentially get into problems. So, it is this idea that is that in a hyperbolic case only part of the initial conditions not the entire value, not all values of over the entire domain will be influencing any point in inside, only a certain portion of the initial condition will be influencing is something that would be a characteristic nature of a hyperbolic problem. If you look at we need to a do the editing just one minute.

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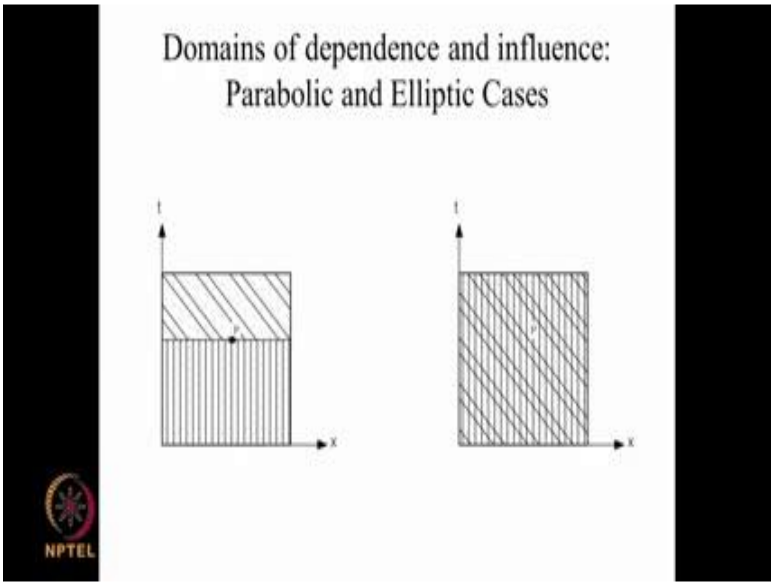

We will start again. So, if you look at the other cases parabolic and elliptic, we do not have two distinct roots, we do not have two distinct directions, we have the case of parabolic this only one direction and that line will be a parallel to this, and p here is influenced by all the points below this and it influences all the points above this. So, it is like a both are evolving both hyperbolic and parabolic are evolving, but the difference between a parabolic case and hyperbolic cases is that p here is influenced by all the values in the in the initial condition here and there also its also influenced by the values of all the boundary conditions right up to the time t that we are considering here. So, p is influenced by boundary values over this part of the time boundary, this part of the space boundary and this part of the boundary condition here. So, it is influenced by these boundary conditions and it in turn influences all the points above it.

Now, if you look at in elliptic case then the point p is influenced by not only by the boundary conditions at t equal to 0 and up to the point of t at which p lies, but it also influenced by these potentially what we can see as per this diagram as future boundary. So, in that sense it looks odd to have a future boundary condition influence the present. So, if that happens it is not like evolving time, evolving problem; it is not an initial problem, it is a boundary value problem and in an elliptic cases are boundary value problems and the solution here depends not on just this boundary or this boundary or the closes boundary it depends on all this boundaries here. So, this is also known as a jury problem because this person, this criminal or accused is being investigated by jury members all round it and so all of them will take a look at this particular case.

So, this is a boundary value problem, it has a domain of dependence of on all the boundary values so that means, that p should be sensitive to boundary any change in any part of the boundary value here, whereas in this particular case p should be sensitive to only these boundary conditions and not these boundary conditions and in the case of hyperbolic thing, p here would be sensitive to only changes in the boundary conditions between a and b and not all these, not any change here and not any change here.

But if we go further up and time such that the characteristics intersect part of this, then they are sensitive to these boundary conditions and this boundary conditions. So, hyperbolic cases are like very fast moving cases and it takes time for these boundary points to have an influence on this and boundary value problems are for a very slowly evolving system or it is almost slow evolving it can be considered as quasi steady and everybody in this room here can have the essay and then p will listen to all this boundary conditions and then its value settled. So, in that sense if this becomes this is a time independent problem and its values is influence by all the boundary conditions here.

