Computational Fluid Dynamics
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Lecture – 15
Equations Governing Fluid Flow in Incompressible Fluid

In the last lecture, we studied the kinematics of fluid motion so as to understand the deformation that a fluid element goes through as flow takes place. And, we have considered three different possible deformations components of deformation: one of which is a pure solid body rotation.

For example, if this is a solid body; you can have pure translation in the x direction or in the y direction. This is one form of a motion that a fluid element goes through. And, in the case of pure translation in any direction, the fluid element does not get any deformation, it remains the same. You can have pure rotation along the diagonal of this. And, here again we can see that, the shape does not change there is no noticeable difference – deformation in this. But, we can also have a pure stretching, so that the volume it becomes elongated in this direction or it becomes elongated in this direction. And, there can be a change in the cross-sectional area. And, there can be a change in the shape from an initial rectangular thing into more rhombus like thing or a general quadrilateral like thing. So, these are different kinds of deformations that are possible.

And, we have seen that, the rates of these deformations can be expressed in terms of velocity gradients – in terms of linear combinations of velocity gradients. And, this is an important aspect because it means that, in the general three dimensional case, all the nine velocity gradients; so, that is, duo u by dou x, dou u by dou y, dou u by dou z; and, similarly for the three v components and the three w components; all the nine velocity gradients – matrix together can define the general rate of deformation that is possible for a fluid element. And, we would like to relate this rate of deformation in a linear way to the stress that is induced by relative motion; and, so that the idea would be that, the stresses that are present in the conservation of linear momentum can be replaced by these velocity gradients. And, since the velocity components are already part of the variables, we can then replace the stresses by velocities and thereby create a system of equations, which is closed in the sense that, we have as many equations as there are the number of variables.

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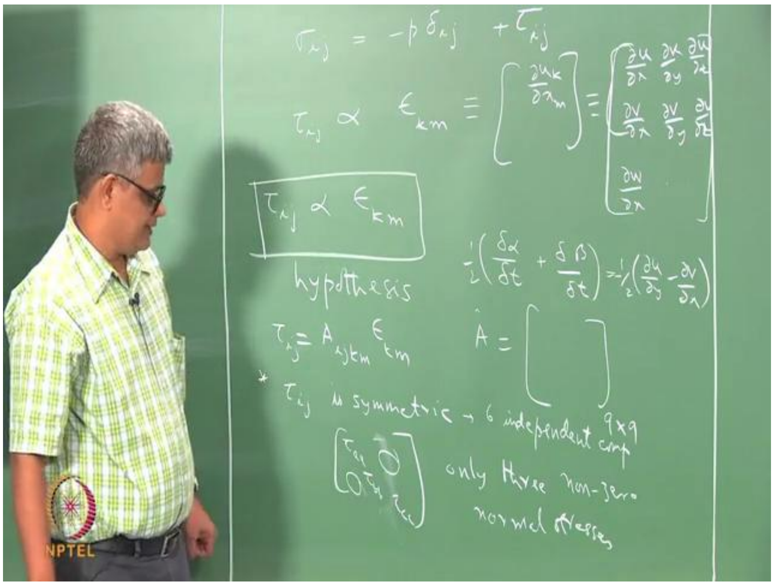

So, here to this extent, in the last class, we decomposed the stress tensor into a hydrostatic component and a viscous stress component, which arises only because of relative motion. The hydrostatic component is present; this pressure is compressible is present, even when there is no relative motion; whereas, this is supposed to be present only when there is relative motion. And, given that, relative motion produces strains – strain rate, which is given by dou u by dou u k by dou x m in the general case. So, this can be written as dou u by dou x dou u by dou y dou u by dou z. And similarly, dou v by dou x dou v by dou y dou v by dou z and dou w by dou x. These are the nine components that are occurring here. And together, this is the strain rate components. And, the idea of putting it here is to make proportional relation between the stresses that are induced by relative motion and the strain rates that are a result of relative motion. We note that the distortion of the rectangular element is possible only when the four corners of this – the abcd's have different velocities. If they have the same velocities; then, they are just translating in a certain direction without any distortion.

When there is relative motion within the fluid element within the fluid continuum; when abcd have different velocities; then, there is relative motion and that relative motion is present in the form of nonzero velocity gradient components here. And, these give rise to different deformations like we have mentioned that, delta – alpha plus – delta div – the rate of deformation is given by delta beta and the average of these two. And, this is expressed as half of dou u by dou y minus dou v by dou x and so on like this. So, this is the shear rate and this is expressed in – in this particular form here – half – a negative sign here. And so, this is a linear combination of two elements that are coming here – dou u by dou y and dou v by dou x.

