Computational Fluid Dynamics.
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Lecture – 13
Forces acting on control volume

So, in the last lecture we have seen, we have introduced the concept of stresses and how we need to have two indices for this. And we also identified, what is the stress components acting in the x direction, in our cubical box like control volume, and the idea of this are to see, how these stresses acting in the direction will contribute to change in the rate of change of linear momentum. So, we are going to do that.

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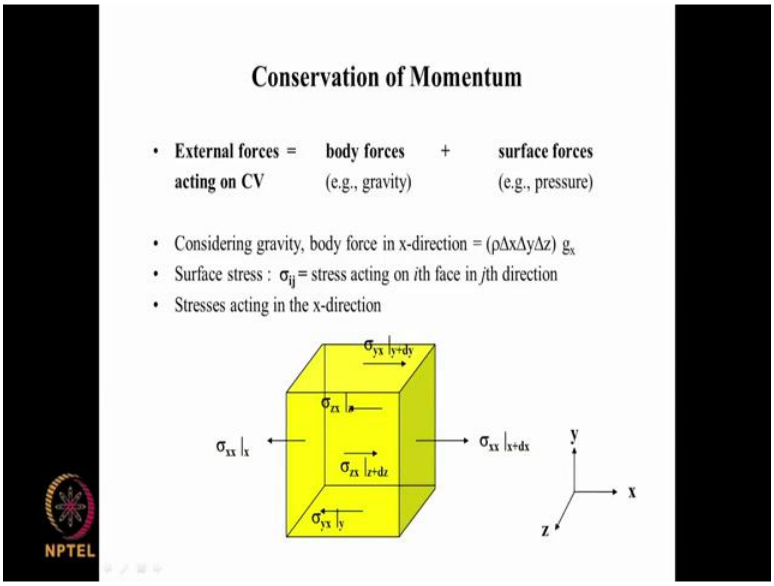

This is our box here, and we identified the six stress components, acting in the x direction on each of the six faces. And they are whether in which direction, they are acting in x, whether in the positive x direction or negative x direction, depends on whether it is a negative face or a positive face as explained previously, and that is also not here.

So, now we would like to multiply each stress component by the corresponding area here. So, if we say that what is the net force acting in the x direction, from the stress which is acting on this positive face? This will be given by sigma x x at x plus delta x times the area of this face, which is delta y times delta z. Similarly the stress which is acting in the negative x direction by the stress acting on the left face will be sigma x x at x times delta z times delta y, but it is negative. So, this is minus delta x like this. Now if you look at the top face; the area of the top face is delta z delta x times sigma y x at y plus delta y, and that is acting in the positive x direction, and the bottom face. Again you have the same area delta z delta x times minus sigma y x at y minus, because it is acting in the negative direction.

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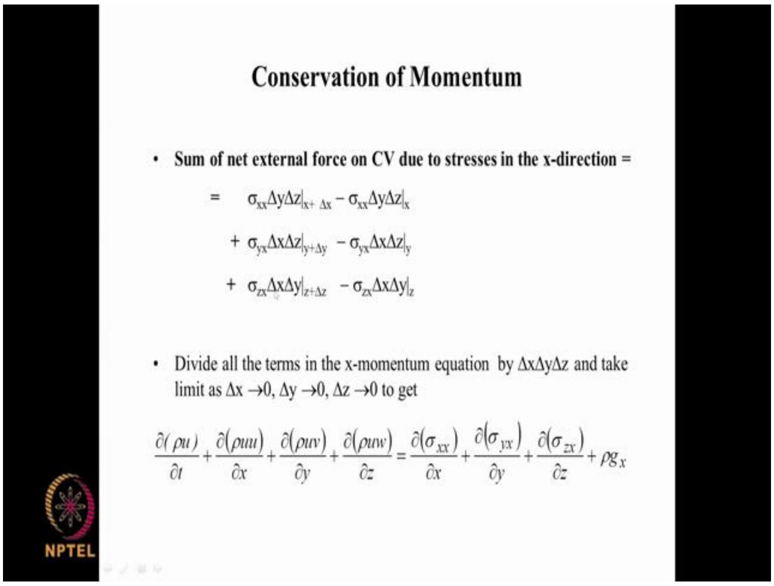

So, we can write down the external forces; some of net external forces on the control volume due to stresses, acting on the out of faces, enveloping faces of this control volume. Stresses in the x direction only, because only the x any force acting in the x direction will lead to change in the linear momentum in the x direction which is what we considering. So, we can say that this is equal to sigma x x at x plus delta x times the area which is delta y delta z minus sigma x x at delta y delta z at x plus sigma y x at y plus delta y times area minus sigma y x times the area by y like this. So, we have here mathematical expression for the sum of net external forces, and we have this component from the net external body force. And then we have expressions here for net momentum flow rate in net momentum flow rate out, and the rate of accumulation of x momentum.

