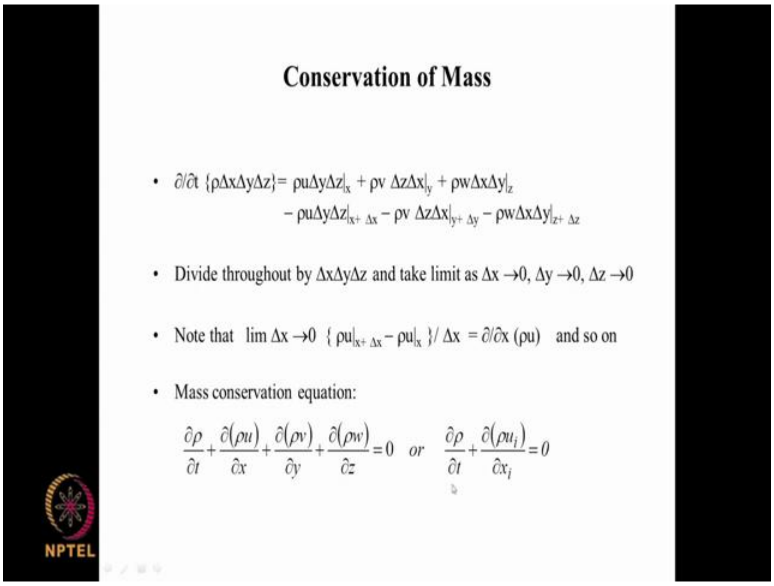
Computational Fluid Dynamics
Prof. Sreenivas Jayanti
Department of Computer Science and Engineering
Indian Institute of Technology, Roorkee

Lecture - 12
Momentum conservation equations

(Refer Slide Time: 00:32)



We have seen in the first lecture of this module on equation governing fluid flow. We have seen how we can formulate the conservation of mass statement; and how from that we could get mathematical expression and equation of this type dou rho by dou t plus dou by dou x i of rho u i equal to 0. So, this is mathematical statement of the conservation of mass. At this stage, let us just take a look at this particular equation, and we would like to pointed out here that way using index notation to make ah things easy for us. And what we mean by index notation is that variables which have components like the displacement vector x has x, y, z components, velocity has u, v, w components, and we will see later on stress has nine components like that.

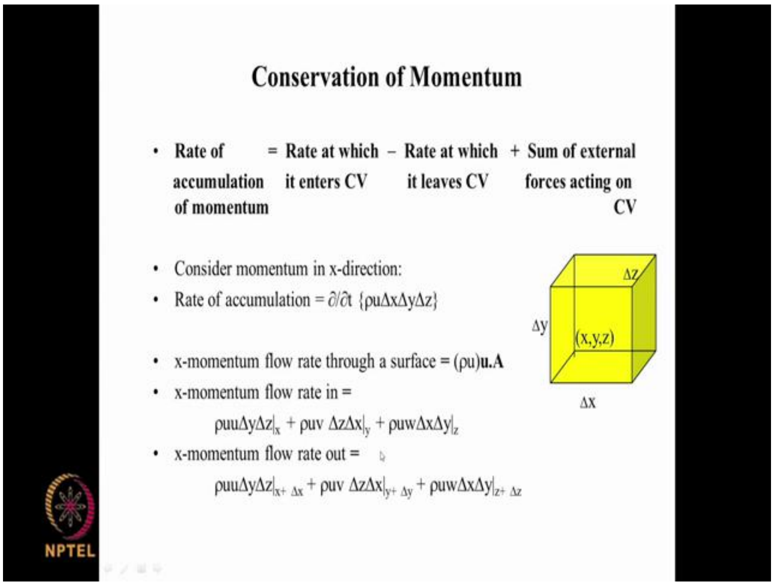
So, when you have those components, the corresponding component is expressed as a subscript using an index and this i here is a general index, whereas x and y are specific indices. So, when you say x 1, x 2, x 3, these are specific x i can be it can be 1 or x 1 or x

2 or x 3 depending on what the value of i is. And in a term, so in this equation, there are two terms, three terms the right-hand side is 0, it is a separate term. First term is dou rho by dou t; and the second term is the entire thing here dou by dou x i of rho u i. So, in a term like this if there is one index which is repeating then this automatically implies sum over that particular index. So, here i is repeated in this term in two variables u and x. So, you have rho u i in the numerator and x i in the denominator. So, this actually means that it this particular term is sum of ah three parts, one of which is with i equal to 1, and the other is i equal to 2, and third is i equal to 3.

