Computational Fluid Dynamics
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Lecture - 11
Equations Governing Fluid Flow

Welcome to the second module of the CFD course. In this module, we are going to look at the equations which describe fluid motion. This is an important aspect of CFD because we are trying to solve the equations numerically. And, we have to make sure that we are solving the correct equations, the equations which represent the physics of the problem. There is a another aspect to the solution of equations using CFD. In the sense that when we are looking at analytical solutions, we may have to make certain simplifying assumptions. But, when we are doing numerical solution, then those simplifying assumptions need not be made.

For example, if you look at for example the viscosity variation with temperature, only certain types of, certain forms of, certain analytical forms, for example, the exponential form, may be something that would yield as an analytical solution. Whereas, in while dealing with a numerical solution, we are not necessarily restricted to the exponential form. We can bring in any other form. And, it is also possible that along with fluid flow, there are number of other phenomenon that are happening and this need to be represented mathematically. And that is possible in CFD. It is possible to solve these equations together with other equations in CFD. And, we often make use of this facility to deal with a practical problems.

For example, turbulent flow or a flow with a turbulent flow with reaction or a two phase flow; in all these cases, we have an extended set of equations. And, it is this extended set of equations that we are trying to solve. And, we have to see that this set of equations that we are trying to solve is correct. And, it takes advantage or the facility that is offered by CFD, which is that we can solve many more cases, many more different types of equations than what is possible analytically. So, it is in that sense we have to look at a problem formulation, mathematical formulation, which is the most generic possible. So, that is one of the things. And, which is why the equations that we solve in CFD or not like the Bernoulli’s equation or not like one dimensional equations. It is the general three dimensional unsteady equation, set of equations, which describe fluid motion. And, we would like to look at those equations. And, what they mean and how to mathematically pose the problem in such a way that we can then attempt a CFD solution.

We can (Refer Time: 03:10) together a format for discretizing these things, and then converting these into algebraic equations and solve that. So, that is the essence of this particular module. What are the equations which govern fluid motion in the most generic case? And, in that general case, we would like to; we would like to take those equations and then you do the discretization, so that by changing the flow domain, we can apply these two different cases.

For example, the equations which you describe the flow through a rectangular duct are the same as the equations that we described the flow through a triangular duct or may be unsteady flow. So, those are the kind of general case equations that we would like to derive here. So, when we look at the equations which would describe motion of fluids, then there is no reason to disbelieve that the equations of motions that we have for solids are not applicable for liquids or gases.

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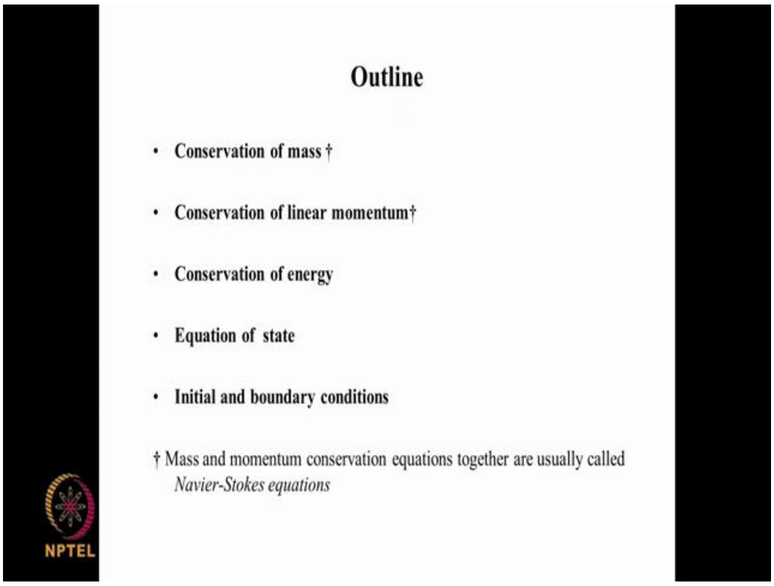

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So, in that sense we are still looking at the basic frame work of a equations, which are, which we are familiar with for Solid Mechanics. For example, we are expecting in a fluid flow for certain fundamental laws to be appealed, which is the conservation of mass, conservation of linear momentum, conservation of energy. And, for fluids we have special kind of things that are coming here; the equation of state and also the boundary conditions and initial conditions. So, these are the ones which together make up the mathematical formulation. And, it is these equations that we are going to derive in this particular module.

