Introduction to Time-Frequency Analysis and Wavelet Transforms
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Module - 03
Lecture -3.2
Continuous- time Fourier transform

Welcome to lecture 3.2 of the course on time frequency analysis and wavelet transforms. In this lecture we are going to learn the mathematical details of continuous time Fourier transform, obtain interpretations of the same, and also learn an exciting result which is very important in the time frequency analysis which is a duration band width principle; we are not going to prove it, but we are going to just obtain glimpses of this results.

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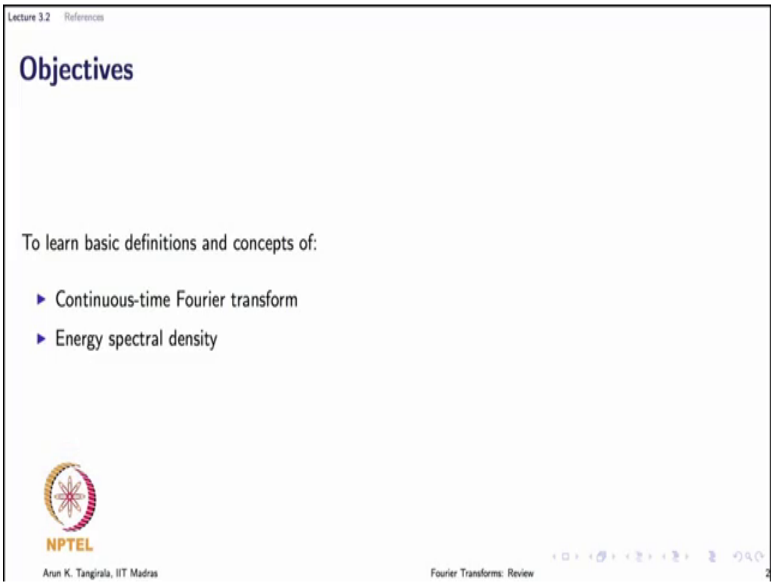

The objective of this module is to go over the mathematical details of the continuous time Fourier transform, and also understand what is energy spectral density, as well as obtain glimpses of the duration band width principle. In lecture 3.1 we reviewed the concept of continuous time Fourier series and the mathematical details as well, where we dealt with periodic signals, continuous time periodic signals.

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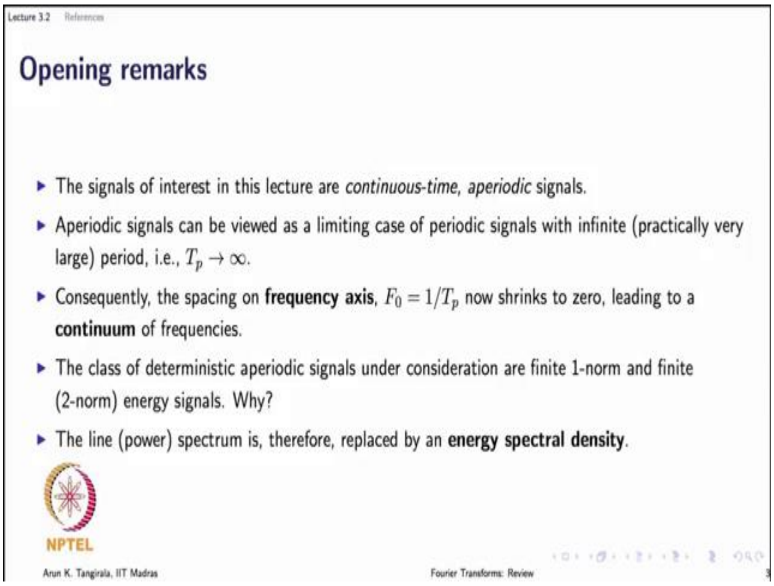

Now, we move on to the class of continuous time aperiodic signals which is where the Fourier transform arises. And, aperiodic signals can be viewed as a limiting case of periodic signals as I had also mentioned in lecture 3.1. They can be viewed as a limiting case of periodic signals with infinite period. Practically what this means is I never get to see the reputation of the signal. And, this is the view point that is used to derive the Fourier transform itself from Fourier series.