So, we are now saying that, we are seeking a relation – a linear relation between tau i j and epsilon k m. This is a hypothesis. And, why are we seeking a linear relation? Because a linear relation is a simplest possible that we can have, other than having no relation at all. If it is a non-linear variation; then, it means that, there will be this plus some multiplication of this and that involves addition constants and so on. So, this is the simplest possible. And, the simplest possible appears to work for very common fluids like air and water. So, that is a advantage of this. The simplest common fluids that we can come across obey empirically; based on empirical observation, we can say that, they obey this hypothesis; and, which is therefore – this hypothesis is very useful practically. But, there are number of other common fluids, more complicated fluids like blood, which contains lots of white corpuscles and red corpuscles; and, those kind of additional things, which do not obey this – this linear hypothesis.

Similarly, many polymeric liquids, which have elongated oriented molecular structure with long chain molecules, do not obey this – this particular assumption. But, this is a very useful assumption and it is an empirical hypothesis – it is a hypothesis, which is back by empirical observations of the goodness of this hypothesis. So, with this thing, we would like to have a mathematical formulation for this, so that we can rewrite this tau i j in terms of this deformation strain rate tensor involving the velocity gradients, so that if you substitute this into this; then, the stresses would disappear and you will have only velocity gradients; and, the velocity gradients are not new variables. So, in that sense, that is where we are heading.

And, we mentioned that, this is a tensor and this is a tensor. A general relation would have A i j k m epsilon k m. And, this is a nine by nine matrix. And so that – and we also mentioned that, this is – these are material properties like viscosity that we are familiar with. But, this many material properties are very difficult to get empirically. And, we also would like to may study this particular thing further and take advantage of special features of the matrices that we are trying to relate. And, what are the special features. We know from angular momentum that, tau i j is symmetric. So, that means that, there are only six independent components. And, in the special case, where we do some matrix operations and rotate this and then write this in the principal coordinates system; then, we will only have tau a tau b b and tau c c. And, all these things are 0. So, that is, we can – this is in the general case of x and y directions. And, when we do principal component analysis and we put this in a diagonal form; then, we will have nonzero components in the shear stresses and only normal stresses.

So, this is a special coordinate system aligned not in the i j, but in the – in principal coordinate direction here. And, here we have only three normal stresses compo – three normal stress components – only three nonzero normal stresses. And so, that means that, here in this special thing, which we are doing by matrix multiplication or matrix operations on this, so that we are not introducing any new features into this; we are looking at three independent components here. So, instead of six independent components, we have three independent components.

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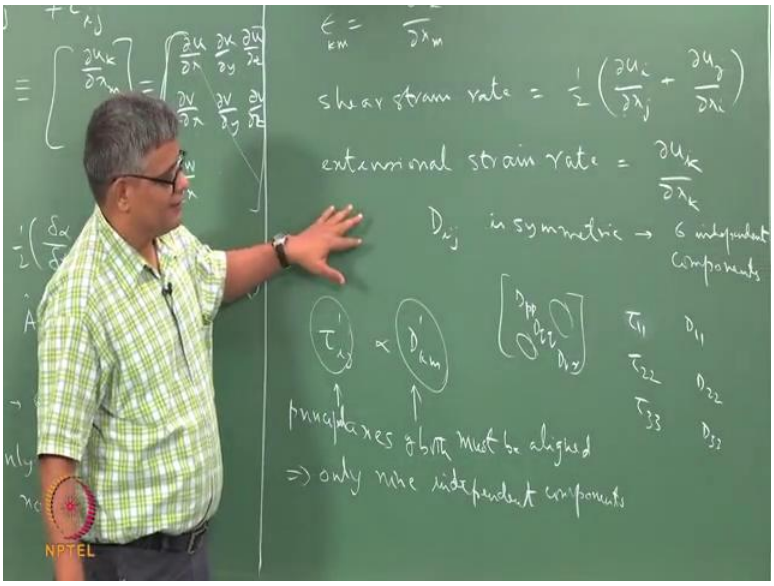

And similarly, the epsilon here k m is dou u k by dou x m. And, in general, this is not symmetric because it is not necessary that, dou u by dou y is the same as dou v by dou x and – in that. But, when we look at our objective, is not to take relation between this and this, but to seek a relation – linear relation between stress and the deformation rate tensor. Now, we have said earlier that, if there is a pure solid body rotation here, the shape remains always rectangular and there is no deformation in this. So, you could say that, pure solid body rotation does not give rise to any change of shape; it does not deform the rectangular element. Similarly, pure translation will not deform this. So, the only thing that is actually changing the shape and thereby creating strain on this is the shear strain rate and the extensional strain rate.