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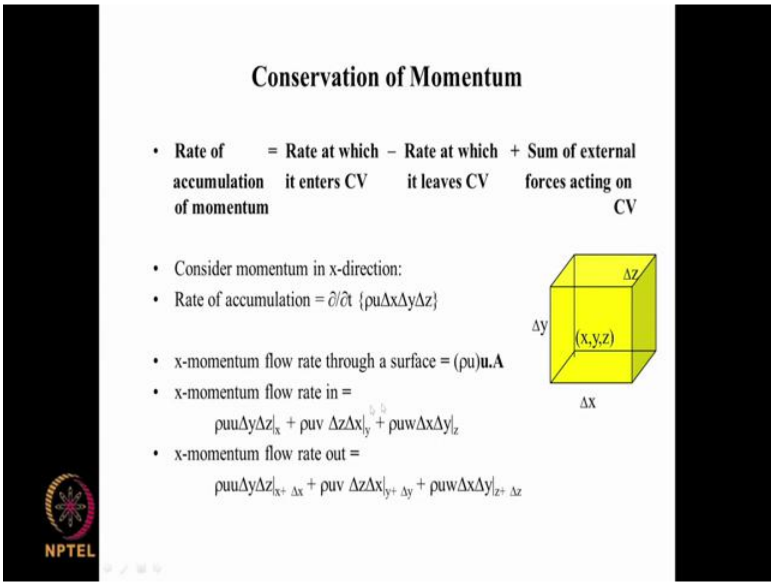

So, all this can be substituted here, and we can get an overall expression. That overall expression is like what we have seen, it will have the delta y delta x delta z. and we do the same thing as what we have done before. We divide all the terms by delta y delta x delta z, and take the limit as delta x tends to zero delta y tends to zero and delta z tends to zero. And that will give us this particular thing will give us dou by dou t of rho u. now you have to. This is plus and this is minus, because this is entering and this is leaving, which we consider only these two terms here and if you divide everything by delta x delta y delta z. Then this delta y delta z will cancel out, and these two will give us rho uu at x. when we say rho uu it is equal to rho u square, because this is a velocity components scalar. So, we can multiply to that. So, we can get rho u square at x minus rho u square at x plus delta x divided by the delta x which remains from division by the delta x delta y delta z. In the limit as delta x tends to zero, and that is equal to minus dou by dou x of rho u square, and because that is on the other side of the, equal to thing we bring it to the left hand side you get dou by dou x of rho is square.

Similarly, these two when divide by delta x delta y delta z, and then we have delta where remains in the denominator, we take the limit as delta by tends to zero. Those two will give us minus dou by dou by of rho u v, and when we bring it to the left hand side, we get dou by dou y of rho uv. And we get this from the net momentum flow rate that is coming in, is given by dou by dou x dou by dou z of rho u v. So, these are the net momentum flow rate x momentum flow rate in when they are on the right hand side. On the left hand side when they are brought in. So, this is the net x momentum flow rate out of the control volume. Now you take these two things and then you divide by delta x delta y delta z, and take the limit as delta x tends to zero; that gives us rho x x at x plus delta x minus rho x as at x divided by delta x as delta x tends to zero. You get dou by dou x of sigma x x. These two will give us dou by dou y of sigma y x. why is it dou by dou y? Because when we divide these two terms by delta x delta y delta z delta x delta z cancel out and only delta y remains in the denominator.

So, when you take in the limit of delta x tends to zero delta y tends to zero delta z tends to zero, as you take the limit as delta y tends to zero this becomes partial derivative dou by dou y of sigma y x. And similarly this will give us dou by dou z of sigma z x, and you have divide by this by delta x delta z you get rho g x. So, this is our momentum balance equation. So, we have got a mathematical statement for the conservation of mass conservation which is like this, and we have got conservation of a momentum equation here.

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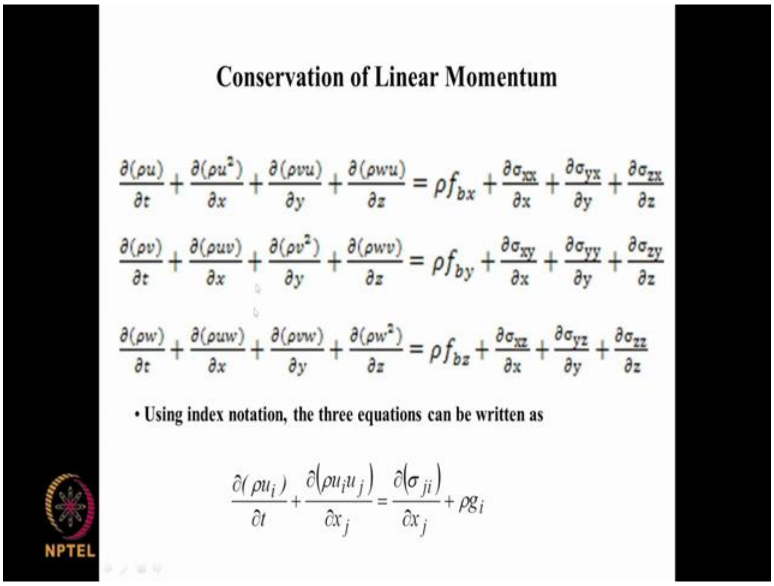