The reason for discussing Fourier series first is because historically the Fourier series came first. Now, when we let, T p, go to infinity that is the period go to infinity, the spacing on the frequency act access that existed in Fourier series which is 1 over T p now shrinks to 0. As a result we have a continuum of frequencies unlike the discrete set of frequencies that we have in Fourier series.

Moreover, the class of deterministic signals that we are going to consider are either the finite 1 norm or the finite 2 norm which is a slightly weaker requirement for the existence of Fourier transform. The main reason for considering only these signals is as we will learn shortly, these are the requirements for the Fourier transform to exist, that is to make sense.

And finally, the line spectrum, the line power spectrum that we learnt in Fourier series, now is going to be replaced by energy spectral density. The reason is we know that aperiodic signals with finite 1 norm or finite 2 norm have finite energy, but they have 0 power, average power, and therefore, they are energy signals. And, because we are going to decompose energy in frequency domain and that the frequency access is a continuum, we can now concede of an energy spectral density.

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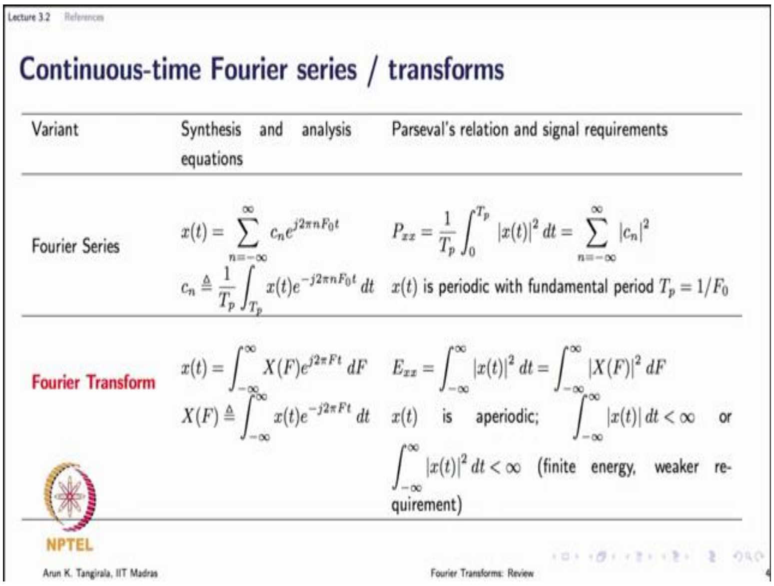

So, referring to the table that I showed in lecture 3.1, now the focus is on Fourier transforms. So, let us quickly go over this table before we dwell into mathematical details. Again, the second column gives me the synthesis and analysis equations. The prime difference between the synthesis equation for the Fourier series and for the Fourier transform is that the summation in the synthesis equation for the series is now replaced by an integral; naturally so, because now the frequency access is a continuum.

The integral expression for the Fourier coefficients in the Fourier series also now remains as an integral. The only difference is now there is no notion of T p because the signal is no longer periodic. And, we integrate from minus infinity to infinity because now we assume that the signal exists, of course, even the periodic signal exits forever, but now the aperiodic signal has to be evaluated over its entire existence, whereas, for the periodical signal it suffices to evaluate the integral over single period.

If you move on to the third column it gives us the energy spectral decomposition result, once again due to Parseval, very much alike to the power spectral decompression result for the periodic case. Now, the notion of power energy spectral density arises because of this integral on the right hand side because the area under squared magnitude of X F gives me the total energy. I can think of mode X F square as an energy spectral density. And, the requirements on the signal as I mentioned earlier are given here. So, we will, of course, discuss this again at a later stage.

