Computational Fluid Dynamics

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Lecture – 16
Navier-Stokes equation for simple cases of flow

In the last tutorial lecture, we have derived the constitutive equation for a Newtonian fluid. Let us summaries the arguments for this.

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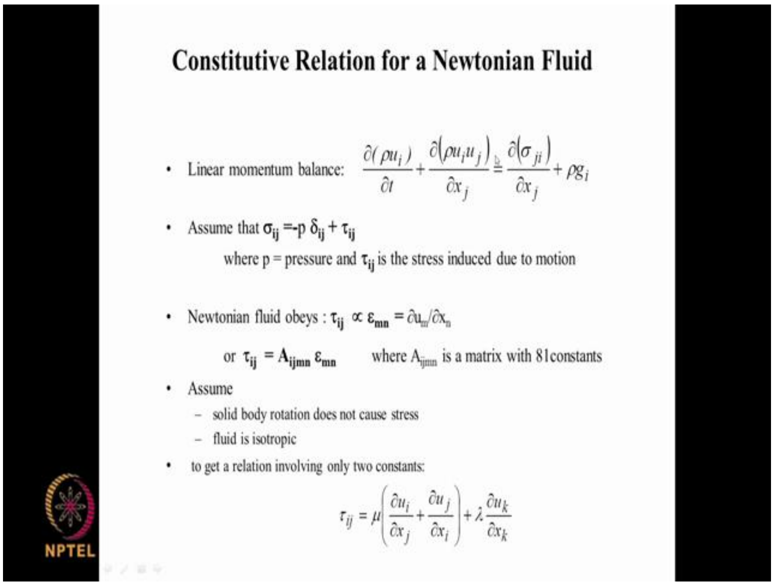

We have linear momentum balance equation applied gives us this equation here, dou by dou t of rho ui plus dou by dou xj of rho uiuj equal to dou by dou xj by sigma ji plus rho gi, which is a body force. Now, in this case this has three equations, but too many unknowns here and even if you invoke the anglo momentum balance equation, we will have too many of the stresses here that are unspecified.

So, we said that we break up the stress into hydro static components involving pressure as a comprehensive force, which is always normal and compressive, which is why we have the delta ij plus shear stress viscous stress which is induced due to relative motion and we studied the kinematics of deformation and we suggested that the deformation rates can be expressed in terms of linear combinations of this a deformation tensor and we seek a linear relation between the stresses that are induced by relative motion and the strain rate that are produced as a result of this relative motion, such that since this strain rate are expressible in terms of linear combination of the velocity gradients you could replace the stresses here by the velocity gradients here into this, so that we can get rid of all these unknowns.

In the general case, we have this linear relation having 81 constants but, if you assume the solid body rotation not causing any stress because there is no deformation it is just rotation and stress translation, fluid translation does not produce any deformation rate. If we say that this is not causing any stress then this tensors here becomes asymmetric tensors expressed in terms of the d mn and there are other arguments also given raise to the same idea there and if you make the further assumption of fluid being isotropic then we have seen that the stress verses strain rate relation distortion strain rate relation can be expressed in terms of only two independent components constants which we call as mu and lambda here and the relation can be by expressed as tau ij as being given by mu dou ui by dou xj plus dou uj by dou xi plus lambda times dou uk by dou xk.

Now, we substitute this into this expression here and this into this expression and we finally get the conservation equation like this.

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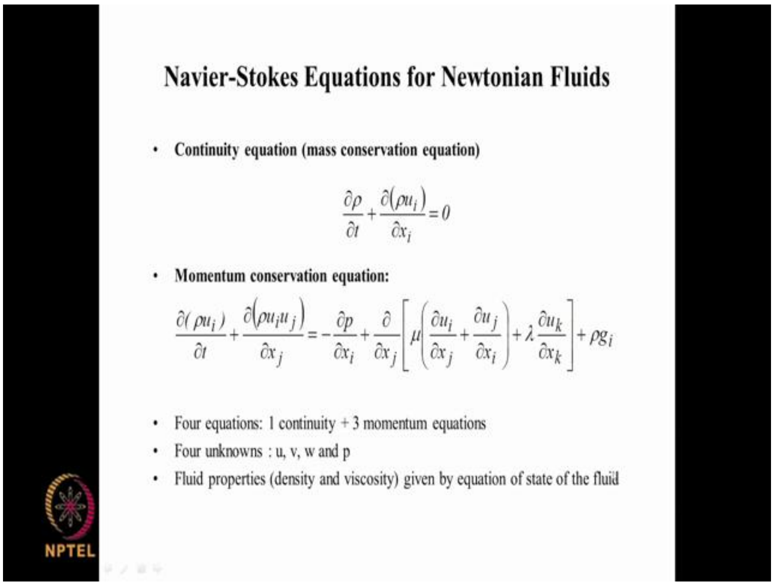