So, in the first module we will be looking at the first two equations along with some basic introduction about fluids. And, in the second module we will be looking at the other part of the outline here.

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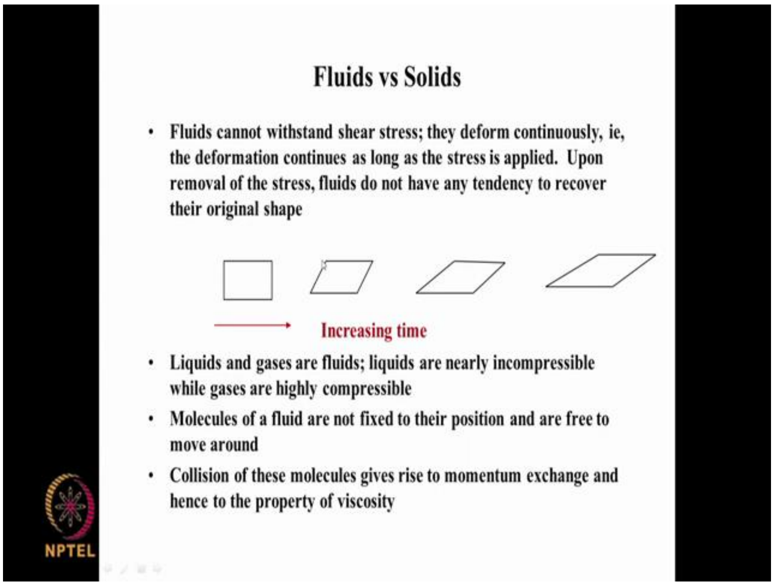

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Let us first of all establish the difference between fluids and solids. We have an instinctive feel for what fluids are and what solids are. But technical way, we can say that fluids are those physical substances, which cannot withstand shear strength, shear stress. And, by this way mean that they cannot withstand means that they deform continuously, that is, the deformation continues as long as stress is applied. Unlike, in the case of a solid which deforms, it also deforms upon the application of shear stress. But, it soon reaches a condition of equilibrium between the force that is applied; the shear stress that is applied and the internal resistive force that is generated so as to reach shape, which is unchanging with further passage of time.

Whereas, in the case of liquids as shown in this slide here, in the case of liquids as shown in the slide here, if you consider an initially rectangular fluid element. So, by this we can mark it as; we can mark point a, b, c, d within the fluid continuum. And, these are such that these are separated in the fluid in such a way that if you draw straight line to them, you will have a rectangle at time equal to 0, at when you start observing this. And, with time they are flowing. So, that is go, moving along. And then, if you apply the shear stress, then a is coming here and b is coming here and c is coming here and d is coming here. And, in such a way that now if you join them by straight lines, you get a shape which is like a parallelogram as shown here for a small time. And then after some more time, this parallelogram is more distorted and even more distorted and so on.

So, in the case of fluid the initial rectangle keeps deforming continuously with time. It does not stop. Whereas in the case of a solid, once it reaches a certain level of deformation like in the second picture here, then there is already a resistive force which is generated internally, which will eventually reach in equilibrium with the applied stress so as to maintain a shape in this distorted fashion. So, it is this nature of fluids that they continue to deform with the application of shear stress. That differentiates fluids from solids. And both liquids and gases, they exhibit this kind of a behavior.

So, we can say that liquids and gases or fluids. And in a sense they flow, they start moving related to each other upon the application of a shear stress. One of the differences is liquids are nearly incompressible, while gases are highly compressible. And, there is also with in contrast to solids. Solids are essentially fixed in their arrays or in their relative position to each other and they remain fixed. In the case of liquids, they go around, they move around within that liquid continuum, but they cannot go everywhere. There is an interface, which is separating the liquid from the gas space.

In the case of gases, there is no such interface and the gas molecules are free to move around in the whole available continuum, until they are blocked by an impermeable wall or an impermeable liquid surface. So, solid, liquid, gas molecules will tend to occupy the variable volume over a period of time. Whereas, liquid will retain its volume. But, it may change its shape depending on the forces that are acting on it. And, a solid can, will retain its shape and volume with passage of time, unless there are some special forces acting on it.