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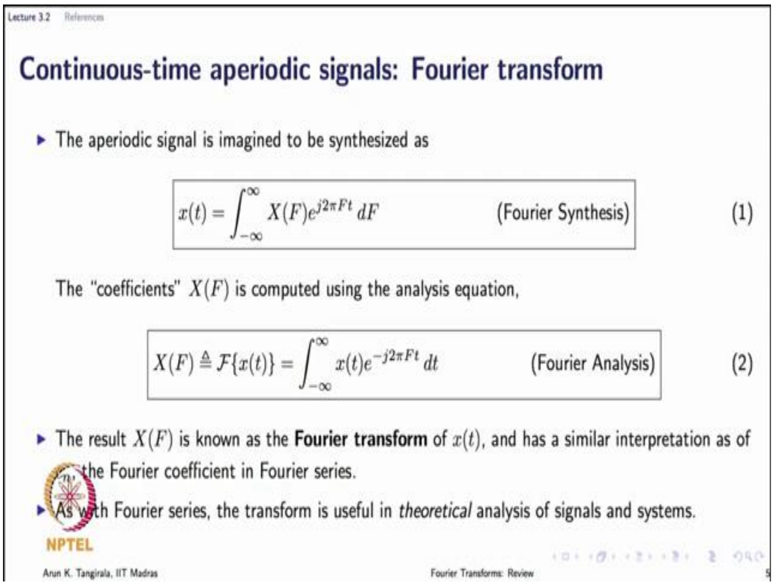

Now, going into the details a bit more. I have now the Fourier synthesis equation for the continuous time aperiodic signal, is given in equation 1, and the analysis equation as given in equation 2. As I said earlier, the prime difference between the Fourier series and Fourier transform is that the summation for the synthesis equation is now replaced by an integral, and the reason is as follows. If you go back to the Fourier series expression and substitute the expression for c n using this integral and evaluate the summation in the limit as T p goes to infinity, you obtain an integral. And, I will just briefly discuss this on the board.

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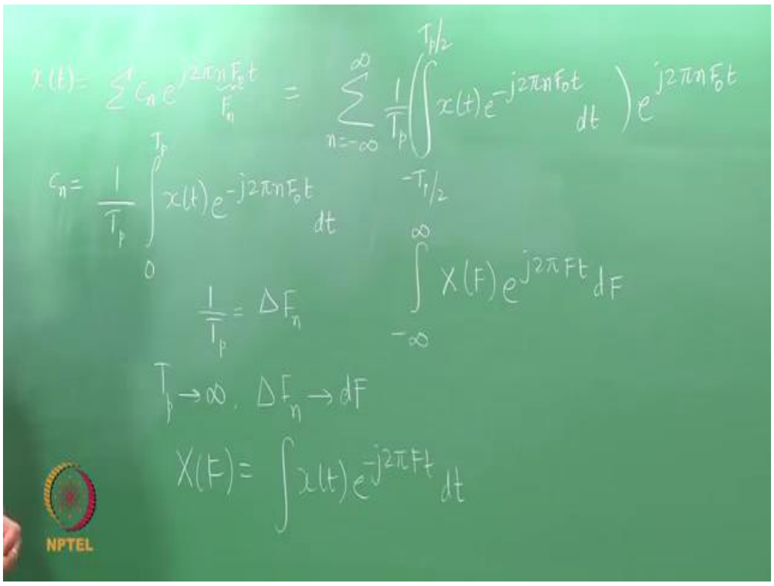

So, start with the Fourier series expansion of x t, I have here c and j 2 pi n F 0 t, we shall call this n F 0 as F subscript n which is nothing but the frequency of the nth harmonic. And, we have the expression for c n via this integral x t e to the minus j 2 pi n F 0 t d. So, I have this integral. What I am going to do is I am going to plug in this expression for c n here, so that I have here summation n running from minus infinity to infinity and then I have this 1 over T p integral x t e to the minus j 2 pi n F 0 t d t times, e to the j 2 pi n F 0 t.

We shall also introduce this notion of this spacing between two successive harmonics, is 1 over T p is nothing but delta F n which is the spacing between two successive harmonics. Now, this limit here runs from 0 to T p, but we will make a slight change without causing any change in the overall result itself; instead of evaluating the integral from 0 to T p, I can evaluate it from minus T p over 2 to T p over 2, as long as the integral interval is 1 over 1 time period it really does not make any difference.