So, these are the Navier-Stokes equation for a Newtonian fluid and any flow of a Newtonian fluid is governed by these equations. We have a continuity equation or the mass conservation equation which is dou by dou t of rho plus dou by dou xi of rho ui equal to 0 and linear momentum conservation equation with the angular momentum all thrown in with the Newtonian fluid assumptions thrown in that is the relation between stress and deformation strain rate is linear and the medium is isotropic.

So, those are the two assumptions of the Newtonian fluid and once we do that we can substitute that sigma as minus p delta xj. So, that it gives us minus dou p by dou xi here plus dou by dou xj of the stress components here gives raise to this equation here and here we see that there are terms involving repeated index j here and repeated index j and repeated index i here and repeated index k here and all these things will give raise to a set of 4 equations; 1 is the continuity equation and 3 momentum equations here and these equations together have the three velocity components that is u, v, w in the Cartesian coordinate system and p pressure as the unknown variables and they also involve three material properties; the density, the first coefficient of viscosity and the second coefficient of viscosity.

So, if the fluid properties are given through an equation of state then we have four equations and four unknowns. So, this constitutes the set of Navier-Stokes equations and the formulation of the fluid flow problems. So, these four equations are the ones that we need to solve in order to describe any fluid flow situation. We also need to have a proper fluid domain in boundary conditions we will come to that later, but we will let us take a simplified form of this which is the case of incompressible constant property flow which for example, for most of the water flows without any heat transfer obey this kind of assumption.

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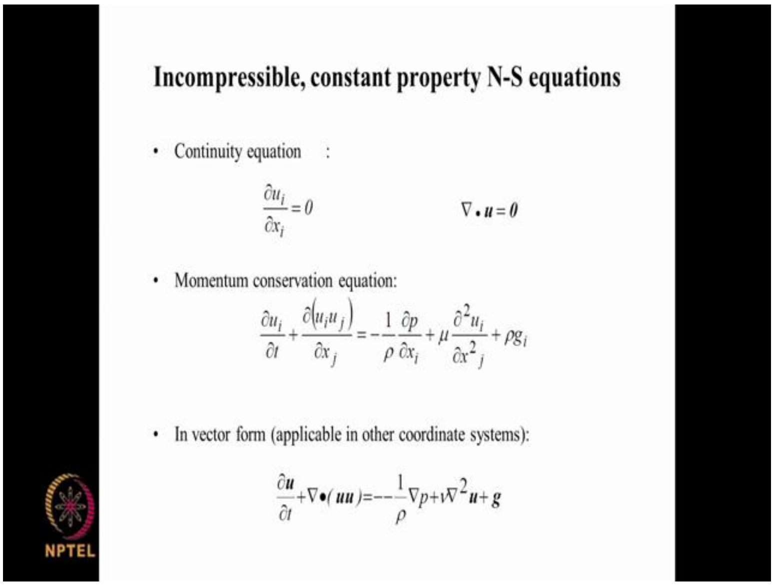

So, in which case if you say that incompressible then density is constant. So, this becomes 0 here, rho can be taken out and then taken to the other side. So, we have dou ui by dou xi equal to 0 that becomes the continuity equation here rho can be taken out and put here one by one by dou here and mu can be replaced with the kinematic viscosity and since dou ui by dou xi equals to 0 therefore, dou uk by dou xk equals to 0. So, this whole second coefficient drops out and you have a simplified momentum conservation equation which is like this. So, you have a simplified continuity equation and momentum conservation equation here for the case of incompressible constant property Navier-Stokes equations.

So, this can be written in the vector form as del dot u equal to 0 and the momentum conservation equation can be written as del dot uu which is a dyadic tensor minus is an extra minus here one by rho gradient of pressure plus laplacian of the velocity vector plus the gravitational vector here.

So, this vectorial from here is good for us to write the corresponding equations in other coordinate systems like polar coordinate systems, cylindrical polar coordinate system or spherical coordinate system and you can go to standard text books to have the definitions of the divergence operator and the del square operator and the gradient operator. To rewrite this in any orthogonal coordinate system that you that you fancy or that supplicable for a specific problem, for example, if you are looking at flow over a sphere then instead of using the Cartesian coordinate system, you might want to use a spherical coordinate system then you can take this from and then define from books on mathematics, what is a divergence operator in spherical coordinate system? What are the correspond components here and here and the gradient here? And you can write down the governing equations for the case of incompressible constant property flow.