So, if you want to look at the distinction between these, since you are all students, you can consider the students in a primary school. The kids stay in the class room and then the teacher comes out and go. Every period, the teacher comes out and then a new teacher comes in, new subject is taught. Like that they retain. They sit in the seats, most of the time. Liquids are like college students. You are going from one course to another course, may be you are changing your rooms, may be you are going in to a different labs, but the batch as a whole, retains its cohesiveness. So, it is unless somebody fails or gets suspended or has some other problem, the batch remains the same. But, they are not fixed. They are not there all the time in their class rooms. So, they do not retain that particular shape that set of school students, primary school students have.

Now, if you go to even higher studies, then or may be in your fourth year and final years and pre final years and when you are start taking electives, then the cohesiveness of the batch also gets lost; because some students may be taking this elective, some students may be taking some other elective, some students may be doing a semester in some other institute kind of program. So, it is no longer; they are no longer confined to that school compound. They are everywhere. They can be everywhere. Ok. So, in that sense.

And, what is making the difference between a primary school and first or second year bachelors program and then senior years and all that, it is the freedom that is there. The maturity that is there within you, which allows you to be given that freedom to go around and move around and still be safe and responsible.

So, in the same way as you increase the energy of the substance, you go from solids to fluids, solids to liquids to gaseous state. We are dealing with not those energies. We are dealing with essentially fluids and how they flow and what is effect of the flow. Now, we would like to make a distinction between liquids and gases. And, we would like to point out at this stage that the fluids have a couple of special properties that solids do not have; one is the concept of pressure. Ok.

So, you have, you can visualize an imaginary plane, which is put in the liquid continuum. And, the liquid molecules are free to go around this way, that way, in random directions. And therefore, some of them will be hitting the wall, hitting this imaginary surface. And then, so that will give rise to potential exchange of momentum. And, that is, that can be considered as an equivalent to a pressure. So, a fluid molecules will exert pressure on an imaginary surface, which is present in the continuum. And, this gives rise to the concept of pressure. And, the pressure is always compressive stress which is there, which is normal to a surface immersed in that fluid, whether it is a gas or a liquid. The other important property about fluids is the concept of viscosity.

So, fluids are not like plasma or something like that. They still resist relative motion. In the sense that if you have a fluid molecule and then if it want, if it is going to be dragged away by the applied shear force, then there is some kind of cohesive force which is trying to resist this. And, the layers which are, the fluid molecules in the layers below that will try to resist this and set up your resistive force. So, this means that if you want to create a velocity gradient, and then if you want to maintain it, you need to do some work. And, all these gets translated into the property of viscosity. And, we will see bit more about this. But, there is a property of viscosity which is similar to the property of elasticity that is present in solids. Whereas in solids, the stress and the deformation that are produced are proportional to each other. Here, it is a stress and the rate of deformation that is produced. And, this relation between the two (Refer Time: 14:47) in the form of viscosity. So, viscosity is a property that is present in fluids, both gases and the liquids.

And, liquids have one more special property which is the inter facial tension or a surface tension. The surface between a solid and a liquid or between a gas and a liquid is a special kind of surface. And, there is effectively a surface tension which acts as a force on it. It arises in some special cases. Definitely very important in some industrial context, but usually in our regular Fluid Mechanics we neglect that. So, we do not go too much into that. But, it is an extra thing that can be added to the set of governing equations, which you are going to describe, which we are going to discuss in the next classes.

So, in this course we are not going to deal with surface tension. So, that is left out of the forces that we are going to deal with. So, we understand the major differences between solid and fluid and gases. And, what we have done is just to recap those things. And, we need to keep these ideas in mind when we discuss the governing equations.