So, when you put i equal to 1, you get rho u 1 by rho dou x 1, and u 1 is u and x 1 is x, so you get this term. When you put i equal to 2, you get u 2, u 2 is the velocity component in the second direction which is y. So, the velocity component in the second direction is v, and x 2 is y. Similarly, when you put i equal to 3 here, you get x 3 which is the z-direction, and the corresponding u 3 component is w. So, this equation in this equation this term is coming alone that this term is sum of three components which are given by this.

So, this longer equation is written in an abridged form using this index notation here, and this is applicable for Cartesian coordinates, and we make use of this. And this idea of this repeated index is also refer to as Einstein's summation convention. And in a term if there is repeated index then it implies some over that particular index alone. If there are other indices which are not repeated then there is no summation implied over that and we will see these kind of cases later on. So, this index notation and allows us to write the equations in a simple form, but it is nothing difficult you just put i equal to 1, and then you get this. So, it is not a difficult thing to comprehend, it is a convenience for us.

(Refer Slide Time: 04:28)



So, with this kind of introduction, we will move to the second fundamental law, which is the conservation of momentum. Within momentum, we have linear momentum and angular momentum; we have conservation of linear momentum and conservation of angular momentum. And here we are looking at conservation of linear momentum.

We can make a statement, verbal statement of conservation momentum, which is similar to what we have made earlier for the mass conservation, which is that in our control volume which is shown here, with box of sides delta x, delta y, delta z, the rate of accumulation of linear momentum in that particular box is again equal to because we are dealing with the flowing situation and fluid can come in and the fluid molecules have certain momentum. So, when they come in they bring in this momentum into the control volume, when they leave out they take away the control volume in they take away there the particular linear momentum that they carrying.

Because some particles are leaving away in the process of going through this domain, there will be taking away some of the linear momentum and the incoming particles will have thereon momentum. So, the rate at which the fluid enters a control volume with extra linear momentum minus rate at which the linear momentum leaves control volume carried away by the fluid particles that are leaving the control volume and momentum can also change from external forces acting on it.

So, if there is an external force, then there can also be a possibility of doing some changing linear momentum and that is essentially what Newton's second law is m times a - mass times acceleration is equal to the sum of external forces acting on it. So, if there is an force acting on it, it can lead to acceleration, so that can lead to rate of change of linear momentum, if whatever external forces are there, they can contribute to change of momentum of the fluid particles which are there inside. So, we have here four components, rate of accumulation momentum, rate of entry flowing in of the linear momentum, rate at which the fluid momentum goes out, and external forces rate of change cause by external force acting on the control volume.

So, if momentum is a is a vector quantity, so it has three components linear momentum in the x-direction, linear momentum in the y-direction, linear momentum in the z-direction. And each of these needs to be balanced, so you can say conservation of momentum in the x-direction, conservation of linear momentum in the y-direction like that. So, here for the sake of convenience to begin with we are starting with the linear momentum in the x-direction. And when we talk about momentum, it is mass times velocity; when you talk about linear momentum in the x-direction, it is the velocity component in the x direction that matters.

So, linear momentum in x-direction is the total mass of the fluid which is there in the control volume times the x velocity is the total momentum which is contained by the fluid which in within this control volume. And the rate of change of linear momentum possess by all the fluid particles within this control volume is given by dou by dou t of rho times u times the volume which is delta x delta y delta z. So, here we are making formulation of the fluid problem in terms of mass being given by the density and the volume. Since, for this particular case, we have fixed the volume change in density can lead to change in momentum. Change in velocity can also change lead to change in momentum.