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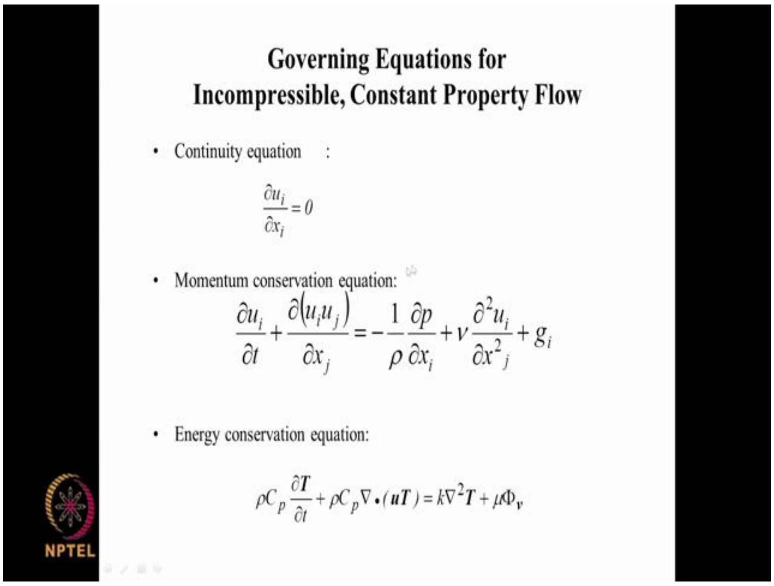

So, this in a sense is the said to governing equations. We will come to the energy equation later.

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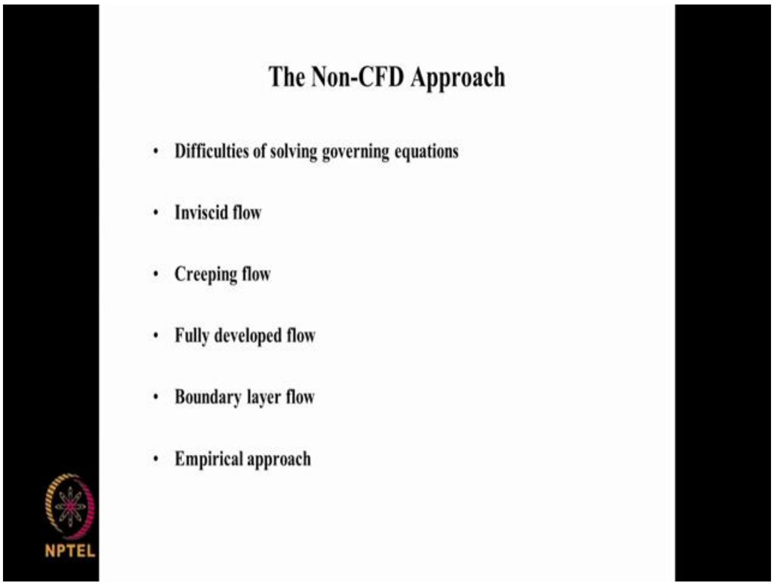

And before we consider this further we would like to draw the distinction between the CFD approach and the non-CFD approach in terms of the problem formation. So, that is, we know these equations, the equations that we solve often in CFD are not the same as the equations that we would solve in your analytical solution in the non-CFD approach, and that is because the equations which govern the fluid flow even though they look fairly simple like this only second order partial difference equation, but this is non-linear equation and you have three coupled equations and you have more problems in the sense of their not being of the same mathematical character all the time. So, we will come back to that. So, despite the apparent simplicity it is not possible to get analytical solution to the exact equations.

So, when we are doing the non-CFD approach then we make certain simplifying assumptions. These assumptions can be like inviscid flow, creeping flow, fully developed flow and boundary layer flow and all these things and we do not have to make these assumptions when we do the CFD. So, which is why it is important to see what we solve in CFD and what we solve in that in the analytical approach and that is the highlight that we are making here.

