Computational Fluid Dynamics
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Lecture - 14
Kinematics of deformation in fluid flow

So, in the last lecture, we have examined the conservation of linear momentum principle, and we have come up with this statement a verbal statement and mathematical expression, which encapsulate the conservation of linear momentum. And we also applied angular momentum to reduce the number of unknowns in that; still we are left with too many unknowns and we decided to bring in some empirical loss, which do not have the same sense of fundamental applicability in the most general case. But still fortunately, we could find some statements of widely applicable, what can be called as constitutive relations, which described the medium that we are dealing with. So one such constitutive relationship, which describes the properties of the medium in the response of the medium to an applied stress is what we are going to look at.

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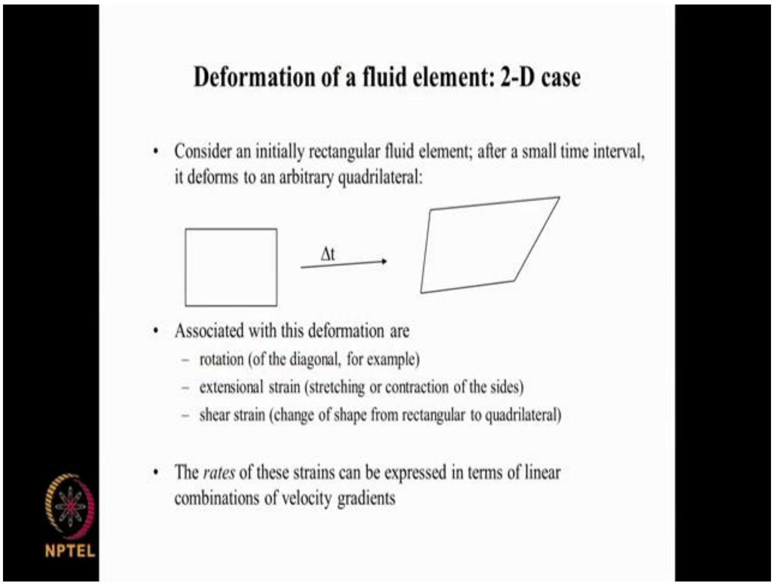

For a fluid element, by looking at the deformation of the fluid element and deformation fluid elements and all these things is in general 3-D, but in order capture the essence of it we are going to consider the two-dimensional case here. So, we are looking at in this picture here, we are looking at a rectangular fluid element. This rectangular element actually it denotes four particles here and over a short time the particles are moved and if you join them by lines, you get a shape, which is not the same as the first case. This will be the same as the first case. Only in the special case, where all the four points you have exactly the same velocity different component. That is the special because, that means, that there is no change in the velocity either in the y-direction or in the x-direction and that is not a it is more like a trivial case.

In the general case, where their velocity is changing and you have velocity profiles and all that. Then you expect these four corners to have different velocities and as a result of their different velocities you can have a different shape after a small time delta t. So, now, if you forget about the fluids and then if you look up look only at this rectangle, which has been deformed into this quadrilateral here. Then you can identify certain types of deformation. So, associated with this deformation are rotations. So, means that this fluid body can rotate and. So, one can also see the rotation for example, by looking at this diagonal here and is the angle of this diagonal with the horizontal has it changed.

If it has changed then you say it is rotation, because in a rotation this is going up like this this is going up like this coming round and round. You can also monitor that rotation by that change of angle of the diagonal. Extensional strain, so which also in plain terms it is stretching or contraction. If this stretch is in the x-direction and y-direction together then, the area of this rectangle changes and we can also see that this become bigger, so that means, that there is a extension. We can also think of a shear strain, so where there is a change of shear is what we normally call as shear is the change of shear and you can see that this is rectangle,. So, you can see that the angle is 90 degrees here and here this a definitely a different shape here.

So, as a result of particle motion, fluid motion where within the fluid you have velocity gradients, if the velocity gradients are there then there are all the three possibilities there can be rotation, there can be extensional strain, there can be shear strain suffered by the fluid particles. So, the idea is that these rates of strain are what we would like to relate to the stresses that are acting on the fluid, which are causing this velocity changes and it is what is important is that these rates of strain can be expressed in terms of velocity gradients.