So, the shear strain rate is expressed as half dou u i by dou x j plus dou u j by dou x i in the general case. And, we can see that, this is symmetric. And, extensional strain rate is dou u k by dou x k. So, that is dou u by dou x dou v by dou y dou v by dou z. And, those are these components here. And, shear strain rate has this plus this going together and this plus this going together and this plus this going together. So, the same component – it is a summation of – it is a summation of these two and summation of these two and summation of these two. So; that means that, a combination of these two – the deformation rate tensor D i j is symmetric. So, if D i j is symmetric. Then, this is also has six independent components.

Now, this also can be decomposed into the principal components into – this can also be rotated and – so that we have only D p p, D q q and D r r; all others are 0. So, we are looking at essentially a matrix rotation – matrix rotation operation in such a way that, a general tensor with all the nine components are converted into a special orientation in which only the diagonal elements are nonzero and all the others are zero. So, this means that, now we have a set of coordinate axis in which we have only three stress components. And, the another set of coordinate axis in which only three nonzero components of the strain rate – deformation rate tensor. And, we would like to have a relation between this tau i j versus D prime k m; where, these are these – special rotate – specially rotated operations here.

Now, when you are seeking a relation between these two, which is linear; then, it is necessary that, the principal coordinates of these and the principal coordinates of these are the same. So, principal axes of both must be aligned. If these are not aligned, then you cannot have a linear relation. So, that means that, we can now say that, we are seeking a relation between not 9 by 9 components, not even 6 by 6 components; we are – we can only have – if we say that we are going to have a linear relation between this and this; mathematically, we can only have three independent components here and three independent components here. So, that means that, we can only have nine independent values, which will describe this general relation, which is applicable in any coordinate system. But, the same constituents must also be applicable in the principal axis coordinate system here. So, in that sense, we have only nine independent components. And what are these nine independent components? We can put these as sigma tau 1 1, tau 2 2, tau 3 3, because only diagonal terms are there; and similarly, D 1 1, D 2 2 and D 3 3. Now, what kind of linear – generically linear relation that is possible?

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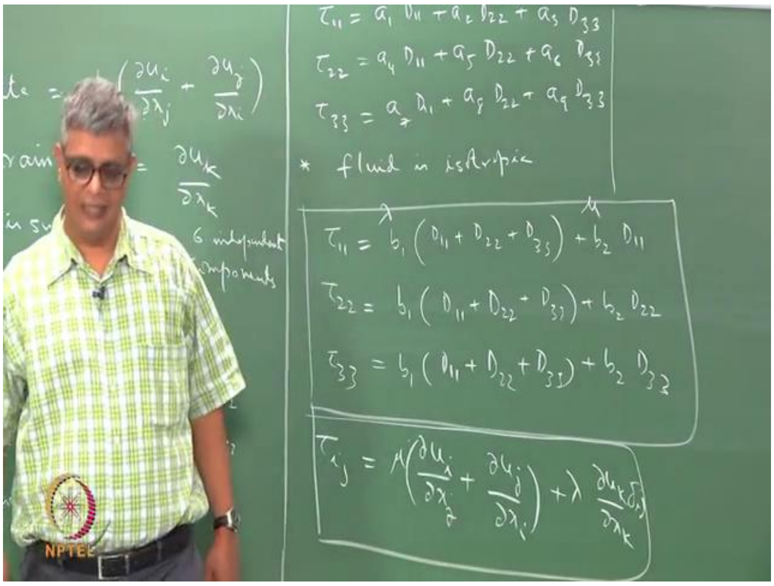

So, we can write for example, tau 1 1 is equal to a 1 times D 1 1 plus a 2 times D 2 2 plus a 3 times D 3 3. So, this is for different values of a 1, a 2, a 3, which are constants. We have a linear relation in which tau 1 1 is expressed as a linear combination of all the three possible variables on the right-hand side. So, when you have all the three possible – all the three variables on the right-hand side (Refer Time: 18:56) here; then, that is a most generic relation that is possible. If you put D 1 times D 2; then, that becomes – maybe that is not a linear relation there. Similarly, we can say that, tau 2 2 is also related only to the same D 1 D 2 D 3, but using a different set of constants, for example, a 4 D 1 1 plus a 5 D 2 2 plus a 6 D 3 3. And, tau 3 3 is also written in terms of the same things. So, you have a 7 D 1 1 plus a 8 D 2 2 plus a 9 D 3 3. So, here we have nine constants a 1 to a 9, which define a general linear relation between the three rotative stress components and the three deformation strain rate components.

So, now, these you cannot reduce it further without assumptions. So, here we make one more assumption, which is that, the fluid is isotropic. So, when we say the fluid is isotropic; then, what we are saying is that, application of a force on this produces a deformation in a solid body. And, in a case of a fluid, it produces a rate of deformation. And, what we are saying is that, if we apply a tensile stress in this direction; then, in a solid body, it produces a deformation this direction, in this direction and also in the z direction. So, in an isotropic medium, it does not matter whether you apply the tensile stress here and here or in this direction or in this direction; the response of this surface is the same in all directions.