Similarly, we can write the conservation of. This is a conservation of momentum in the x direction, where f is the body force by unit volume, and the y direction conservation momentum is very similar. Since we are dealing with x momentum we get u component here. this is the momentum conservation in the y direction. So, that gives us v here, and one of the u's becomes v here. So, this u becomes v. So, this u becomes v here and this u becomes v here, so you have this. And similarly, when you consider the z momentum balance, this v becomes w, one of the v's becomes w here, and one of the, this v becomes w here. So, you get w rho w square.

So, these are all the rate of change; the temporal change here, and then the change associated with the flow; that is the difference between what is being, what is coming in to the flow and what is going out with the flow, is advection, this called the advection term here, and then the body force term, and then these are the stresses acting on this. So, using index notation, we can write in a short form like this; dou by dou t of rho u i plus dou by dou x j of u i u j equal to dou by dou x j of sigma j i equal to rho g i. Now here in this term we have one repeated index. So, there are three terms, but they are not summed, because it is only the, it is not a repeated index there are. So, there are three individual terms and therefore, this is representing three equations; this equation, this equation, this equation. In this term here there are two indices i and j and only j is being repeated so;

that means, that this term is actually sum of three terms, in which j can take the value of one and two, one two three, but i remains the same. So, what is the value of I; that is define by this term here if i equal to one here it is also equal to one here. So, when you put i equal to one you are looking at momentum conservation the x direction. When you put i equal to two you are talking about momentum conservation in the y direction and z direction for i equal to three.

So, now if you consider for example, i equal to two. So, that is momentum conservation in the y direction. So, u two is v. So, this term here represents dou by dou x j of rho v times u j, and when this implies sum over three numbers; j equal to one two three. So, when you put j equal to one you get u here index. So, that is rho u v by dou x, when you put j equal to two you get u two u two is v and then x two is y, so you have this, and that we multiplies here to give you v square. When you put j equal to three here you get x three is z and then u three is w you get this. In all the cases u two that is equal to v which has remained the same. So, one part of this is remaining the same, even here, and the other things are changing, because j is a repeated index and i is not a repeated index. now when you come to this again here in this short term there are two indices here, and one index here and it is being repeated over j here, and what is not changing is the second index i, why is it. So, if you consider these three terms here, in this x the second index is unchanging, only x y z the first index is changing taking the values of x y z here.

So, when you consider the x momentum equation, then i is not changing. So, the direction representing the force acting is not changing. So, that is why you put i index here and j index here. And we would like to point out at this stage that in a term like this, if there is a repeated index that j is a dummy index. So, it does not matter whether it is j or k or m. It gets washed out when you expand this, but this index remains the same. So, for example, when you put i equal to one, this has to be x x x here, and it is the same the corresponding x component x component x component in x component. So, that i identity is maintained throughout here, but whether it is j or k, it does not matter, because it is anyway going to be x y it is going to change x y z x y z like this. So, the j here is a dummy index and i is proper index with substance attach to it. So, sometimes for convenience we put it has dou by dou x m of rho m i. it is a same as dou by dou x j of rho j i. So, this is the conservation of linear momentum, but we have a problem, because a conservation of linear momentum in itself given like that is not very useful, because if you are considering a flow in which there is no heat transfer, just air flowing over a car.

We are not really interested in the temperature; you are not interested in any kind of reaction that may be happening and all that. So, you should say that they should be given only by the mass conservation and momentum conservation. So, if you do that, then the numbers of equations that you have are four, because you have one continuity equation or the mass conservation equation, and three momentum conservation equation in the x direction y direction z direction. So, you have four equations, and how many variables are there.