So, density and velocity the u, v, w are properties possessed on the average by this fluid within this control volume. And we are taking it to the extreme limit of becoming very small, so essentially we are looking at fluid properties of density and velocity components changing in the entire fluid continuum, the entire fluid domain. And we are looking at what is happening, what is a property in terms of density and velocity of the fluid possess by the fluid within which is occupying this particular control volume here. So, we can say there for this particular fluid in this control volume, the rate of accumulation is given by dou by dou t of rho times density times the x momentum velocity component u.

So, now rate at which linear momentum enters is equal to the rate at which the fluid particles are coming in times the velocity that deposits in the x-direction so that means at this rate of flow in flow out is happening because fluid particles are coming in, and those fluid particles have a certain property of linear momentum in the x-direction. So, it similar to a lorry coming into the city limits, through the city gates, it is coming in and then it is not only bringing lorry, it is also bringing with it all the other quantity that are there in that.

We have making a balance of that just as it may be carrying a bag of tomatoes and lots of potatoes, and rice, and milk and other things; and each of them needs to be conserved as the lorry goes in and goes through the city exchanges with lots of traders and then goes out. So, linear momentum in the x-direction is one such property, linear momentum and the y-direction is one such property, the density is one such property all this properties need to be conserved in a certain way. So, conservation of mass is one such property that it been conserved. Similarly, the conservation of linear momentum is also of fundamental property that needs to be conserved and that is what we are looking at here.

So, wherever of fluid particles enter at that point we have to see how much of momentum, it is carrying it is bringing into the control volume. So, the rate at which x momentum is brought in by the fluid entering the control volume is equal to the rate at which fluid is coming times the x-momentum that it possesses. So, that is what we have here u times a times density is the mass flow rate of the fluid, and it possesses the x-momentum of u small u ah the velocity component, and this describes the momentum flow rate which is being brought in by a fluid which is entering. And similarly if it is right phase and it is leaving the control volume, rho u times u dot a the flow rate at through this particular phase times the velocity at that particular phase will constitute the amount of momentum in the x-direction that the fluid particles are taking away. So, if we consider, for example, the bottom phase here, at the bottom phase, the mass flow rate of fluid which is coming in is rho times v times the surface area which is delta x, delta z, so that is the mass flow rate. But the fluid particles which are coming through at this particular rate, so that is rho v times delta x delta z have u momentum which is given by u at y that is the bottom part.

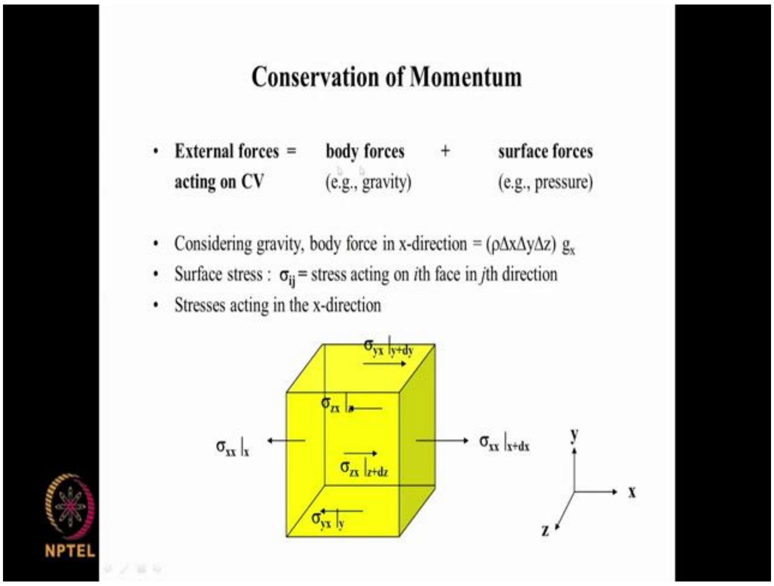
So, the flow rate the momentum flow rate in the x-direction that is coming through this is which is being added to the control volume is rho v times delta z delta x times v through the bottom face. And similarly through the back face there are particles that are entering through the back phase, and they constitute a flow rate of rho w delta x delta y; and each of them has a velocity in the x-direction, which is given by the velocity at that location, and what is that location it is a midpoint of this particular plane so that plane is at z at the center of this.