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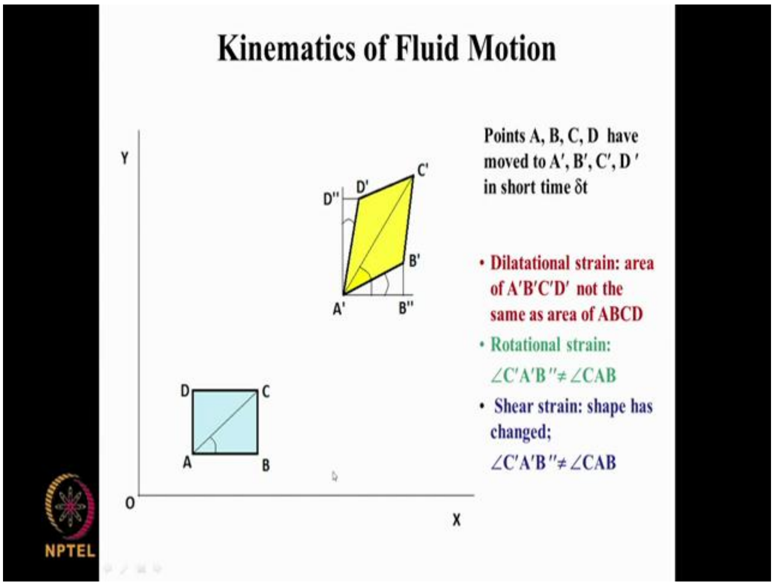

So, this is what we will see in the kinematics of fluid motion and we have put the essentially the same thing here and some color. We have x y coordinate systems and you have points A, B, C, D with some defined diagonal angle here and after short time A has gone to A prime, B has gone to B prime here and C has gone to C prime and D has gone to D prime.

So, these are the new positions of the same particles after short time delta t. Now if you join them by linear lines then you can see that the initially rectangle cross section has become more like a parallelogram, but it can be anything it is quadrilateral. Here we can, now become quantitative, we can say you have dilatational strain, dilatational or extensional strain, which means that the area of A prime,, B prime, C prime, D prime of this quadrilateral is not the same as the area of A, B, C, D. So, rotation strain is given in terms of the change in the angle C prime, A prime, B double prime with respect to the angle C A B both are angles with respect to the horizontal, this much here and this looks like it slightly more. So, when the two are not the same then you say that this fluid element has suffered rotational strain. Similarly, the shear strain is change of shape and the shape is one measure of the shape is the angle between the two vertical sides and the horizontal side.

So, the angle D A B, which is 90 degrees here, has become shrunk and this has shrunk, because this A B has moved up. So, that is B prime has moved up and it is come like this and D has moved to the right here. If D has moved to the left by exactly the amount that D has moved up here, in such a way that there is 90 degree is maintained they would not be any shear strain. But while D has moved in this direction towards the diagonal even D has moved to the towards the diagonal, which is given rise to this particular change of shape. If D moves away from the diagonal by the same amount of this then, there would not be any change of our shape. So, shear strain is essentially dependent on by how much this has moved in this direction and by how much this has moved in this direction. So, it is all related to the angles that are present here and this provides a quantitative definition. Now how do you make it how do we relate it to the velocities and velocity guidance.

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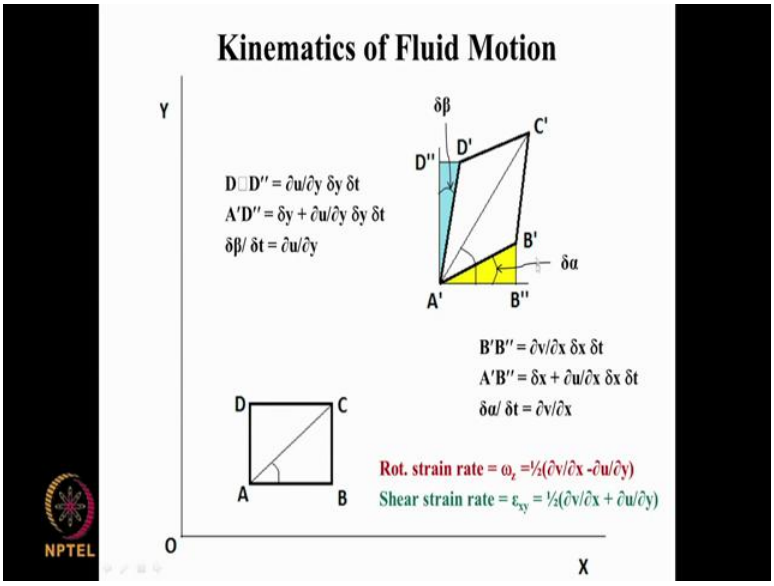