Now, when I let T p go to infinity then the spacing becomes infinity small. I am going to replace the finite spacing within infinity smell spacing. And also, I am going to drop the subscript n because now the frequency access is a continuum, alright. And, we are also going to replace, sorry, interchange the order of the summation and the integral assuming that it can the done. It can be done for this situation.

So, what we have here is, in the limit as T p goes to infinity, I am just writing the final result, I have taken this integral outside and I have minus infinity to infinity. 1 over T p can be now replaced with d F. And, if I define X F as integral x t e to the minus j 2 pi f t d t, then I can rewrite this expression as x f e to the j 2 pi f t d f, so that is the final result that I have for the synthesis equation.

Of course, this is an adhoc derivation, a formal derivation, it is not completely adhoc, but the formal steps that are involved are left out. There is a slightly more detailed derivation of these equations in many standard texts such as the signal processing text by Oper, Heim and Schafer, and so on. So, I would advise you to refer to this text for more detail derivation of the Fourier synthesis equation for the aperiodicals.

Okay. So, now, we return to the interpretation of the results; the rest is all about interpreting these results. So, I have this Fourier analysis equation and the interpretation of X F is on the same lines as that of the Fourier coefficients for Fourier series. X F is a complex quantity. It denotes the weighting associated or the weights associated with the analysis function, or the building block which is a complex sinusoid, and also it contains the phase information.

So, if I look at the magnitude of X F, it gives me the weight associated with the corresponding building block. And, the phase of X F of will tell me when that particular frequency started to exist in the signal. And, the other point that I would like to make is as with the Fourier series the transform is, this particular transform is useful in theoretical analysis because in practice I have only sample data. So, these equations are not straight away valid for sample data.

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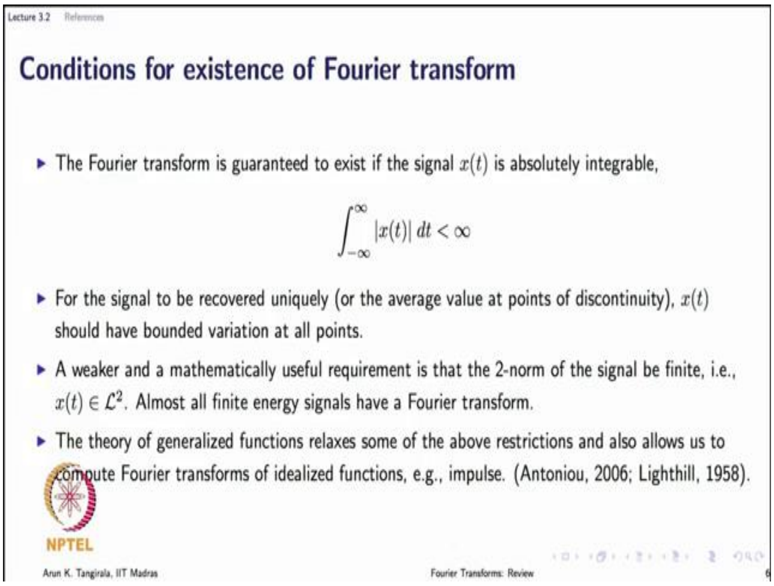

Before we move on to the discussion on energy spectral density, it is useful to know the conditions for the existence of Fourier transform because then that will tell us that we are dealing with energy signals. The main condition for the Fourier transform to exist is that x t be absolutely integrable. When we say Fourier transform should exist looking at equation 2, I want X F to be a bounded value that is the integral should yield me a bounded value, that is a primary requirement.

And, with that requirement you can start of, by require the magnitude of X F to be less then infinity, and then use the triangular inequalities or the, and so on, certain inequalities, and then you can arrive at this result or the requirement that x t be absolutely integrable. Also, it is important to ask when this integral converges and whether it converges to x t. Both are tied together, and the common requirement is that the integral be absolutely converge, the integral of the, signal to be absolutely convergent.

