Introduction to Time-Frequency Analysis and Wavelet Transforms

Prof. Arun K. Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 3.1
Continuous-time Fourier Series

Welcome to lecture 3.1. This is the first lecture in the unit on review of Fourier transforms. In this lecture, we will particularly review continuous-time Fourier series, but we shall begin with general overview of Fourier transforms, the conceptual aspects of it, and then we will slowly dwell into the map.

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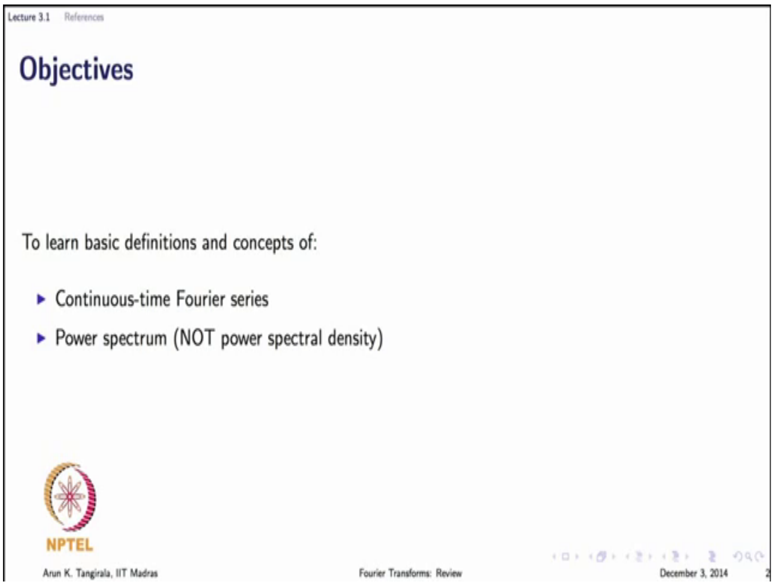

So, as I said the objectives of this module is to review continuous-time Fourier series and the concept of power spectrum, not power spectral density, because the notion of power spectral density does not apply to continuous-time periodic signals.

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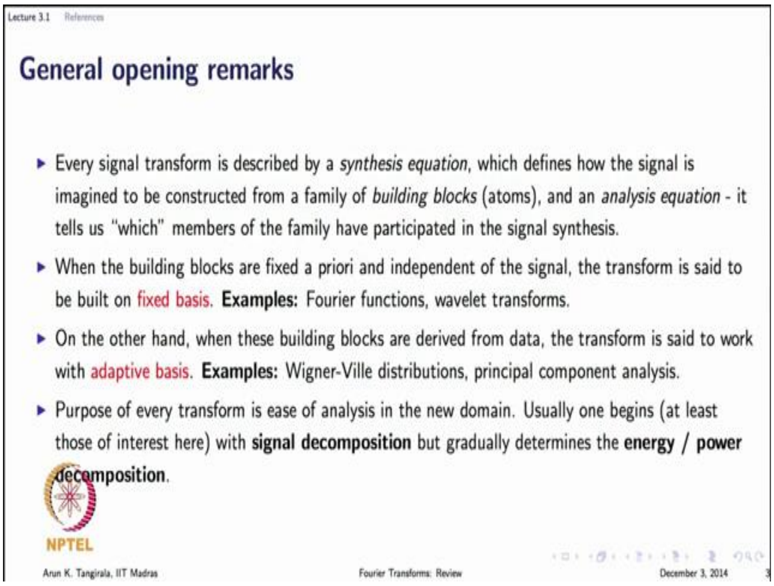

So, before we dwell into the continuous-time Fourier series, a few opening remarks that I would like to make. Any signal transforms that you study in the literature, be it Fourier or wavelet or short-time, and so on, is described by what is known as the synthesis equation. So, the person who propose this transform would first propose a synthesis equation, that is imagine how the signal is made up of in terms of what are known as building blocks. And then analyze the signal, so that the procedure, line of procedure for proposing a transform, but pedagogically what we would be interested in and even practically we would be interested in the analysis equation, because I have the signal already. So, I do not have to imagine how it is made up of rather I am interested in analyzing the signal.