And it has an x component which is x plus delta x by 2 and y coordinate is y plus delta y by 2. So, since we are looking at changes here and this we are identifying all these things at the midpoint of that particular phase. So, through the back phase, it is a velocity in the x-direction at the center of the back phase, the density at the center of the back phase, w component the back phase, all these things go together in making up what is the amount of momentum flow rate that is coming in.

So, in the sense, we can write an expression, we can calculate, what is the amount of momentum x? Momentum that is being brought into the control volume by the incoming flow and what is being taken out by the outgoing flow. If these two are the same then there is no net change in the momentum because of this. If the velocities are different, if the x velocities are different at x plus x and delta x plus delta x or either v or u are different at y plus delta y or either u or w are different at z and z plus delta y is z plus delta z then or density is changing at this point, this point, this point, this point then all these things are potential re sense by which the overall momentum x momentum that is contained in this is changing.

So, it is like we can imagine this to be a bank, then deposits are coming in and depositing money. So, then that leads to an increase the total amount of money that is deposits in the bank. And there are depositors who are taking away money or it is being given in the form of loans, and that will lead to a decrease in this. And there can be profits, which are coming in and that can also lead to the total wealth possessed by the bank. So, these are the kind of process, but the processes that are associated with this rate at which this is coming in and this is more like the depositors depositing money and then depositors are taking out money and the external forces are the processes which are actually leading to profit generation by whatever needs. So, we have quantified the rate of accumulation, rate at which x-momentum enters the control volume, rate at which it leaves the control volume, but we also need to quantify the sum of external forces acting on this control volume. So, here we need to take a pause, and see what do we mean by this external force.

(Refer Slide Time: 16:41)



So, what kind of external forces can be there? So, external forces when we consider in the most general case, these can be volumetric forces spread over the entire volume anywhere in the volume like gravity or it can be just surface forces like the stresses that may be applied or the pressure that is applied on a phase like that. So, when you consider our box here, when we are looking at external forces, external forces enter act on only on the external surfaces that is exposed left face, right face, bottom face, top face like this. So, external stresses can influence only through the surfaces, whereas body forces like gravity can act on all the particles that are contained in this. And we would like to make this distinction here and so we divide the external forces acting on the control volume into body forces like gravity plus surface forces like pressure and other stresses we will see later.

So, if we consider gravity body force in the x-direction, gravitational force is m g, where m is the mass and g is the gravitational vector. And in the general case, the g has the three components g x, g y, g z even in the simplest case, we say that it is vertically downwards, but we do not know what whether the x, y, z coordinate system that we have taken is such that it is aligned in the y-direction, gravity is aligned in the y-direction, if it is not taken be an x-component of g. And it is only the x-component of the gravitational vector that contributes to the change in the momentum in the x-direction. Gravity as a vector will have the three components g x, g y, g z, and g y will act to change the momentum in the y-direction and g z will be acting in the to change the momentum in the z direction.

So, what matters to us when we are looking at conservation of momentum in the x-direction is the gravitational force that is the total mass which is rho times delta x, delta y, delta z times the gravitational component in the x-direction. So, this gives us one type of body forces. And in this course, we are considering gravitational force as the only external force that is acting, you can have magnetic forces, electric forces, electromagnetic forces all those things may be there in the general case, but in this case we are only dealing with gravitational force. So, we have only this has the body force.

Now when we talk about surface forces we have to take more caution in describing the surface forces, because when we talk about surface force, and we looking at box three-dimensional box we have to ask what surface, which surface, I have many surfaces. And on this surface what type of stress because stress can be compressive or tensile, like this is tensile it is acting normal to the plane, and it can also be acting in the plane like a shear stress. It can be acting for example, like this trying to distort the end, it can distort in the x-direction, it can distort in the y-direction, so that means, that the stress on a face has three components in the x-direction, in the y-direction, in the z-direction. And it can act on any of the six faces.