And, for the signal to be recovered uniquely x t should have bounded variation; that means, you should not have infinite amplitudes at any point in its existence. A weaker requirement is that the 2 norm of the signal be finite. Why we say weaker is, the same reason that we had said for the Fourier series. In the Fourier series we said finite 2 norm over 1 period then the series itself may not converge exactly to x t, but it converges in a integral square sense.

So, here also the same situation exists. When I have a finite energy signal, the integral here in equation 1, converges to x t, not strictly but in a integral square sense. That is the error between x t and this integral will diminish in a, the squared error will diminish and not the error, that is what it means. Now, there exists a theory of generalized functions that relaxes some of the above restrictions. So, as to evaluate Fourier transform or some ideal functions such as impulse and so on, I advise you to refer to these 2 texts to know more about this theory of generalized functions.

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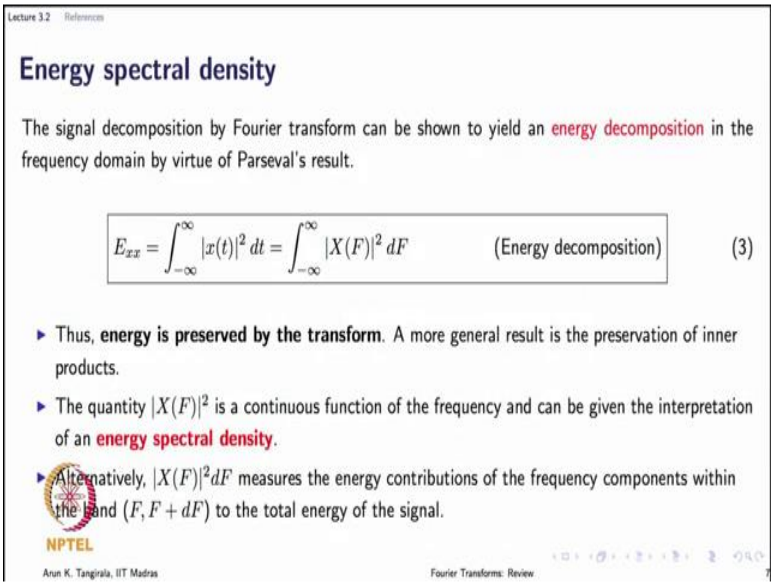

So, now coming back to the practical part of the Fourier transform. We are interested in this energy spectral density. As I mentioned earlier, this result is again due to Parseval where we say that the energy is preserved in both domains. That is, whether I evaluate the area under x t square, mod x t square, or mod X F square, I recover the energy of the signal. We call this as energy preserving transforms and so on.

Now, this quantity mod X F square, unlike the mod c n square, is a continuous function of frequency, and therefore, I can give it the idea of energy spectral density. Of course, not just because it is continuous because the area under it gives the energy, I can call it as a energy spectral density. The alternative way of looking at it is, the fact, quantity here mod X F square times d F, gives me the energy contribution of the frequencies present in the band band F, F plus d F, that is an alternative way of looking at it. Overall, I have an energy decompression in the frequency domain. So, let us work through an example.

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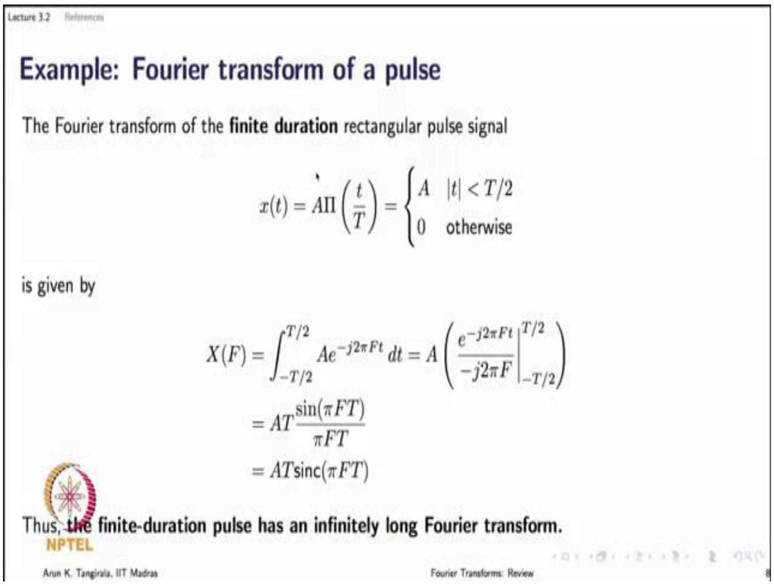