So, normally when I am learning transforms, I learn the analysis equation first, and then worry about synthesis, but both are important, in terms of the development the synthesis is followed by analysis, but in terms of utility analysis signal and then you use the synthesis equation for reconstructing the full signal or the part of the signal and so on. So, the analysis equation will tell us which building blocks really participated in the signal transitions and remember that all of this is just imagination. So, it is very hard to claim and verify in many situations whether truly the signal must have been synthesized that manner, in that manner.

So, what is the reason for assuming a transform? It is simply because it makes my analysis easy. In the new domain signal has such a representation that certain features that I am searching for become obvious, a classic example being the Fourier analysis.

Oscillatory features of a signal are highlighted very well in the Fourier representation of the signals. So, that is the primary reason we look to transforms.

Now, the second point that I want to make is, we said that synthesis equation, essentially tells, tells you how this signal is imagine to be constructed in terms of this building blocks. If these building blocks are fix a priory, like in Fourier and wavelength and so on, we say that working with fixed bases or fixed analysis functions.

However, why should a signal be made up of what I imagine a priory signals can be made up on their own, and I should first study the signal to really know what they are made up of and that is the philosophy in so called the adaptive bases decompositions of signal, and so on. And Wigner-Ville distribution and principal component analysis and so on, even the Hilbert ((Refer Time: 03:40)) transforms or the empirical mode decomposition really fall into this category. The final and the third point is, in any transform that you study, first we will study the signal decomposition, that is imagine how the signal must have been made up of, break it up, and so on. But finally, we move to the energy or the power decomposition of the segment.

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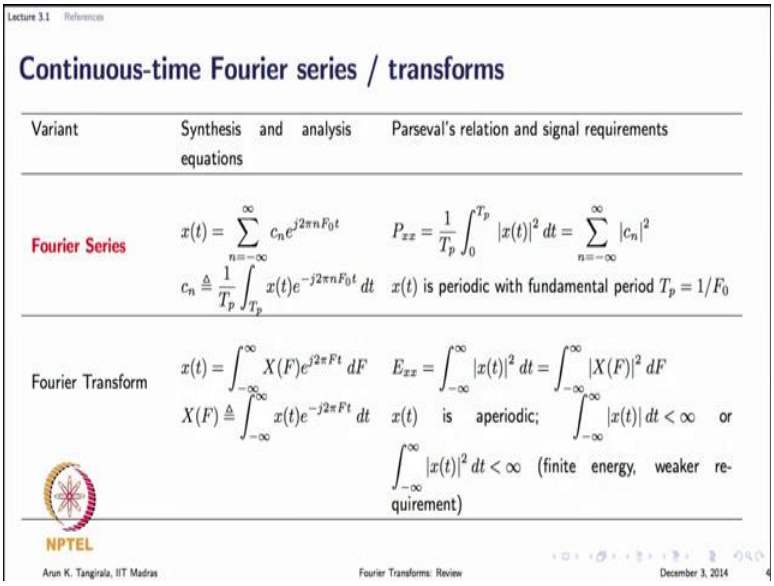

So, now coming to the Fourier transforms, there are many version of the Fourier transform that you see of Fourier series and so on. But primary among them are four different types, the Fourier continuous-time Fourier series or Fourier series itself, which is applicable to periodic continuous time signals. And then you have the Fourier transform for the continuous-time signal, which is applicable to a periodic finite energy signals. And then you have the discrete time versions of each of this.

Now, in this table what you see or this, these three columns, of course, first column is name. In second column I have given you new synthesis and analysis equations and in the third column I have given you how the energy and power decomposition occurs. So, as I said earlier, you start off a signal decomposition and then develop the synthesis equation and decomposition equation, but quickly move on to energy decomposition. That is very important to me because that is what I would like to use in practice and so on. So, in this module we will be primarily interested in the Fourier series and look at the synthesis and analysis equation.

We assume, that signal is periodic, that is the major assumption and what is happening is, that the signal is being expressed as an infinite weighted combination of complex exponentials. And these complex exponentials are periodic with the same period as the signal itself. And beneath you see what the analysis equation is and we will go into more detail very soon. And in the third column you see what is known as the Parseval's relation, which essentially tells you that power that you calculate, average power that you calculate in the time domain and can also be calculated and in the Fourier domain using the Fourier coefficients Cs, and it states clearly the assumption that is made on x of t.