So, we come up with the two index notation for the stress. So, we define the stress surface stress as sigma i j; and here by convention we are defining stress as sigma i j as a stress acting on the ith face in the jth direction. So, what we mean by i th face is that both i and j can take indices of 1, 2 and 3. When i equal to 1 that is x, when j equal to 1 that is again x, when i equal to 2 it is y; i or j equal to 3 is z. So, if you consider i equal to 1 here, so that means, that as a face whose outward normal vector is aligned in the x-direction. So, we are looking at the right face or the left face because in this the normal vector outward normal vector is along the x; in this case it is along positive x, and in this case it is along negative x. So, when you say sigma 1, you are a referring to forces acting on the right face and forces acting on the left face.

Within this, whether the stresses acting in x-direction or y-direction or z-direction is given by the second index. So, here we have sigma x is stress acting on the x-face; and the second index, x here indicates it is acting in the x-direction here. Now if you go to the top face here, in the top face, this is stress, which is an area a face, whose outward normal vector is aligned in the y-direction, so that is why we call this as sigma y, i is equal to two. So, sigma y, this y indicates that is acting on the top face or the bottom face. And but this particular stress is acting within this i plane within this j plane here and it is acting in the x-direction, so that is why we call this as sigma y x.

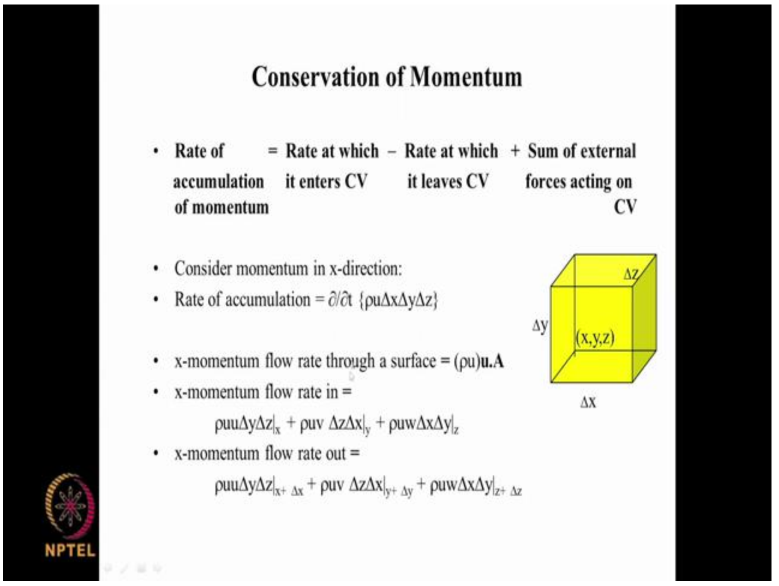
And if we consider this particular front face, it has a center right here and from this center we have stress which is shown and because shown by this horizontal arrow here. And what is the stress component it is acting on the z-face, so that is sigma z i is equal to 3; and it is acting in the x-direction, so j equal to 1. So, we have sigma z, x is acting in the x-direction on the z plane sigma x x is acting in the x-direction in the x plane and sigma y x is acting again in the x-direction, but it is acting on the y plane.

Now, we have one more convention. You have face which is aligned either in the positive coordinate direction like positive x or it may be aligned in the negative x-direction. So, by convention, according to rules of transformation and other things, we define this stress when it is acting in the positive face, so that is where the outward normal vector is aligned in the positive coordinate direction which is this one here. Then it is the stress is acting in the positive x-direction. When we say sigma x x equal to 100 on this particular face here then that means, is stress a tensile stress acting as hundred like this. When you say sigma x x tensile stress of 100 acting on this face, it is a tensile stress. So, it must be acting it is pulling away this face. So, it is acting to the left side. So, this sigma x x is always aligned in the outward normal direction of that particular face here, sometimes it can be on a left face it is acting it is positive in the negative x-direction; on the right face, it is acting it is in the positive x-direction.