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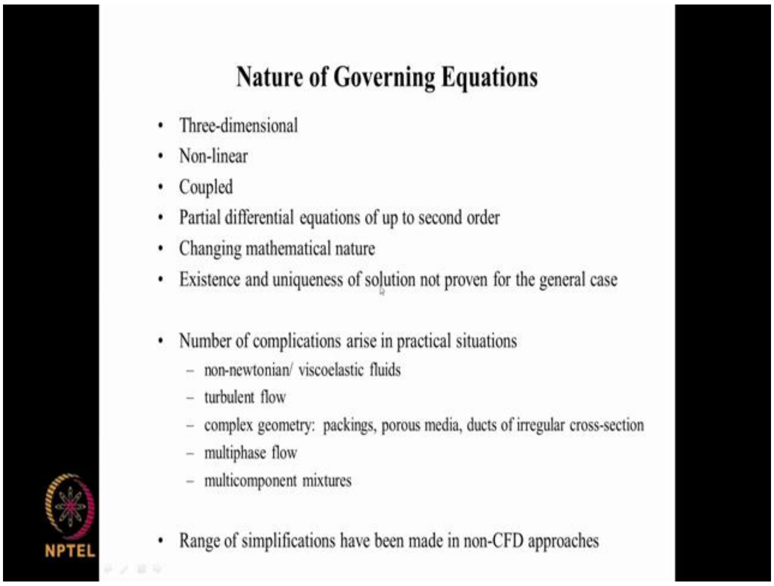

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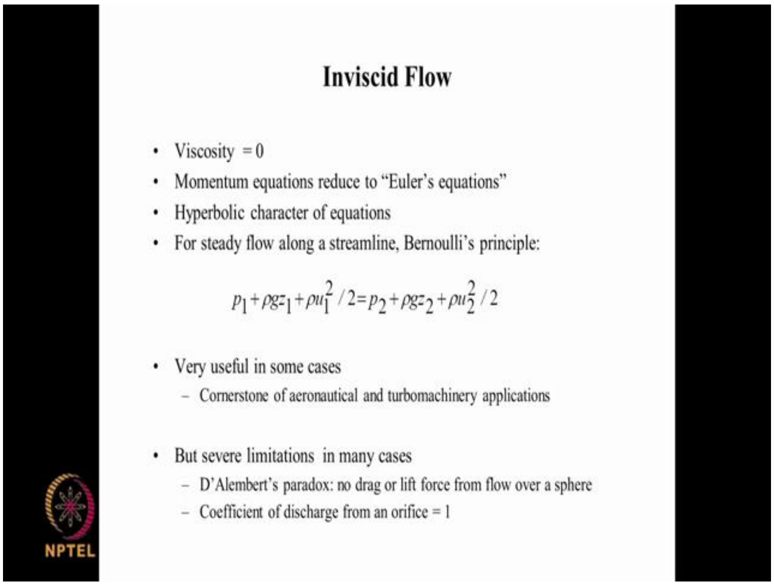

For example, in the distinction that we are trying to draw here, in inviscid flow we make the assumption of viscosity being equal to 0. So, then the momentum equations reduce to Euler's equations and they become hyperbolic type of equation and for steady flow along the streamline you can have the momentum equations being written in the form of Bernoulli's equation, Bernoulli's principle p 1 plus rho gz 1 plus rho u 1 square by 2 equal to p 2 plus rho gz 2 plus rho u 2 square by 2.

So, this is another form of the governing equation which is very useful in many cases and it is been the cornerstone of aeronautical and turbo machinery applications long before computers were invented and widely used, but the inviscid flow assumption is not a practical assumption, for example, in many cases in cases you have the most famous of this is the D'Alembert's paradox which means that if you have inviscid flow then you cannot have any drag force for flow over a sphere and similarly the coefficient discharge from an orifice will be equal to 1. So, those kinds of things are not practical and we know that when fluid flow is dragged and that phenomenon cannot be explained by the inviscid flow assumption.

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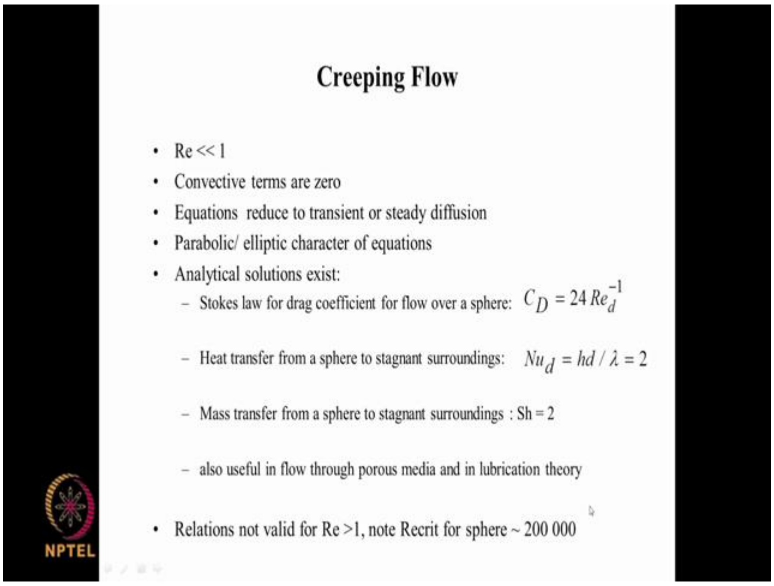