Here I have taken a finite duration rectangular pulse signal because it is a finite duration signal, it will have finite 1 norm, it will have finite energy because amplitude is also bounded as you can see here. Therefore, Fourier transform exists, that is something that you should ensure even before you compute the Fourier transform. The mathematical expression for the Fourier transform is worked out here, and this is something that you can also work out fairly easily, the details I have given.

What we have here is what is known as sinc functional or sin c function. So, the Fourier transform of a rectangular pulse is a sinc function; and, the sinc function are exist for ever. So, the quick observation that we can make is whenever I have, well, it is not necessarily that I can generalize straight away, but this is true whenever I have a finite duration signal the Fourier transform of that signal will be infinitely long.

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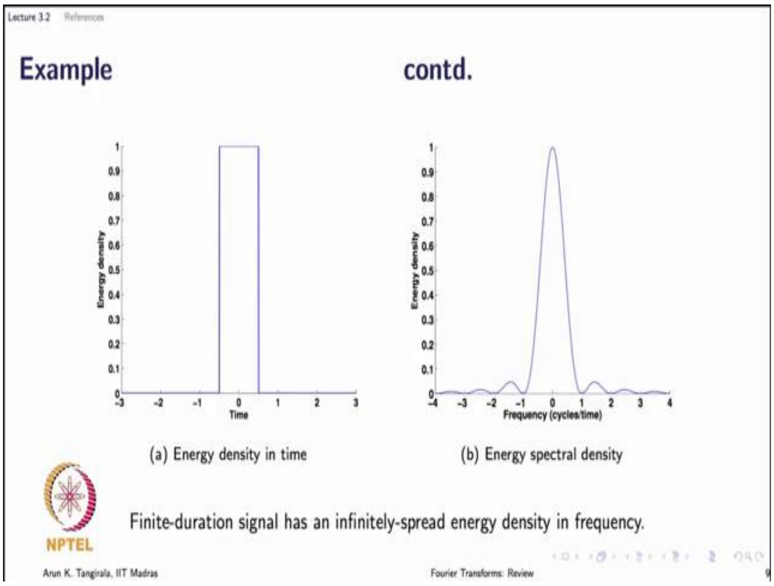

Let me illustrate the same example here with the specific value of, a; I have chosen, a, to be 1. And, I am showing you here the energy density in time; this is not the signal itself, this is energy density. And, on the right hand side, here you see the energy density in frequency. How do I calculate energy density in frequency? Very simple; here I have X F, I take mod X F square, and I obtain the energy density in the frequency domain.

Now, an observation here, because a signal is symmetric in time the Fourier transform turns out to be a real value function. In general, the signals that we work with need not be symmetric then your X F will turn off to be a complex quantity; that is something to remember. So, what do I observe here? The energy density exists only over a finite time; whereas, the energy density in frequency exits forever. Well, we do not say exits, but it is actually infinitely spread.

Rather than talking of energy densities we can talk of what are known as duration and band width. The duration is essentially a measure of how long the signal exist in time and band width is a measure of how much spread is present in the frequency domain, that is what is the spread of the energy, spectral density. Now, the formal expressions for duration and band width in unit 4, but the qualitative field should already be there, alright.