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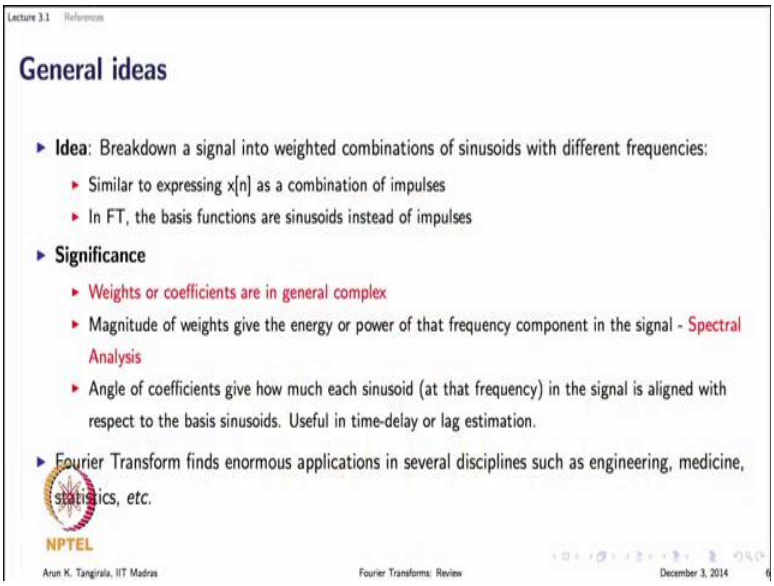

So, let us move on and look at the continuous-time Fourier series more in detail. As I said, the idea is to break down or express the signal as weighted combination of sinusoids with different frequencies. The given signal has a certain period, therefore, but it need not be a sinusoid, it could be a square wave and so on.

So, what I am trying to do here is, I am expressing the periodic signals in terms of sinusoids of the same frequency. But remember, if a signal has a frequency, fundamental frequency f naught, it also has or a time period tp, fundamental period tp, it also has a period 2 tp. It repeats after 2 tp, 3 tp and so on. Therefore, it is natural to consider all signals, all sinusoids with these periods tp, 2 tp and so on and these are called harmonics plus the so called DC components, 0 frequency components account for the non-zero mean of the signal. So, there is a certain significance to the weights that we use in this expansion and we will explain this more in the detail.

But what these weights give me is to the extent to which the building blocks are appearing in the signal and a nice analogy can be given by thinking of a song. So, if you think of persons, of a music composer who is actually conducting an orchestra and composing a song. Basically, this song contains, this song has to be synthesized by many instruments, which give out sound at different frequencies and so on. Now, not all instruments are going to be of this same loudness. So, that means, that some will be much more audible than and others and so on. Then, we say, the weight for that particular instrument is more and so on in terms of Fourier language, alright.

So, the other part is the phase, the phase of the Fourier transform will tell me when a particular building block or sinusoid started to participate in signal. So, again you go back to the music example you can think of, when a particular instruments starts to play and time-frequency analysis takes you a step further, when it stops as well, alright. So, that is the analogy that you can keep in mind. There are, of course, analysis you will find in several other texts.

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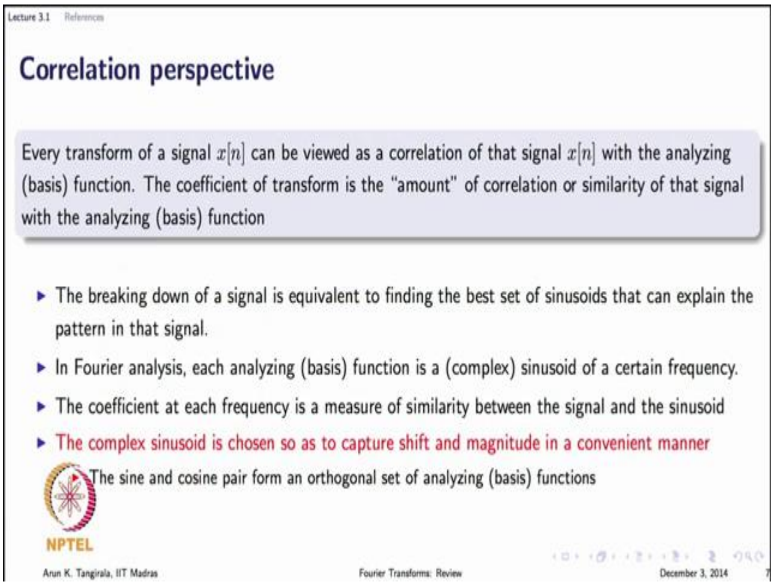