So, that means that, if you apply a tensile force here; then, the body elongates here and then it also contracts on this side and contracts on this side. The contraction along this side – the z direction must be the same as the contraction in the y direction. And similarly, if you now apply the stress in this direction along the direction, which you applied the stress, you have a certain extension stretching. And, in the other two directions, there is a small contraction. And so, the contraction that is produced by the application of a stress in this direction in the z direction; so, that is, you apply a stress in the y direction, that is, a tensile stress; it produces a contraction in the z direction.

Now, similarly, you apply the same stress – same tensile stress along the x direction. It again produces a contraction in z direction. If the body is isotropic, it does not matter whether the contraction produced is produced by application of a tensile stress in the y direction or x direction. If the stress is the same, then it produces the same amount of contraction in the z direction. Similarly, if you apply in the z direction certain tensile stress or a compressive stress, and that produces a corresponding deformation in the transverse direction; then, the amount of deformation that takes place, that is produced, must be the same if the same tensile compressive stress is applied in the other direction; so, in the other orthogonal direction.

So, what we are saying is that, in an isotropic medium, we can – we are allowed to distinguish between a deformation, which is produced in the direction of application of the stress and a deformation, which is produced in the transverse direction. In the transverse direction, you cannot distinguish between y and z. So, there can be one deformation, which is in the direction of the stress and there is the same deformation, which is produced by the same amount of stress in the other two directions. The two transverse directions are similar. So, that is the idea that we can take it here. And, that kind of deformation, that kind of meaning, can be rewritten in this way – that is, tau 1 1 is b 1 times D 1 1 plus D 2 2 plus D 3 3 plus b 2 times D 1 1; tau 2 2 is b 1 times D 1 1 plus D 2 2 plus D 3 3 plus b 2 times D 2 2; tau 3 3 is b 1 times D 1 1 plus D 2 2 plus D 3 3 plus b 2 times D 3 3.

Now, you look at this relation here. In this relation here, it is possible for tau 1 1, tau 2 2 and tau 3 3 to be different, because although they are related to the same set of variables on the right-hand side – D 1 1, D 2 2, D 3 3 here; even though for a given set of values of D 1 1, D 2 2 and D 3 3, this contribution is the same. There is also a contribution coming from this; so, tau 1 1. These three stresses will be equal only if these three are equal. If these three are different; then, you can have different stresses that are possible. So, this is a generic description of a linear relation between the principal stress components and the principal deformation rate tensile components involving only two constants: b 1 and b 2, and which is isotropic, because you have a deformation D 1 1 associated in the direction of stress and you have something like a transverse component, which is coming here. So, in that sense, it – this is a relation between the same three components on the left-hand side and the same three components on the right-hand side, which is written with only two independent constants. And, this kind of relation is applicable for an isotropic medium.

So, now, what we are saying is that, this linear relation between a symmetric stress tensor and a symmetric deformation rate tensor here for an isotropic medium, would have only two independent constants. And normally, we put this as lambda here and this as mu. This is our dynamic viscosity. And, this is known as the second co-efficient of viscosity. And, this relation expressed in – in the principal coordinates will be like this. But, when it is transformed from the principal coordinates into general i j coordinates, can be written as tau I j equal to mu dou u i by dou x j plus dou u j by dou x i plus lambda times dou u k by dou x k. This is a relation between the stress induced by the relative motion and the strain rates – deformation strain rates that are induced by the stresses. And, that relation is linear and it is for an isotropic medium, involving the two coefficients mu and lambda here; and so, delta i j here – the canonical delta function. So, this is the expression that we have.

Here usually this particular thing dou u k by dou x k is very small for most fluids. So, this becomes negligible. And so, we do not need to really know the value of lambda in most cases. So, this – this can be obtained for some simple gas species as having certain value. But, otherwise, this is not possible to; it is not possible to get an accurate estimate for this; whereas, viscosity is something that can be measured in the special cases of where we impose a velocity gradient by rotation between two cylinders and so on.

We can get estimation for the velocity. So, this is easily measured. So, we measure the viscosity and then we make use of this relation here. So, this relation now can be substituted into this. And, that gives us minus p delta i j plus mu times this and lambda times this. And, what we then have is that, the stresses on this side are expressed in terms of pressure and velocity gradients. So, velocity gradients are not new variables; and, material properties, which are the two viscosity coefficients: the first coefficient of viscosity and the second coefficient of viscosity.

So, in the next lecture, we will see how when these are substituted into the governing equation. We will end up with the conservation of linear momentum, which has only extra variable as a pressure, so that the three momentum equations for the three velocity components plus the continuity equation together constitute a set of four equations.

Thank you.