So, if you have a second order partial differential equation, the boundary condition that we should specify for a proper solution depend on whether you have an elliptic type of problem or a parabolic type of problem or an elliptic type of problem. If it is elliptic type then the boundary conditions that are required are very clear you should specify the boundary conditions over the entire domain. If it is parabolic then only these boundary conditions can be made to influence the value of p and these boundary condition should not be used at all in the evaluation of solution at point p if the point p is moved forward then this additional condition should come in.

In the case of hyperbolic equations, the solution at point p should be determined only using this initial conditions over this restricted boundary are not everywhere. So, if you where to pose this p to be a function of the solution of the value at a prime, then it would be wrong, it would be physically wrong to do that, this is about what type of boundary conditions and where to be specified.

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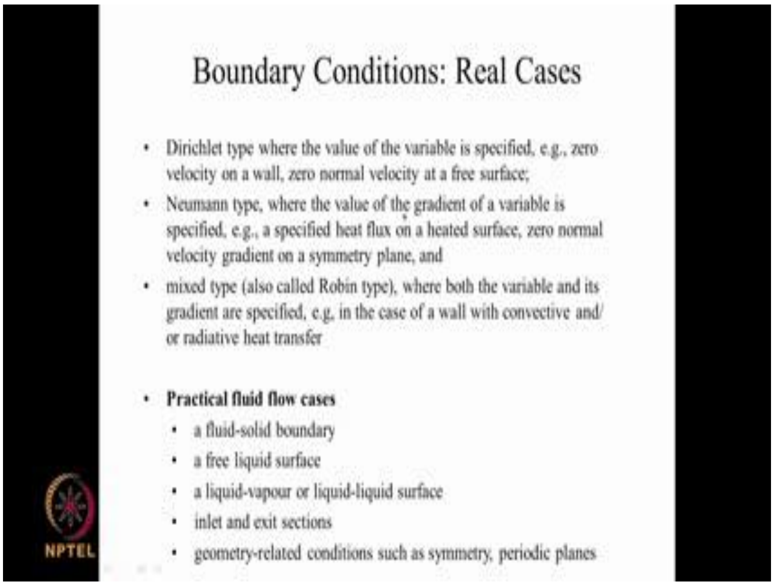

Boundary conditions come essentially in three types here; one is the Dirichlet type where the value of the variable is specified at the boundary, for example, zero velocity on the wall or what we call as a no slip condition, where this velocity of the wall is equal to the velocity of the fluid at that particular wall condition. Similarly, the temperature is of the wall is same as the temperature of the fluid which is adjacent to that. You can also have a slip condition at which the value of the temperature of the fluid at that particular location is different from the value of the wall by a small amount it may be because you have an interface and across interface there is a small temperature drop.

And similarly when you are looking at microfluidic conditions you usually have slip boundary condition at the wall. So, the velocity at the wall even for fluid flow is not exactly equal to 0, it is somewhat a higher than 0. So, you bring in a non no slip boundary condition there. So, where the value of the variable is specified then you call that as a Dirichlet type and where the value of the gradient is specified it is a Neumann type of boundary condition, for example, a specified heat flux on a heated surface or zero normal velocity gradient on a symmetric plane. So, these are all you are saying that dou phi by dou n equal to 0 or dou t by dou n times k with a minus sign.

So, that is the heat flux and that is equal to some value hundred thousand million watts per meter square is a condition, where you are you are imposing directly on the gradient value that the solution can take minus k dt by dy equal to thousand means that k is a given material property. So, that fixes the value of d dou t by dou y there and a mixed type it is also called as robin type where both the variable and its gradients is specified for example, in the case of a wall with connective or radiative heat transfer.

So, in such a case you say that the wall on the outside of the computational domain you have, for example, natural convection and natural convection means that there is a heat transfer coefficient which where we the order of 5 to 10 watts per meter square degree centigrade and that is the heat transfer coefficient with, which heat is evacuated from the surface and how much heat is evacuated what heat flux it is depends on the temperature differences between the wall and the ambient. So, the heat flux their condition is t wall minus t ambient times h. So, that brings in; that is a kind of heat flux boundary condition.