So, the Fourier series or the Fourier transform can always be given correlation perspective and this is true for other transforms as well. Whenever I am computing or analyzing the signal using a certain transform, essentially what I am doing is, I am taking this signal and trying to match it with the building block. If I find a good match, then I say, yeah, this building block is present in large proportions. And knowing the properties of building block I can immediately infer the properties of the signal as well or the, if I know the properties of the basis. That is the advantage of using of the fixed bases.

In fixed bases I know property of bases function a priory and therefore, the moment the weight associated with that, weighting associated with that basis function is large, I can immediately conclude, that this particular feature of the bases function is present significantly in the signal. Whereas, with an adaptive bases you do not have a priory what bases function go into the making of the signal.

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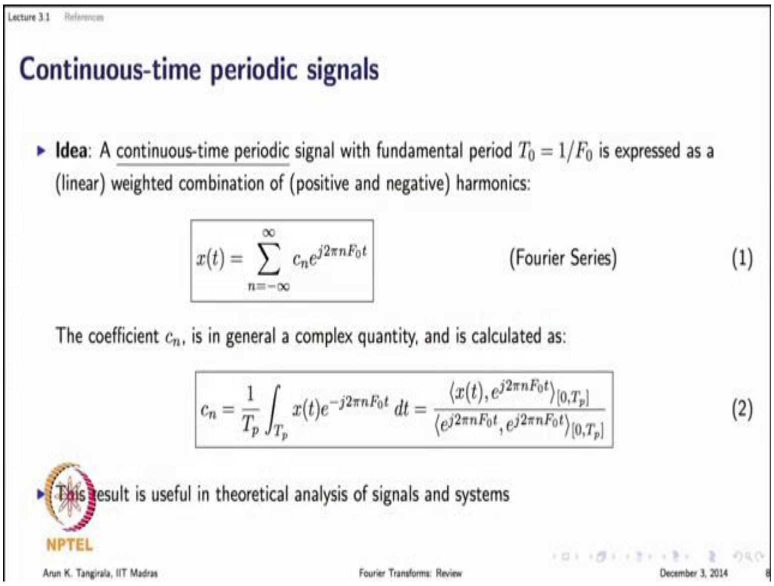

So, coming back to the Fourier series synthesis equation that we have here, this is the same expression that you saw earlier. Weighted combination of fundamentals plus harmonics and the DC components is going to synthesize my signal. That is my imagination or Fourier’s imagination and the goal in analysis is to find these coefficients c n. These are called Fourier coefficient as well.

And the expression for c n is given here in equation 2, which is obtained by multiplying both sides of equation 1 with the complex conjugate of the building block here, e to the j 2 pi n f naught t. In other words, I am going to take an inner product of x of t and the building block, which is a complex exponential, but evaluate it only over the interval 0 to tp because this x of t is a periodic signal, alright. So, this is an inner product evaluated only over this interval and that is what the subscript means.

You may recall the definition of inner product between two functions that we learnt in module or lecture 2.3 and the derivation for this coefficient is given in many texts. Therefore, in interest of time I avoid it, but you can also do it by yourself, it is a fairly straight forward thing. Just substitute this series into, in place of x of t in equation 2 and you should get the expression.

Now, having said all of this, this particular continuous-time Fourier series is useful in theoretical analysis because in practice, I do not have continues-time signals. So, the goal is now to, for the rest of the lecture is understanding what this c n mean to me, what is

Fourier coefficients mean to me, but before we do that we have to understand why negative frequencies are coming in. Originally, in Fourier's proposition negative frequencies did not exist because Fourier propose sines and cosines as the building blocks, but then you could replace sines and cosines mathematically with complex exponential.