Another assumption that we make is the creeping flow assumption and here we are assuming that the Reynolds number is much, much less than 1. So, that theconvective terms are set to 0 and equations reduce to transient or steady diffusion and these are primarily parabolic or elliptic in nature and you have some analytical solutions, for example, Stokes law for drag coefficient over a sphere is expressed in terms of 24 by Reynolds number and heat transfer from a sphere to stagnant surroundings.

So, you have certain analytical results which are useful, but these are useful in some special circumstances that are Reynolds number much, much less than 1. So, these are not valid for Reynolds number greater than 1 and Reynolds number greater than 1 is not such an unusual thing, for example, the critical Reynolds number at which laminar turbulent transition takes place for a sphere a flow over a sphere is a Reynolds number of something like 200,000 not 2100 not 2300 not 1000, it is 200,000. So, whereas, this relation here, the Stokes law assumption is valid for Reynolds number less than 1. So, in many cases we cannot make use of this.

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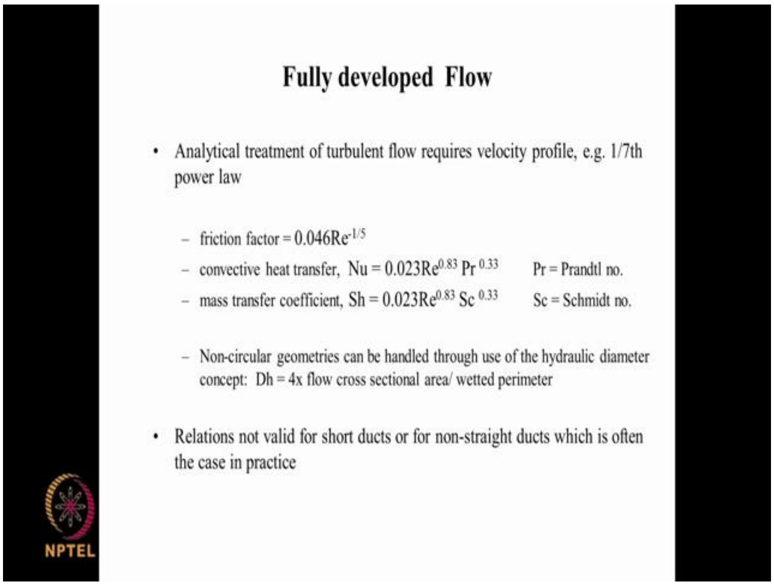

Another assumption that we often make in order to get some insight into fluid flow is the fully developed flow assumption. So, gradients in the flow direction except for driving force such a pressure, temperature and concentrations. We say that the gradients are 0 of other quantities and the gradients of these quantities are constant. So, that we reduce the problem to one-dimensional flow and you have only one non-zero velocity component and you have an elliptic character of equation.

We will come back to that and we have Poiseuille flow in a pipe for which you have friction factor, which is defined which can be obtained as 16 by Reynolds number where the friction factor here is defined as the shear stress divided by half rho u square, where u is the average velocity and you also have the convective heat transfer coefficient expressed in terms of Nusselt number is 4.364 per wall heat flux constant and 3.65 for a temperature wall temperature constant cases. So, you can get these useful relations, but these are valid for laminar flow and only for the case of fully developed flow. If you have developing flow then this not valid, you also have certain relations forturbulent flow, but these are not valid for short ducts or non straight ducts which is often the case as we saw in the very first lecture of this course. We saw certain cases where you do not have fully developed flow and you do not have ducts of a constant cross section. So, in such a case applications of these relations become questionable.