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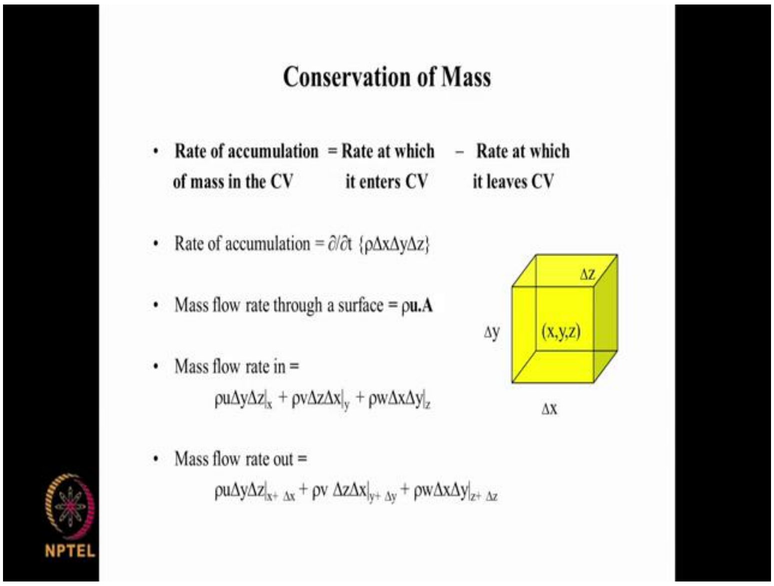

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So, when we talk about the governing equations we already mentioned certain fundamental laws, which must be obeyed by all substances. And fluids, while they are flowing must also satisfy these laws. And one of these, the most basic of this is the conservation of mass. In the absence of nuclear reactions, which is what we are considering, we expect the conservation of mass to be up held, while the fluid is flowing.

So, conservation of mass can be represented mathematically as dm by dt equal to 0. But, when we are dealing with fluids, we are at some disadvantage compared to when we are dealing with solids; because in a solid, it is easy to identify a system. It is easy to identify what are the molecules or what are the particles of the solid that were dealing with. Like, a motor car.

So, the motor car is a system which is comprised of the passengers inside and then it has a definite shape. It is in a bounded volume, which is moving along because you are applying some force through the motor. And then, there are some viscous forces; wall friction, a tire friction and then air (Refer Time: 17:42) all those things are there. And based on this, you can describe the velocity or the change of velocity of that vehicle. And, all the time you are looking at the same set of people and the same set of tires and the same set of motor vehicle molecules and particles that are there.

In the case of fluid flow, it is not really easy to keep track of the moving, the ever changing shape of the liquid, in case of liquids. And in the case of gases, they do not have a shape. It is difficult to keep track of the individual gaseous particles that are present. So, we always talk in terms of; for fluids, we talk in terms of the changes that are happening in a fixed or a moving control volume.

Like, we are looking at a, for example, within a motor car we could be looking at the way that the cold air is coming through the air conditioning system and then it is spreading. So, we are not looking at individual molecule of the air, but we are looking at what is the flow of air and how is it spreading its coldness and how is getting hot; because hot air which hot because they are exchanging heat with the passenger sitting inside. And then, so, what is the temperature field, what is the velocity field, who gets the coldest air first and what is the flow pattern. So, these are all we are talking about what is happening inside the car. And that is what is of interest.

If you are looking at a chemical reactor, usually these are may be some stirred vessel with some reactants coming in and another reactant being fed and some reaction going on, like this. Again, we are looking at how does a concentration change and how do the reactants mix, where do the products go. And, so these type of things you are concentrating on what is happening inside that reactor. So, we are always looking at what is happening inside a control volume, rather than asking what is happening to this particular particle. And, these two are equivalent.

Again, coming back to the educational example. One can learn about the system of knowledge sharing, knowledge giving; that is happening in an engineering program by tracking the increasing level of knowledge of a student as he or she goes through the bachelor’s program.

If you have some way of measuring what is the knowledge in a particular domain, and then if you start measuring it that at the end, at the time of entry and every month after entry as the student is going through the classes, then you can do that. And then, you can do that for hundreds of students and get to know what is the knowledge transfer process that is happening in this particular program. Or, you can look at a class room and then you can see over a period of time what kind of knowledge exchange is taking place inside the class room and then how the students are benefiting. And then, you do it for all the different classrooms and which different subjects have been taught and then you can get an idea of the knowledge transfer process.