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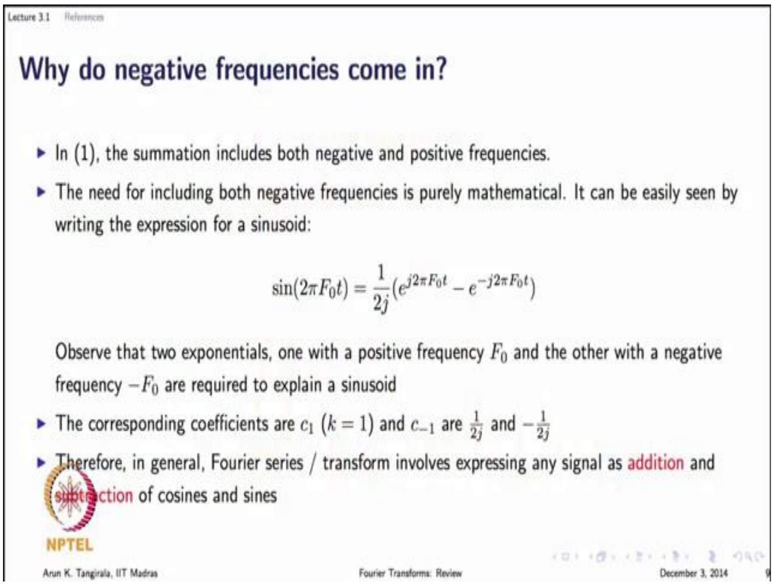

And a quick way of understanding why negative frequencies coming, come in because even to express a pure sinusoid I require two complex, two complex sinusoids, one being the complex conjugate of the other and that is the only reason why I have the negative frequencies coming in. No particular reason that should be attributed to this appearance of negative frequencies and the same argument holds for cosine as well. You need two complex exponentials that are complex conjugates of each other. So, there is a certain symmetry for this basis functions expansion.

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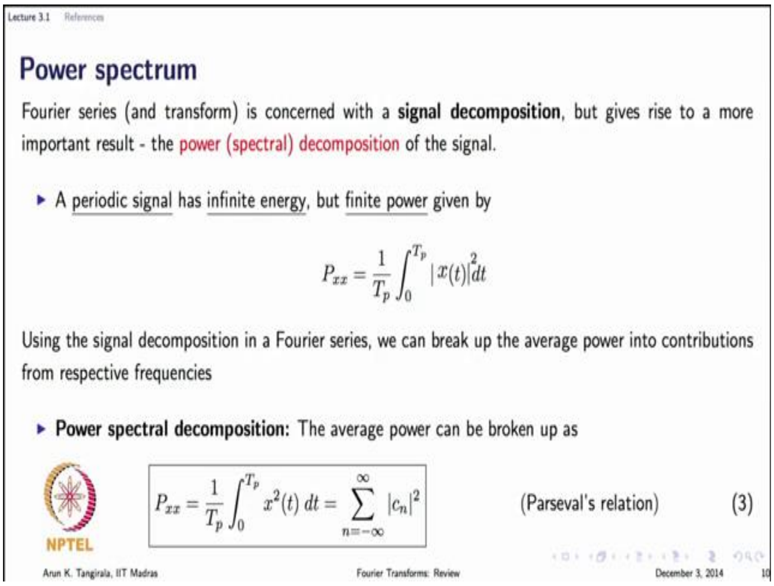

Now, one of the first interpretations for the c n comes from the power spectral decomposition. This is what we refer to earlier. We started off with the signal decomposition, but we quickly move on to power spectral decomposition. Now, Parseval showed that and even this comes from (Refer Time: 12:33) Plankeral’s formula, that the power, average power calculated or defined in time this way, you may refer to one of the lectures in unit 2 is nothing but sigma mod c n square. Now, here is where you have to be carefully interpreting c n. Mod c n square denotes a contribution of the nth harmonic to the overall power of the signal.