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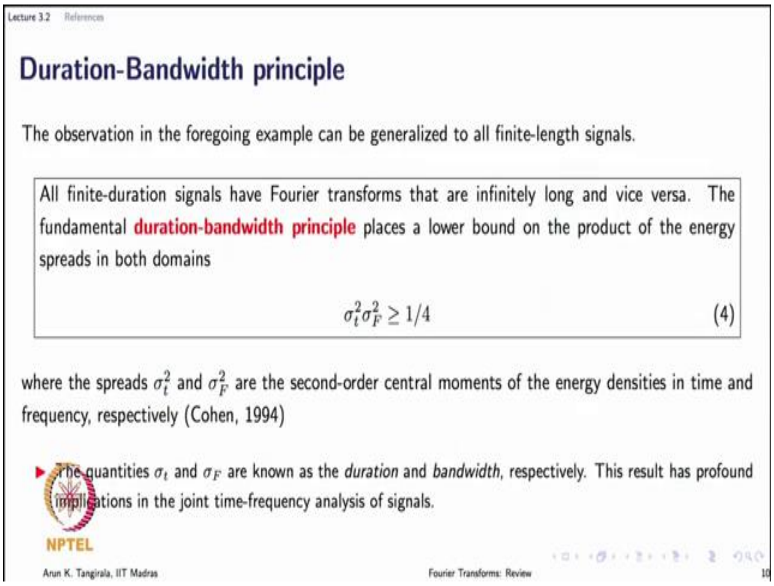

So, with this qualitative field for duration and band width, and without going through a formal proof, I am stating the duration bandwidth principle which we will revisit later on. It states that for all finite duration signal setup Fourier transforms, they have infinitely long Fourier transforms. And also, it states that if the signal exist for infinite time then its Fourier transform or the energy density will actually exist only over a finite interval in frequency.

Well, the main result of the duration band width principle is that the product of duration and band width is bounded below by this number 1 over 4. Rather than attaching so much importance to this number, the main interpretation of this result is that whenever the duration of the signal is finite the bandwidth is going to be very large, that is if duration is small, sorry, if the duration is small then the band width is going to be long.

And, these quantities sigma square t and sigma square f are not exactly the duration; the sigma t is the duration, sigma square t is the second order central movement of the energy density. Now, we start to see the utility of defining energy densities. The energy densities can be viewed in a similar fashion as probability density functions; sigma square t is the second order central moment of the energy density in time, and sigma square f is the second order central moment of the energy density in frequency.

All you have to do is view the energy densities as analogues to probability densities. Then, sigma square t will take the place of variance when you think of random variable and sigma square f likewise. The only difference is the density functions that you are using to arrive at this quantity sigma square t and sigma square f. Of course, as I said we will go through this more formally in the next unit, alright.

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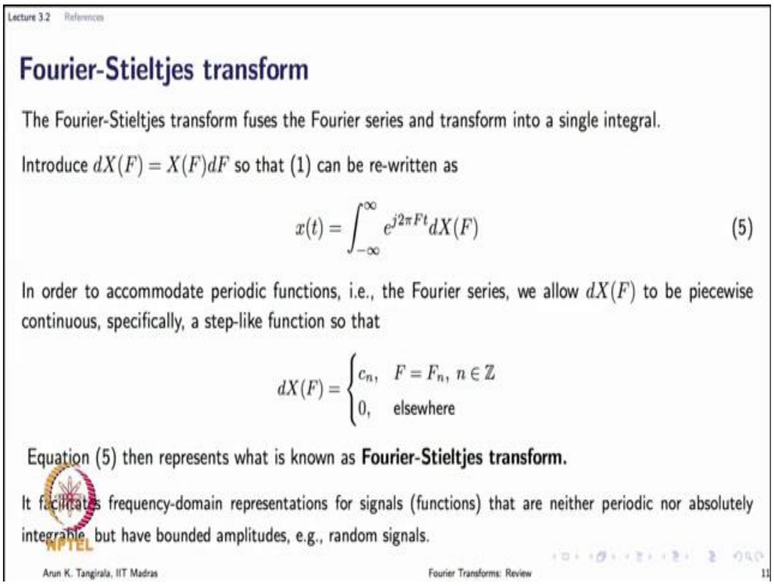

Finally, the concept that we would like to discuss is a Fourier stieltjes transform. This may not be so important in the time frequency analysis, but it is a useful result to know in the general analysis of, frequency domain analysis of signals. The Fourier stieltjes transform essentially fuses the Fourier series and the Fourier transform into single integral, and how does it achieve it?