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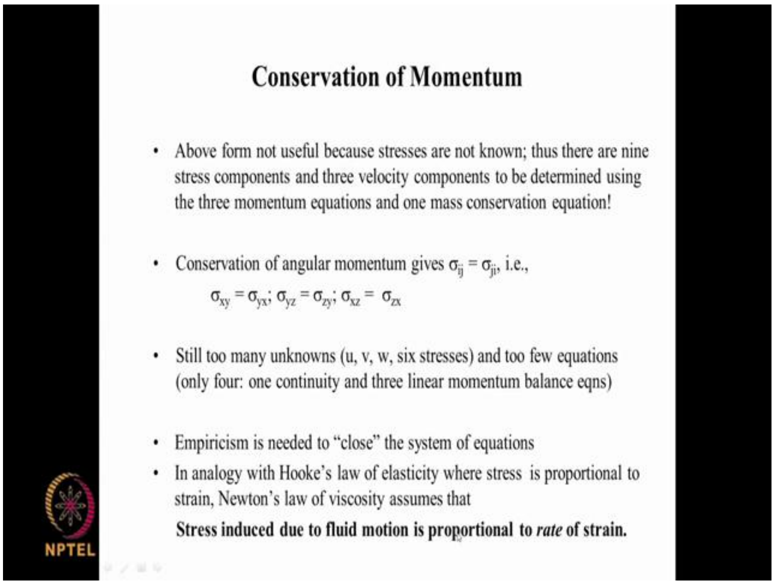

Here you have three variables, because dou is u v w. if you look at this set of equation and the continuity equation here. In the continuity equation you have rho u v w. We can say rho is at density of the fluid and I should know it. Sorry you could say that or it is given independently and. So, we have three variables u v w here and in these three equations you have u v w. So, good, but on the side f b is not known, and we can say that we are only considering gravity and you know the g; gravitational constant is known. And we know the direction of the components, because we are aligning it depends on what control volume which is once we choose a control volume, then the g x g y g z are known. So, you can say that as long as we restrict ourselves to gravitational component alone, then this is known you already have u v w here, but these stresses there are nine of these, these are also not known, and you are looking at gradients. So; that means, that we are saying that these stresses can change with position. So; that means, that these are variables. So, when you look at the number of variables that are present in these four equations these are u v w and the nine stress components.

So, you have thirteen variables, and you have only four equations. And you are looking at flow without heat transfer and all that and you do not see why you have to consider energy equation and all that. So, in that sense you have a problem, because you have an indeterminate system. You have two few equations and too many unknowns, and that gives us a problem here. So, you can try to bring in other equations. You can say what I have considered is linear momentum, but I can also consider the angular momentum, and that has nothing to do with heat transfer and all that. So, I can invoke the angular momentum balance equation. You could do that and if you play it and go through it is systematically. It gives you extra relations, it tells you that the stresses that we are considering here, must be symmetric so; that means, that sigma i j must be equal to sigma j i. So; that means, that sigma x y is equal to sigma y x sigma y z is equal to sigma z y and sigma x z equal to sigma z x.

This is useful, because instead of the nine components that you have here, there are only six independent components, because if you know sigma y x then sigma x y is known. So, we can substitute here, we can eliminate sigma x y and just keep sigma y x here. So, you have six components here. So, in that sense angular momentum conservation is good, but it does not solve the problem, because we still have too many unknowns. We have u v w and six stress components, and two few equations only four; one continuity equation or the mass conservation equation, and the three linear momentum balance equations. We have already used the angular momentum balance equation, there is nothing more to get from it. So, this means that we have a system of equations here, which does not lead us to a useful solution, because we have too many unknowns. So, we need to bring in, there is no way out, because we cannot think of any other relevant fundamental loss that need to be abide by this. So, we need to bring in additional empiricism more about the natural the fluid that needs to be brought down to close this system of equations. So, this is where we look at the analogy with solid mechanics, where when we are considering forces and displacements in the deformations and all that. We bring in the Hookes law of elasticity where the stress is made proportional to the strain. For example, if you imagine spring and then you stretch it, and the amount of stretch the strain that is produced in the spring, is proportional to the stress that you apply, and that is Hookes law. So, there you have a relation and empirical relation, because it is not always correct, and it is more. It is a law which is obeyed by many solid substances under certain limiting, but still useful conditions. So, we take that Hookes law as a useful gift to us, so that we can continue with our analysis. So, we take that Hookes law of elasticity, and we would say that here we are talking about stresses, and we should somehow then we able to make use of a similar kind; that is what Newton's law viscosity says. You have the viscosity as special properties of fluid, and we know that based on observation, that you have viscosity resist this deformation.

So, you can bring in similar law in which the stress induced fluid motion is proportional to strained it. So, stress induced by fluid motion is proportional to the rate of strain rather than strain. So, we can make an assumption like this, and we will in order to truly understand this, we need to study the kinematics of fluid motion. So, this is what we will do in the next lecture. What we have done in this lecture, in this short lecture, is to force this conservation of linear momentum in global statement, and we have worked out what each other terms are. And based on this term we have been able to come up with conservation of linear momentum principle, but we find that it is not sufficient enough, and we therefore, pose the angular momentum principle, and we have reduced the number of unknowns, but we still do not have enough. So, we are now trying to pause and empirical bring in an empirical relation. This empirical relation has to be brought in a proper way. So, that it is applicable in the general case not in the specific case. So, that is what we are going to do in the next lecture.