Now, this statement is correct because the complex exponential remember the mods. This c n is associated with the building block, which is e to the j 2 pi nf naught t. So, each c n is telling us how much each basis function has contributed to the construction of x of t. The question is, whether they are unique or there is some overlapping information between the c n’s. Now, you can show that is a nice property of this complex exponentials, that they are orthogonal to each other. What this means is the coefficient c n bring in new information, each c n is unique.

What one building block explains other building blocks cannot explain. That is a nice property of having an orthogonal family of analyzing functions and immediately. Therefore, we can say, mod c n square is the contribution of the nth harmonic to the overall power and that, that for, give rise, gives rise to the, also known as line spectrum because remember, the index and frequency domain is discrete. It is, it is kept track by n, whereas the index and time domain is continuous. So, you have a continuous-time signal, continuous time. You have an energy density, a power density in time, but you do not have a power density in frequency here, we only have a line spectrum.

And when I show you an, the example it will be more clear to you and remember that this coefficients are going to be complex value. So, the magnitude of this coefficient will tell me how much of this building block has gone, basis function has gone into synthesizing x and the phase of this will tell me when this basis function actually started to exist in the signal, that is called the phase spectrum when I plot phase as a plot of n.

Now, very importantly this spacing between harmonics depends on the fundamental period itself. Remember, the spacing between harmonics is the fundamental frequency itself, which is 1 over the period when the period of this signal goes to infinity, which means, I never get to observe the periodicity. We call this as aperiodic signal. What happens in the Fourier domain is, that the spacing starts to shrink and the frequency domain now becomes a continuum and this is what we will see in the analysis of aperiodic signals in lecture 3.2.

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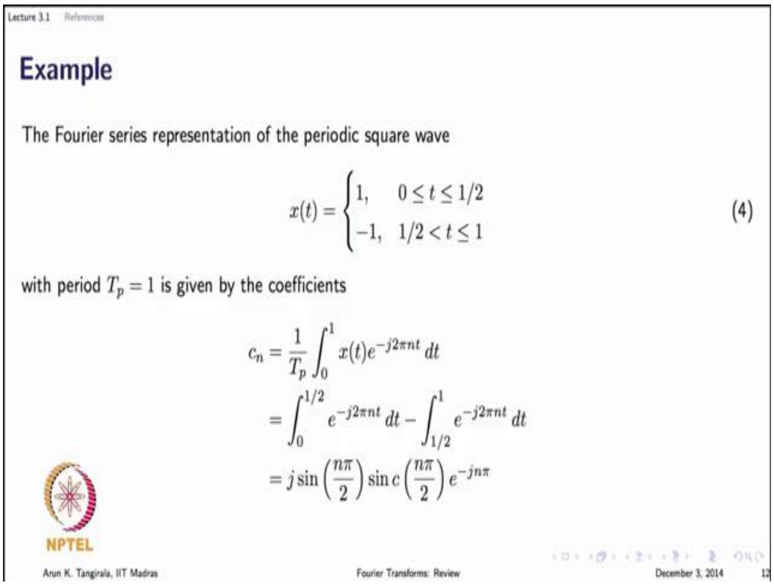

So, let us walk through an example and understand the continuous-time Fourier series in a better way. So, I am taking here a periodic square wave, which is the equation for which is given here in equation 4 and it has a period of 1. Now, I want to analyze this signal. I have this c n here, just working out the math, gives me this expression a.

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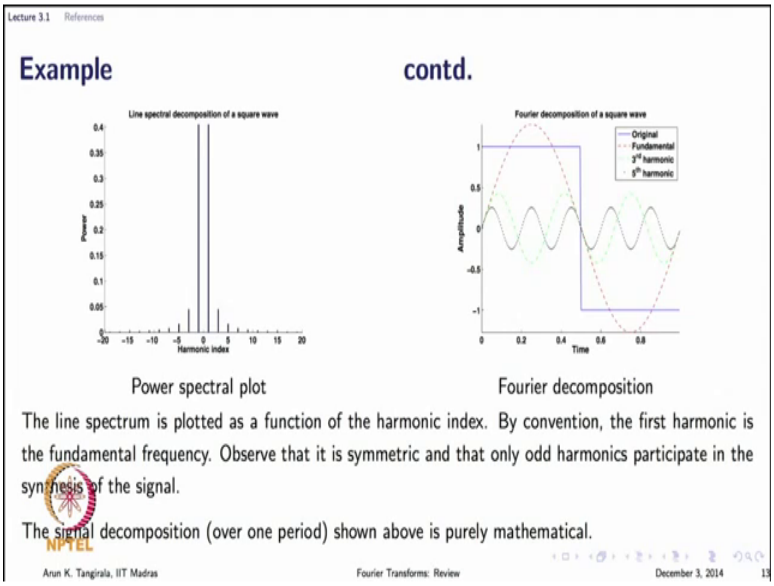