So, look at this equation 5 that we have, I have x t as minus infinity e to the j 2 pi f t d X F. Compare this equation with the synthesis equation for the x t. What we are doing, that is to understand Fourier stieltjes transform I am going to replace X F d F with d X F; I am going to define d X F as X F times d F whenever x t is periodic. When x t is periodic, I will have d X F behave like a piecewise continuous functions, that is specifically a step like function, alright.

And, the expression here for d X F shows the step like behavior that we want for d X F, so that it also represents a Fourier series. How does it represent the Fourier series? Well, plug in this d X F into this equation 5, then the integral is not defined everywhere; it is only defined at specific points, therefore, we replace this integral with a summation.

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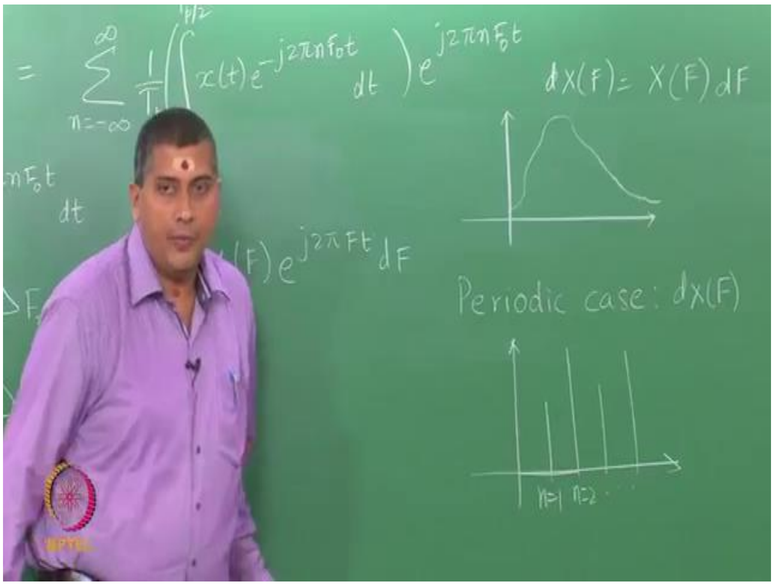

So, let me show you quickly how d X F would look like for aperiodic signals and for periodic segments. So, for aperiodic signals d X F is a continuous quantity, and it can have any shape that you want; it could be any continuous shape. Whereas, for the periodic case, d X F is not continuous, but has a step like behavior as you see on the slide.

It is only defined at specific points on the frequency. So, it could, it essentially looks like, it could be something like this, alright. So, it is essentially 0 at other points and non- 0 at this specific points, let us say this is n equals 1, n equals 2, and so on. As a consequence, the integral that you see that is Fourier stieltjes integral now becomes a summation, that is the point.

Well, why is this Fouriers stieltjes transform useful? It is useful in handling signals that are neither periodic nor absolutely integrable. Of course, we are not going to deal with such signals here; a classic example of such signal is a random signal. Anyway, so, this was just to give you an insight into the Fourier stieltjes transform, not that we are going to use it extensively in this course. So, with this we come to the close on this module on continuous time Fourier transform.

So, we have dealt with continuous periodic signals and aperiodic signals. In the next module we are going to look at discrete time periodic signals, the expressions simply because the discrete time sinusoids are unique only in a finite interval as we discussed in unit 2 when we talked about sampling and when we talked about periodic discrete time signals; we recognize that discrete time sign waves are unique only in the fundamental frequency range minus point 5 to point 5. So, that will make a difference to the expressions for the discrete time Fourier series. So, these are some of the useful references that you can read up to obtain more theoretical details.

Thank you.