So, that is what we have saying here. Now we need to here we have identified angles delta alpha and delta beta and what is delta alpha delta alpha is the new angle that the side A B makes with a horizontal here it is zero and here it is delta alpha. That is because B has moved up with respect to a relative to A both A has moved up to A prime and B has moved up to B prime. So, both of them have moved, both in the y-direction and also x-direction. So, A has moved this much in the x-direction it has come up here and B has moved this much in a x-direction this come up here. But it has moved relative to A more in the vertical direction so and how much more it has moved is B prime B double prime.

So, this distance is how much D has moved up relative to A in the vertical direction. This was because the vertical velocity, v velocity component at B is greater than the v velocity component at A. If the two vertical components are the same then, they will be moving up at the same right they will reach the same height in the time delta t. But because the v velocity component is higher this has moved up here. So, you can say the distance B prime B double prime is dou v by dou x times delta x times delta t. Now what is this dou v by dou x. So, dou v by dou x is rate of change it is a change of v with respect to x. So, that is particles A and B are displaced relative in the x-direction. So, you have a velocity v here and you another velocity at this different x. So, dou v by dou x is indicates how much the vertical velocity has changed as you move from A to B in the x-direction,. So, that is dou v by dou x. So, this is rate of change of v with respect to x in the x-direction and that expands of x A B that distance is delta x. So, dou v by dou x times delta x is the delta v the difference in the vertical velocity component between point A and point B. That relative velocity has acted for a time of delta t and. So, it is the delta is more than this would go even higher if the delta t is less than this would be only here. So, this B prime B double prime is arising, because of the change in the vertical velocity delta v between point B and point A. Over a time delta t this has led to a displacement of D relative to a of B prime B double prime, which is given by dou v by dou x times delta x and delta t. Similarly, A prime B double prime because we are looking at this angle delta alpha, this angle delta alpha is supposed to be small because we are looking at small delta t small change in time.

So, you can say the delta alpha is roughly equal to B prime B double prime divided by A prime a double prime. So, that sine alpha for small alphas is tan alpha is given by this. So, we can estimate this delta alpha as B prime B double prime divide by a prime B double prime and we have already got an estimate for what B prime B double prime is that is given by this. A prime B double prime would be having been the same as the distance A B. If both A and B has the same horizontal velocity. If a is traveling slowly in the horizontal direction B is traveling fast in the horizontal direction, after a small time delta t we will have gone way to high. So, that is the original horizontal displacement will be different from the original value here. So, this A prime B double prime this distance is the original distance delta x plus dou u by dou x time delta x. So, that is the delta u between particle B and particle A.

So, if u is changing in the x-direction and the rate of change is going by dou v by dou x, then the delta u between B and u is given by dou u by dou x times delta x. So, this would have given rise to a relative displacement in the x-direction of this velocity times delta t and that is this. So, if dou v by dou x is positive, so that means, as you go from a to B the x velocity is increasing then B is moving faster than a in the x-direction. If dou v dou x is negative, then A is moving faster and B is moving less fast. So that means, that the distance will be decreasing and that is what this is saying the original distance with delta x plus this distance that as the change in the distance in the x-direction, which is this term.

So, now in the limiting case of delta t tending to zero delta x A prime B prime B double prime is a same as A B. So, we can say that this is roughly equal to delta x for small values of delta t it is all about small value of delta t. So, you can say that a prime B prime B double prime is still about delta x. So, an estimate of delta alpha is B double prime B prime B double prime divided by A prime B double prime. So, that is dou v by dou x times delta x delta t divided by delta x. So, this delta x and this delta x gives you cancel out and you get dou delta alpha will be equal to dou v by dou x times delta t. So, the rate of change of delta alpha the change rate of change of this angle dou alpha by delta alpha by delta theta is given by dou v by dou x.