So, that involves t wall and the specification of the boundary condition in terms of heat flux means in the temperature gradient. So, that brings in both the wall variable value at the wall and also the gradient. So, that type of thing is a is a mixed type and all this kind of boundary conditions are encountered in real cases and although we have been talking in terms of abstract boundary conditions here, when we look at practical fluid flow then the boundary conditions are much more physical, for example, at a fluid solid boundary in the case of temperature we can say that there is a heat flux or there is a wall temperature, in the case of momentum we say there is a no slip condition or free shear condition so that means, that the velocity gradient is 0 or some shear stress condition.

So, those types of things or in the case of free surface flows free liquid surface, we can bring in the surface tension and we can bring in kinematic boundary conditions dynamic boundary conditions which take into the account the curvature of the interface and it can have a liquid-vapor or a liquid-liquid interface and across this liquid-liquid interface you actually have mass transfer, whereas typically across fluid-solid boundary, we do not have mass transfer. So, that is an invariable wall boundary conditions something that we specify, except in certain special cases where we can specify example of blowing surface, blowing boundary or a section type boundary that kind of thing is also sometimes brought in here.

So, if you look at the three physical realistic boundary conditions here, we have a non permeable rigid interface and a free liquid surface, free liquid-vapour or liquid-liquid interface is one which is not rigid that it can move it can deform and you bring in surface tension and a normal stress balances and all that.

And third one is its not only it may be rigid it may be flexible it may deform, but it brings then this condition here, brings in the non-permeable nature and across interface they can be mass transfer and associated with the mass transfer there are other boundary conditions, and you can also have special geometry related boundary conditions for example, what we have as typically inlet and exit sections. At the inlet you have flow coming in or you have a pressure of our. So, much that induces flow from outside into the domain and exit is that there is a essentially a passage for the flow to go out exit the computational domain. So, there again you can bring in certain constant pressure on the outside or you can say that the flow is fully developed in which case the velocity gradient in the flow direction is equal to 0.

So, these are all special type of boundary conditions that are helpful for us to introduce flow and take out flow and also flow related things with this and finally, you have some specialized geometry related conditions like symmetry plane, periodic plane.

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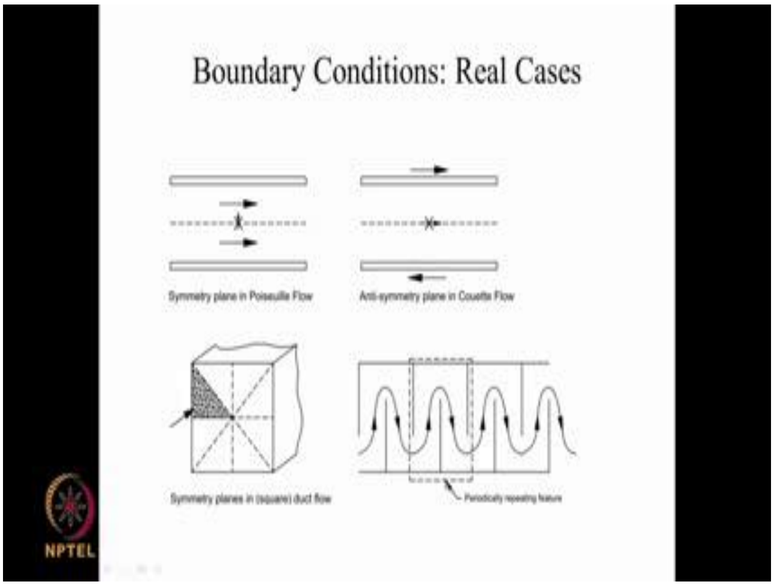

For example in Poiseuille flow, you have at the centre you have symmetry plane so that means, that flow above this and flow below this is exactly the same. So, if you have a profile like this, parabolic profile like this then the upper part is symmetric with respect to the bottom part. It is not necessary for us to for compute bottom part if you know the upper part.