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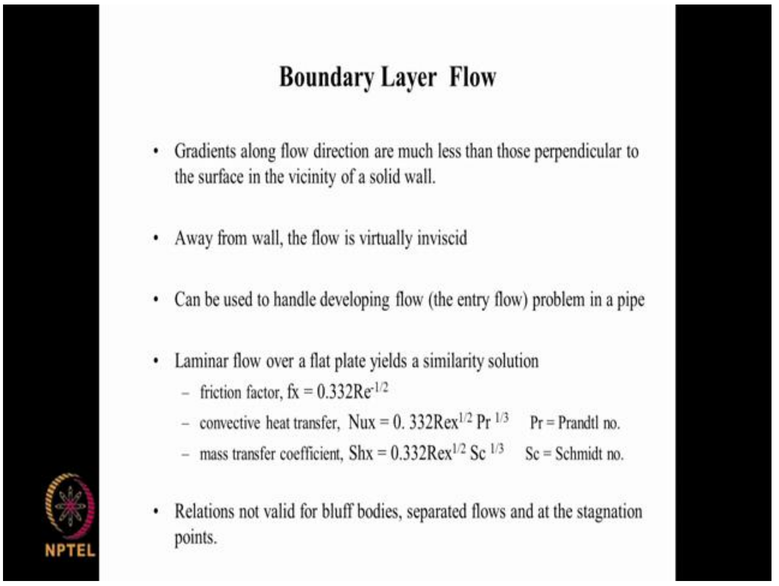

And again one more common assumption that we make is the boundary layer flow in order to get some analytical results and so gradients in along flow direction is much less than those perpendicular flow to the surface in the vicinity of the solid wall and well away from the wall, the flow is virtually inviscid and it can be used to develop, handle developing flow, for example, the entry flow in a problem in a pipe can be done using this kind of thing and you have useful relation for a laminar flow over a flat plate. You have certain relations for the friction factor and then for heat transfer coefficient and mass transfer coefficient, but these relations are not valid for bluff bodies, so that means, in the sense, where you do not have a thin boundary layer and you have strong where a pressure gradients and non-cylinder bodies where the l by d ratio isahsmall its not very large and also for separated flows and very close to stagnation points where the boundary layer thickness is large compared to the distance it covers along the surface. So, these are all useful things, but these are not for the general case

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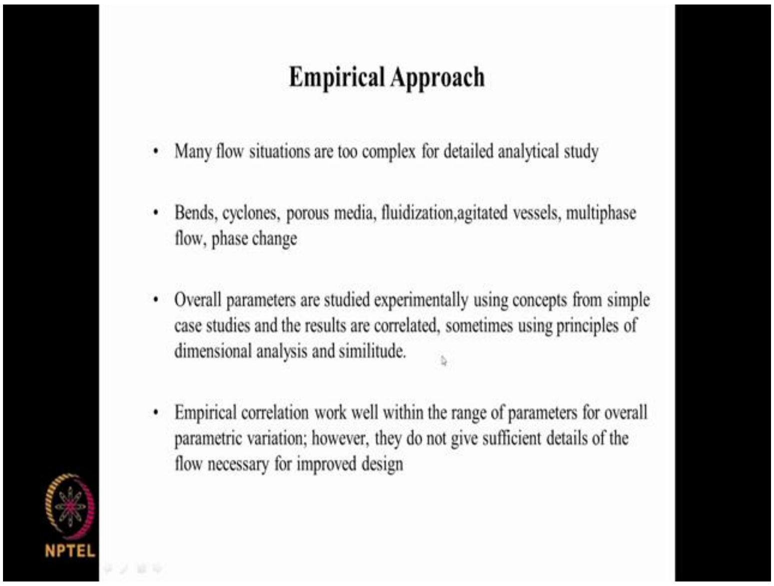

And we can also do the empirical thing.

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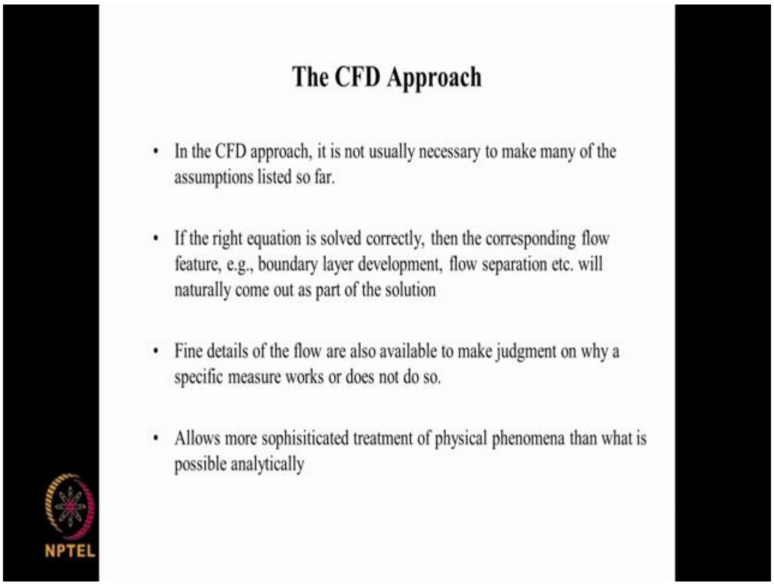

But these are not for the general case in the general case we have only the Navier-Stokes equations that are applicable and these are the things that we try to solve. So, that when we solve the CFD problem, we do not make any assumptions as to whether the flow is the creeping flow, whether it is boundary type of flow or fully developed flow and all that.