Why do we talk about rate of change because are hypothesis is that the rate of change is proportional to the stress. So, that is why and this deformation is given by is related in terms of how much this is changing what at what rate this is moving in this direction, at what rate this is moving in this direction like that. So, delta alpha rate of delta alpha by delta t is dou v by dou x. If you come to this triangle here, again we need to have this estimate delta beta this delta beta is happening because D has a velocity a horizontal velocity, which is greater than a by how much that depends on the rate of change of u in the y-direction, so that is dou by dou y. So, dou v by dou y times delta y will give you delta u between D and A. That delta u has occurred over a time period of delta t. So, that is the horizontal displacement D prime D double prime is given by dou u by dou y times delta y times delta t.

So, in the case, where dou v by dou y positive then D prime D double prime will be positive. So that means, it will be going in this direction. If dou u by dou y is negative, then D prime will be on this side. So, whether it turns to the right or to the left depends on how the velocity is changing. If dou u by dou y positive, then it will be going in this direction. If dou u by dou y is negative, then D prime will be on the left side of this vertical and it all depends on velocities velocity gradients. Similarly, this distance a prime D double prime is the original distance plus dou u by dou y times, this should be dou u dou y not dou u by dou y times delta y delta t. This is again in the limiting case of delta t times zero this equal to delta y and therefore, delta B by delta t the rate at, which this is going in this direction, is given by dou u by y.

So, now we can define rate of rotation strain omega z in this 2-D case as half of dou v by dou x minus dou u by dou y. So, that is half of this minus this and shear strain rate as half of this plus this. So, that is dou v by dou x and dou u by dou y. So, what we have seen here by examining the kinematic of this case, that the strain rates rotation strain rate and the shear strain rate and even the extensional strain rate can be expressed in terms of the velocity gradient extensional strain rate is the rate at, which the distance is increasing. For example, if you say that a prime B double prime is this and a B is delta x then the rate at, which the distance is increasing, is this minus this divided by delta x delta t. So, this minus this will give you dou v by dou x times delta x delta t is the amount by, which the distance between a B has changed over delta x over delta t. When we talk about strain, it is the original length divide change in length divided by original length ok.

This is a change in length original length is delta x. So, we divided by delta x we get dou u by dou x times dou t. So, rate of extensional strain is dou u by dou x times delta t that is change in strain divided by time delta t is that will give you dou u by dou x. So, we can say that extensional strain rate in the x-direction is given by dou u by dou x, shear strain rate in this plain is given by half of dou v by dou x plus dou u by dou y, rotation strain rate is given by half of dou v by dou x minus dou u by dou y. All these strain rates are can seem to be expressed in terms of linear combinations of the velocity gradients and that is what kinematic of fluid motion tells us.

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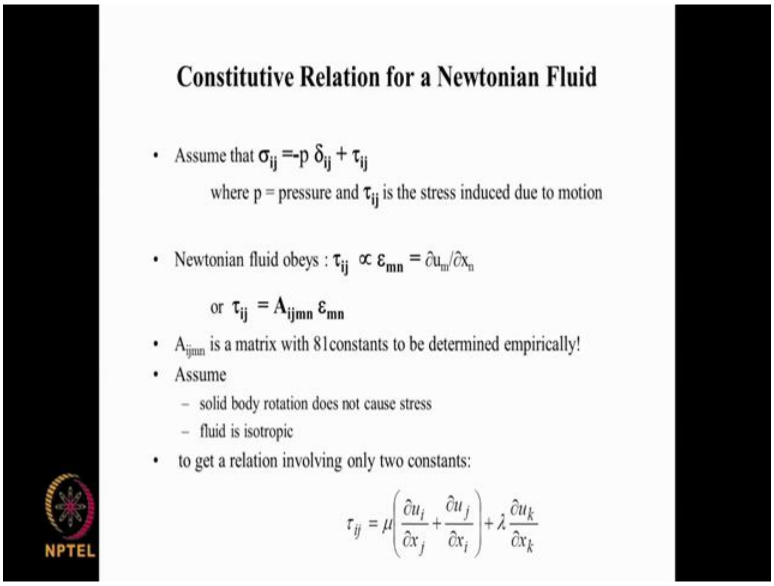

On the way, this we can go back to our original formulation and we can very briefly touch upon how we can derive this Navier-Stokes equation. So, we have the stresses sigma i j and we first decompose this into two components pressure, which is a known property of fluids and we know the pressure is compressive. So, we put it as minus p delta i j we also know that pressure is normal.