So, we can say that we will consider the upper half of the domain and imposes symmetry boundary condition here and you can also have an anti symmetry. So, here it is being in this direction is being pulled along in this direction. So, for example, plane Couette flow you can have this type of boundary condition and you can also have for the rectangular case that we consider right in very first examples in; we consider this whole cross section and then we made this into lots of points here, but in this thing here, the flow is symmetric about this half this half and this half have symmetric, similarly is this half and this half symmetric and this half and this half along the diagonal is symmetric.

So, you can say that this whole thing is symmetric there is a symmetry that one-eighth of the symmetry is there here. So, if you compute flow domain in this then from this you can make use of symmetry plane and then transfer it to the other side and once you know have this whole thing, you can bring it to this side and once you have this half we can bring to this side and get the whole thing.

So that means, that it was not necessary for us to have done simulation over the entire rectangle cross section. We can done over one-eighth of the domain and use shear number of points, if you knew how to deal with boundary conditions here and you can also have this special kind of cases a periodically self repeating unit, for example, in heat exchanger you can have this kind baffle plates and flow comes in like this and then goes out like this. So, this is these kind of baffle plates are used to maintain good circulation of the flow so that there is no by passing and things like that. In such the geometry of the cube arrangement and all that can be quite complication in this.

So, in such a case instead of tackling the full geometry which will take many grid points then you will take one repeating unit here and then we put periodic boundary conditions. So, we specify what is a pressure drop in this and then we can calculate the flow distribution and see whether flow is going well through this, or whether it is just coming out like this leaving a big recirculation loop here and loop here, where the cubes are not properly well irrigated. So that means, that they can be ineffective heat transfer in that.

So, under what kind of conditions we have good flow pattern through this and what will be the corresponding pressure drop in that, and how is the flow cube arrangement in that going to help us all those kind of things can be investigated thoroughly by taking a small part of the domain and imposing periodic boundary conditions on this rather than take all these baffle plates and then increase the number of grid points and increase the complexity solution and all that. So, you can have this kind of special conditions also that you bring into play here. So, to sum up initial and boundary conditions are part of problem specification, part of the problem formulation and there are certain consideration like wellposedness that we have bear in mind while specifying the boundary conditions and while including the boundary conditions and initial conditions in our computation, but those results are essentially for a single quasi-linear second order equation. What we have in our actual condition are a set of coupled equations of this type.

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These are definitely quasi-linear, but these are coupled equations in the sense they do not involve a single variable, they involve a lots of variable and no equation can be solved independently, and these are also this a high degree of non-linearity that is coming in this when we are dealing with high speed flows.

So, in all this kind of things you do not have that clear cut hyperbolic and parabolic and those type nature associated with this. So, these are a mixture of hyperbolic parabolic or elliptic parabolic and those type of situation actually arises and we do not have a clear cut demarcation that yes, this is a hyperbolic problem this not a hyperbolic problem and so these things are some rules and guidelines that we have to keep it in mind for potential problems that may arise in the solution and if we do that and if you apply sensible well thought of boundary conditions, then we should be safe in getting a unique solution in most cases, but we should always question the type of boundary conditions and initial conditions that we specify.

We should also questions ourselves on the proper choice of the flow domain that we consider because along with the flow domain, we have the boundary conditions that are coming in and what are the variables that we are evaluating and what do we know about this variables at these boundaries, we have to really examine that in critical cases and then accordingly we should specify the boundary conditions and the initial conditions.

So, with this discussion we can say that we have come to the end of the governing equations and problem formulation. In the next module, we will take that generic scalar transport equation that we derived in the tutorial problem and then we look at how to go about doing a good solution numerically for that equation, and once we establish the principles by which we can get a good solution then we can come back to the set of coupled equations that we have derived and then look at the solution of all this things. So, that is what is there in the next two modules. So, the first module is how to get a solution for one equation incorporating the generic features of fluid flow that we are that we are encounter and the next module is on how to solve all this equations together.

Thank you.