These things come in the specification of the flow domain and the boundary conditions which we will see later on and we specify those boundary conditions, and we identify the flow domain if the flow happens to be critical creeping flow because the velocity that is specified is small then the creeping flow solution will emerge from your CFD conclusions and if the flow velocity is high and there is a boundary layer type of behaviour is expected then your CFD simulations will show a boundary layer type of flow solutions.

If you were to draw the velocity profiles, you see that the velocity gradients are very large close to the wall and away from the wall. They are very small, you do not have to make the assumption of boundary layer, you do not have to identify the thickness you just have to specify the boundary conditions and identify the flow domain.

So, we do not make these assumptions. We start with the flow governing equations and then we solve this, so that is important point that I want to mention herethat in when we are doing the CFD we solve the fundamental equations which is a consideration of mass momentum and we will see later on energy equations without any assumptions. So, that gives the strength of argument for the goodness of the CFD solution whereas, if we have to make any assumption about creeping flow or flow and all that then we need to have very low velocities.

In the case of creeping flow and very long ducts in the case of the same cross section, in the case of fully developed flow those things are not found in practice and so those assumptions are defeating when you want to apply to practical case, but we do not have that limitation with CFD, but the equations that we have derived are not universally always valid. So, that is another important point that we have to bring out keep in mind.

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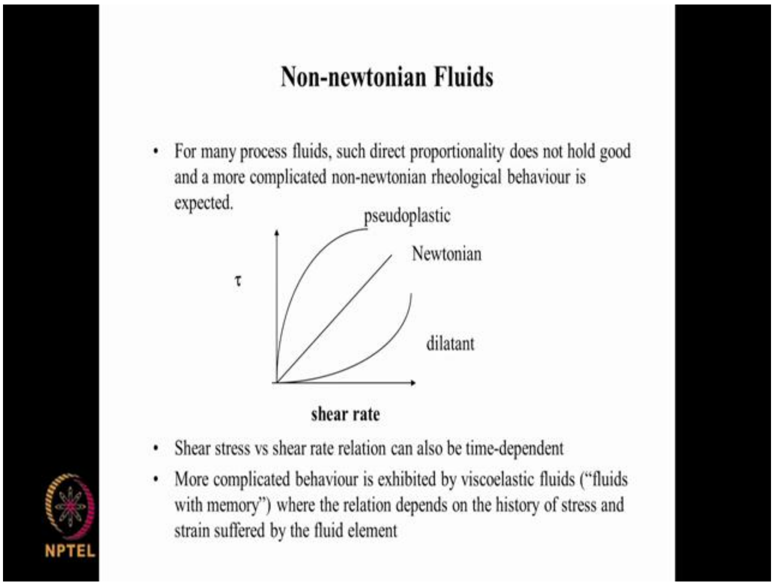

When we apply these equations and one of the simplest assumptions situations where the CFD equations that we are writing down, the Navier-Stokes equation that we are writing down can be questioned is for non-Newtonian fluids.

We have made an assumption of a linearity between shear stress and the shear rate or the strain rate shear strain rate and so that is the linear assumption, but there are number of fluids which exhibits a non-linear variation. This kind or relation where and you can also see this kind of you have a pseudoplastic or dilatant variation and you can have variation with respect to time, and we can also have more complicated behaviour like that of a viscoelastic fluid, where the shear rate or the stress verses shear rate does not depend on the local values.

The stress does not depend on just the shear rate that it is present locally. So, that is train not just equal to mu dou u by dou y at a particular point tau at a particular point xy is not equal to mu dou u by dou y at xy, but it also depends on the previous history, time history of the deformation that the fluid element has gone through.

When we say previous history, we can only handle short distance, short memory kind of things, but it is it exhibits therefore; a solid like behaviour and a fluid like behaviour and these are called viscoelastic fluid.