Similarly, you consider this top face here. So, a top face is a face with a positive outward normal direction, so that is it is outward normal vector is aligned in the positive y-direction, whereas the bottom face here the outward normal vector is acting in the negative y-direction, so you can say it is a negative face. So, on the negative face, sigma y x of 100 is acting to the left side; and on the positive face, it is acting to the right side which is the positive x-direction. So, as depending on in which direction the outward normal vector is aligned, the corresponding stress components may be acting all are acting in the x-direction. But some are acting when you say sigma of 100 on this particular face here, this stress component of 100 then depending on whether it is a positive face or negative face, it may be acting either in the positive x-direction or negative direction and that is determined by whether the outward normal vector of that particular face is in the positive coordinate direction or in the negative coordinate direction.

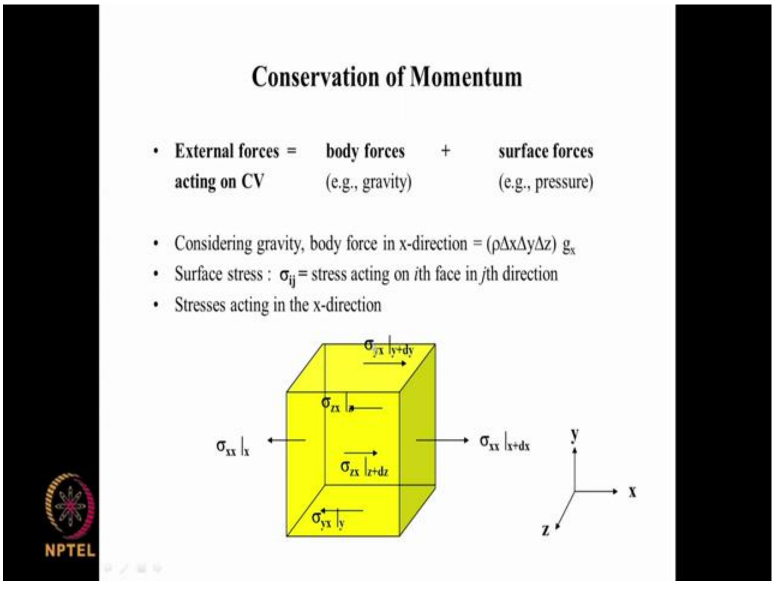
So, this particular thing means that if stresses are equal, if there is no change in the stress, so that is sigma x x at x and sigma x x at x plus d x are numerically the same then there is no net force acting on this, this stress and this stress will cancel out. Similarly, this stress and this stress will cancel out, so there is no net force acting on this they can only be torque. So, this is a convention that we follow with this stresses. And here we make use of two indices sigma i j to define the stress.

(Refer Slide Time: 27:00)



And the stresses acting on the x direction which are important for a consideration of linear momentum in the x direction here are what are shown here.

(Refer Slide Time: 27:04)



So, this particular cube this box has six faces positive right face, left face, bottom face, top face, back faces and front face. So, the stress acting in the x-direction on each of these faces is marked at the center of that particular face; and along with that we have given the direction of action and the corresponding name of that stress component. For example, if you consider this top face here, the center of the surface right here, and the stress acting in the x-direction on the top face is given by sigma y x. And it is acting in the positive x-direction. And on the bottom face here, the center of the bottom face is here, and the stress sigma y x acting on the bottom face is in the negative x-direction. These two can be different if stress is varying vertically as you move up from point to point here if the stress is changing then these two can be different so that is why we define sigma y x at y and sigma y x at y plus d y.

If there is a change in the stress in the vertical direction, these two can be different is as change in the stress in the x-direction in the horizontal direction then these two can be different. Similarly, for the back face, this is center; it is back face, it is a negative face. So, sigma z x is acting in the negative x-direction. And the front face is this, the center is this, and this has stress which is acting in the x direction in the positive x direction and that is given by sigma z x. So, these are the different stresses acting on this. What we are looking at is external force, so each stress needs to be multiplied by the corresponding area and then we will be able to get the sum of all external forces acting on this. And this net external force can give raise to change in momentum and this is where we will take a break and we will continue in the next lecture. So, in the next lecture, we are going to see how we can evaluate the net external force acting in the x-direction.