So, both of them if they done very accurately will give us the same information. But, which we do depends on the convenience. In the case of, for example, if you are looking at a specific subject and if you looking at CFD and how it is, what kind of a knowledge transfer is taking place in a CFD, then we would like to look at a classroom and then we would look at what is happening inside, what is the topic that is being discussed today and what is the next one and all that. So, we are looking at a particular classroom and then see what is happening. We are not looking at a sample of students, who have been sampled every day. And that is a kind of control volume, kind of formulation that we do for fluids, purely for the sake of convenience. And, we look at that kind of formulation to describe mathematically these fundamental laws of conservation of mass momentum and energy, so that we can get some equations out of this.

So, one application of the conservation of mass is what is shown in slide here. We have here conveniently chosen cubical control volume with origin right here, at the back side and then X going in the horizontal right side direction, Y going in the vertical upward direction. And, since we follow the right hand coordinate system, Z is coming out towards us. So, that is the direction that is a coordinate system. And, this particular box here has a length of delta X in the X direction, a length of delta Y in the Y direction and length of delta Z in the Z direction. So, this is our box and our control volume. And within this, we would like to describe in simple verbal statement, this conservation of mass. Ok.

So, we can describe it as rate of accumulation of mass within the control volume is equal to rate at which mass enters the control volume and rate at which it leaves the control volume. Why do we talk about entering and leaving? This particular volume, where is it? It is a small portion in a fluid continuum.

So, in this room there is a box here. And within this, I am looking at what is coming in and what is going out. And, why is it coming in because there is flow. This flow is coming in like this and this flow coming in like this, from all sides there may be flow like this. So, it is coming in through some, it is coming into the control volume and then it is going out through the control volume. And, in the process there are changes. There can be changes. And, it is those changes that we are looking at. And, what we are saying is that as the fluid is flowing through there can be changes. But, those changes will have to obey the conservation of mass. And, what that conservation of mass is that the rate of accumulation mass within the control volume is the rate at which it enters the control volume minus the rate at which it leaves the control volume.

If there is a difference between the two, then there is a rate of accumulation. If what is leaving out, leaving is greater than what is coming in, then it is a rate of decrease of mass within the control volume. If the rate at which it enters is more, then there will be rate of accumulation. So depending on whichever is greater, the rate of accumulation can be negative or positive.

So, we can describe this mass here using the concept of density. And, so we know the density tends volume is a mass. And, we are therefore saying that the density of the fluid in this location, may be at the center of this or integrated over this entire volume is times the volume; which is delta X, delta Y, delta Z is the total mass. And, rate of change of this with time can be represented as partial derivative of dou by dou t of rho delta X delta Y delta Z. So, this is a statement of the rate of accumulation. A mathematical representation of this.

Now, we need to find out the rate at which the fluid enters and then the rate at which it leaves. If you take a surface here, for example, this phase here at this point, then the mass flow rate through that surface is given by rho u dot A. Rho u d A, which is integrated over the entire surface here. So, here this is a dot product. So, that is if the velocity is at an angle to this, then there is less flow which is entering. And, as if the velocity is parallel to this phase, then they cannot be, it cannot be entering the control volume. So, that is why we take a dot product between the outward normal vector of the phase, which is A here, and then the velocity which has a certain orientation.

So, now we have chosen these phases in a convenient way. We have chosen this. If we consider this block, this box here, it has a left phase and a right phase, a top phase and a bottom phase, a front phase, which is this one and a back phase. And, these are oriented in the corresponding X, Y, Z coordinate system. So, that means that the outward normal vectors for each of these phases is aligned either in the X direction or Y direction or Z direction. So, that means that we need to multiply only by the corresponding velocity component here, in order to get this.

For example, if you look at this one here, if you look at the left phase, this is the phase we are assuming that u has components U, V, W. And, these are positive, so that if we have a U here, then that is aligned in the X direction; so, that means it will be entering. Here, on the right phase if there is a U, then it is again aligned in the X direction, positive X direction. So, it will be leaving the surface. So, we can say that through the left phase and through the bottom phase and through the back phase, flow is entering. And, through the right phase and through the top phase and through the front phase, the fluid is going out. So, you can put it like this or you can strictly apply the dot product and then get the same values here. So, we can say that the total flow rate that is entering the mass flow rate is the density times the X velocity component, which is U, times the area of the left phase, which is delta Y times delta Z. And, this is evaluated at X at this particular point here, which is at X and mid Y and mid Z here. And through the bottom phase, you have a corresponding velocity which is perpendicular to the phase direction is V.