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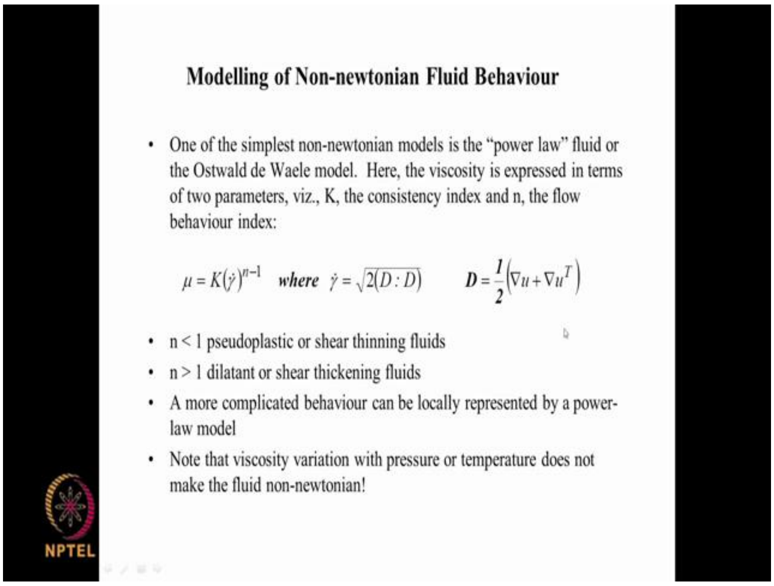

So, that is much more complicated, modelling that is required for this. So, that kind of non-Newtonian fluid behaviour is something that our equations that we are solving in this course are not applicable.

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So, there are also other cases like flow through porous media in which the equations that we have derived are not applicable because part of the flow domain is occupied by the solid only part of bit is available for flow and even then you do not have a simple straight forward flow through that, you can have interconnected porous of varying cross section and all that. So, that requires a special treatment and that treatment is not there in the equations that we have derived, for example, and similarly grid movement calls for extra terms which are related to the velocity of the grids. So, there again these equations in the form that we have derived are not applicable.

Turbulent flow is a very special case where we are getting chaos, chaotic type of flow behaviour from the deterministic equations that we have derived and. So, there is lot of theory behind thatand then we have to have special forms of equations for thatsimilarly when we have the multiphase flow where we do not have a single fluid single phase, but two phases then again we have some special considerations that come into picture.

So, all these practical cases, we have to have an extended set of equations and the equations that we have derived are not any longer fundamental. So, what we claim as fundamental equations are essentially fundamental under certain special cases, but these are generic enough that the solutions of these is still interesting and there is also further mathematical treatment and extra equations that we can derive for all these things, all these physical phenomena and in all the cases the resulting equations can be solved using the techniques that we are going to discuss in the rest of the theory.

So, we would like to look at how to solve this simplified form of the special form of the equations that we have derived, which is the Navier-Stokes equations for a Newtonian single phase fluid and then. So, that armed with that we can then solve the extra equations that come into picture, when we are dealing with all the more complicated physical phenomena.

The objective of this lecture, this part of the module is to formulate the problem or the mathematical problem for especially derive the fundamental equations that we are going to solve for the, rest of the course and these fundamental equations are applicable for single phase laminar flow often Newtonian fluid. In the last part of this last module of this course, we look at the equations that can be used for turbulent flow, but otherwise we are assuming laminar flow and these equations are in a special form, these equations are in the form of coupled non-linear partial differential equations and these equations are 4 in number for the simplest case and involve four variables u, v, w, p in the case of Cartesian coordinate system and they require us to specify the density and viscosity and the two coefficient of viscosity for the general case.

So, with these things we can go into solutions of the equations, but before we do that we would like to bring in the considerations about the boundary conditions and initial conditions because we have these are partial differential equations. So, the mathematical problem is not complete without specification of the boundary and initial conditions and we would also like to look at a allied phenomenon which go along with the fluid flow and the allied phenomenon are the heat transfer the mass transfer and the chemical reactions that are usually of interest to practicing engineers.

It maybe if you looking at a flow over a car then the equations that we have derived are sufficient for very low velocities, but for high velocities, we have turbulence and turbulent flow equations are extra things, but if you are looking at flow of inside a car and if you are looking at how the air conditioner works and all that, then you are bringing it in mass transfer, you need to be able to solve those kind of things. If you looking at what is happening inside the piston and how the fuel that you are putting into that is combusting and then producing power which is generating power to drive the wheels and then all that kinds of things, then you have to consider also the chemical reactions.

So, we are going to look at the mathematical formulation for flows with heat transfer, with mass transfer and with chemical reactions just the very framework of this. So, that we have the basic ideas that are coming in. And then we will briefly look at the boundary conditions, and what you mean by a well posed mathematical problem, where we have a set of equations, enough number of equations for the number of variables that are coming and the appropriate initial boundary conditions which will give us a unique solutions.

So, those are the things that we going to discuss in the second week of the second module dealing with governing equations.