So, that is why this delta i j is known as the conical delta function and it is equal to one when i equal to j and when it is i is not equal this is 0. So, that means, that if you put sigma x x i equal to x and j equal to x so that means, that this take value of 1. So, sigma x x will be equal to minus p plus tau x x, but if you say sigma y z then x i equal to x n i equal to y and j equal to z. So, this becomes delta y z and i is not the same as z j. So, this will be equal to 0. So, sigma y z is equal to tau y z, but sigma x x is equal to minus p plus tau x x. So, in that sense minus p becomes the compressive stress adding to sigma x x sigma y y sigma z z, but it does not appear in the shear stress components sigma y z sigma z x sigma y z and all these things ok.

So, we are decomposing the stress into a hydrostatic stress component and I have discussed stress component, which is induced by relative motion between within the fluid. So, the overall stresses are decomposed into a component, which exists even when there are no velocity gradients that are the hydrostatic component a stress component which arises only, because of relative motion why relative motion. Because we have seen that relative motion induces this shear strain rates and extension strain rates and rotation strain rates. So, whenever there is relative motion whenever there are velocity gradients there is strain rate and whenever there is strain rate we are introducing we are saying that there is a stress. So, that stress, which is produced by relative motion, is the tau and that that stress, which is existing even when there is no relative motion is the pressure. So, we decompose to these two these things and we bring in an expression for this tau i j and we say that this is proportional to the velocity gradients.

So, this is you can see that the velocity gradients are we have dou v by do x here dou u by dou y and here we have dou u by dou dou x, which is coming here so, in the general three dimensional cases, all the nine velocity gradients. So, that is dou v by dou x dou u by dou y dou u by dou z dou v by dou x dou v by dou y dou v by dou z all those nine components are important. It is a linear combination of these things that can give rise to strain rate or shear strain rate and extensional strain rates and all that. So, when we are looking at a linear combination linear relation between the stress and the induced strain rate we can look for a general relation between the stress tensor and the deformation tensor or strain rate tensor, which is deformation rate tensor, which is given by epsilon mn and epsilon mn is dou u m by dou x n. So, that means, that it depending on the value of m and n it will there will be nine components just as we have nine components here.

So, when you are looking at a linear relation between one stress component and another stress component the most generic relation will have a set of a coefficient matrix, which contains nine stress components here and nine strain rate components. So, this is a tensor this is a tensor. So, there are 81 coefficients that appear in this. So that means, that a is a matrix with 81 constants. All this 81 constants are supposed to be properties of the fluid, material properties just as spring constant is a property physical property of the spring. Here we are talking about the fluid property viscosity and that viscosity has there are eighty one types of viscosity. That is what we are saying here. So, 81 constants is almost impossible to determine empirically for any fluid. So, we make certain assumptions and we also see that when you really apply the principles of this relation and mathematical relations that are admissible. Then it is not really eighty one constants by making the assumptions that solve body rotation does not cause any stress, it can be said that it is partly an assumption and partly condition of the existence of linear relation between stress and strain rate. So, it is not possibly an assumption.

So, it is a consequence of the fact that we are seeking a linear relation between stress and strain rates, where stress tensor is a symmetric here that gives rise to this. If you further assume that fluid is a isotropic, then one can show that these eighty one constants are not eighty one independent constants there are only two independent constants and you can apply you can derive a relation between this involving nine individual components here and nine components here. In this very simple form and this simple form has only two constants mu and alpha lambda here. So, we are saying that tau i, j here is mu times dou u i by dou x j plus dou u j by dou x i plus lambda times dou u k by dou x k.

So, in the next lecture, which will be in the form of tutorial we will substitute this into the momentum equation that we have derived earlier and then derive what we have known as Navier-Stokes equations. And we will see this Navier-Stokes equations together that is conservation of linear momentum and angular momentum along with the these assumptions about isotropy and the continuity equation together, we will give us a set of four equations involving four unknowns. That is what we are going to do in the next lecture in the form of a tutorial, so that we have time to digest it.

Thank you.