So, rho V times the area of this phase which is delta Z times delta X is the total flow rate, mass flow rate that is coming in through the bottom phase. Similarly, through the back phase it is a W velocity component. At the mid-point of this phase at a Z location of Z here is A. So, that velocity W times the area of this phase which is delta X times delta Y will give you the total flow rate that is coming in. So, similarly we can say, we can evaluate the mass flow rate out which is rho U times delta Y delta Z at X plus dX because this phase is at a location of X plus delta X here. So the velocity at this point, that is, at a point of X plus delta X can be different from velocity at X because of the changes that are taking place inside the control volume. If that is the case, then the flow rate that is going out has a velocity of U at X plus delta X. And, this U at X plus delta X can be different from U. If it is different, then there is a differential amount of mass flow rate which is coming here, depending on whichever is greater, there can be accumulation. Similarly, there can be accumulation because V at Y and V at delta Y, Y plus delta Y are different and W at delta Z and W at Z plus delta Z are different. So, this is a total mass flow rate going out and this is the total that is coming in.

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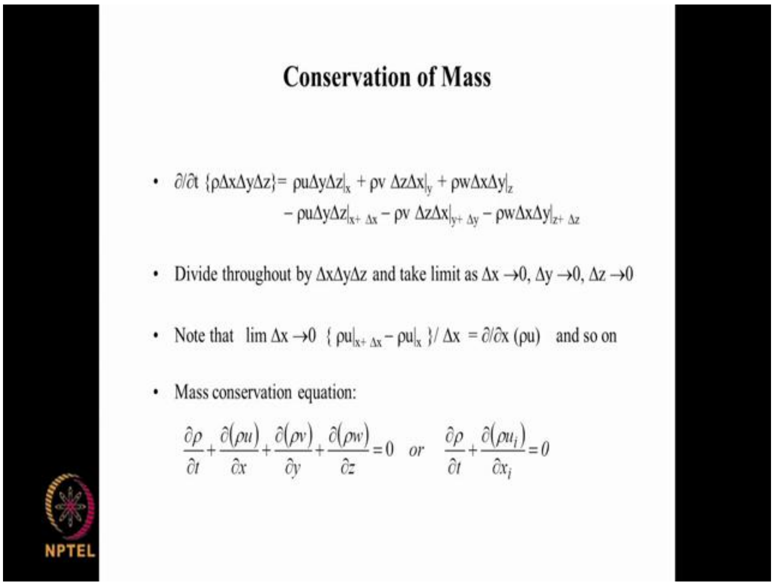

And, you can write a mathematical expression; dou by dou t of rho delta X delta Y delta Z equal to all that is coming in minus, all that is going out. So, we can now do some simple manipulations. We can divide this whole thing by delta X delta Y delta Z. The product, the volume and then take the limit as delta X tends to 0, delta Y tends to 0, delta Z tends to 0; because we are looking at equations which describe change at every point. So, we are looking at a control volume with a volume of a point. Of course, point has no volume and we cannot conceive a particles having no volume. So, this is the when we say delta X tending to 0, we are looking at so small; that it is practically like a point, when we look at our applications. But mathematically, it is still large enough that we can think of billions of particles which are present, so that we can talk about average properties along it. So, that is known as a continuum hypothesis.

So, within that 0 is defined subject to the continuum hypothesis. So, if you do this, then the left hand side becomes dou by dou t of rho because this gets cancelled out. And, if we look at these two, delta y delta z cancel out. This gives you rho u at x minus rho u at x plus delta x divided by delta x. And, if you take the limit as delta x tends to 0, this becomes dou by dou x of rho u. So, these two together will give you minus dou by dou x of rho u. These together will give you minus dou by dou y of rho v. and, these two together will give you minus dou by dou z of rho w. So, these are all minus. So, you bring it to this side and then you have this form here, which is a conservation of mass. So, dou rho by dou t plus dou by dou x of rho u plus dou by dou by or rho v plus dou by dou z of rho w equal to 0 or delta rho by d rho by dt plus d by d xi of rho ui equal to 0 is a index notation.

So, this is the statement, mathematical or a conservation of mass. So, in the next lecture we will look at how to do the, how to derive an equivalent mathematical expression for a conservation of linear momentum and all that.