Better understanding is obtained by looking at this plot. So, what I am showing you on the left is the power spectral plot, which is a plot of mod c n square versus n, alright. Now, in one of the points that you must have seen here I have said, that you will get symmetricity by virtue of the complex, complex conjugate symmetry of this coefficients and therefore, the power spectrum is also symmetric with respect to n. Therefore, it is a common practice not to plot for the negative frequencies. There is no need. You only plot for the non-negative frequencies.

You can see, that the power is very large at the particular 1st, 3rd, and 5th. So, the 1st one is a fundamental frequency, n equals 1 and then you have the 3rd harmonic and then you have the 5th harmonic. Then, after that there is negligible power contributions from the other harmonics to the overall power, which means a square wave, that you see on the right is the power of that square wave contained in that square wave, is predominantly coming from contributions of the fundamental frequency. The 3rd harmonic and the 5th harmonic, the other harmonics also contribute, but in a very insignificant way.

So, we can say predominantly these three frequencies are present. Of course, a question is, whether if I add up these three building blocks, which have been scaled according to the magnitude of Cs, will I recover this square wave? Well, in this example I will recover the square wave except at the corners here, except at the corners because it is a very short discontinuity and that is what is known as a Gibbs phenomenon, you will see.

So, as you add more and more harmonics you will see more and more ripples here at the corners and you can read up. This is a very famous phenomenon with Fourier analysis. So, now the question is, whether this Fourier series exist for all periodic signals or not? What we mean by existence is, when I, can I synthesis any signal here using any periodic signal using this fashion. Does not matter what are the properties. Well, obviously not, there are some restrictions.

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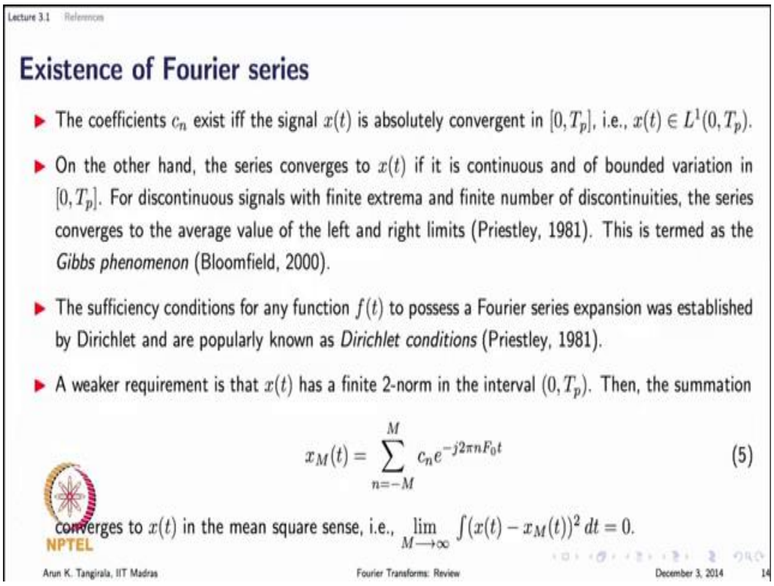

And the restriction is, that the signal x should be absolutely convergent in one time period, which means, its one norm should exist or one norm of this function should exist over that one period. A weaker requirement that you see at a bottom is, that it should be a finite two norm in the single interval, which means, you cannot have really very large infinitely infinite amplitudes in that interval. Why we say weaker requirement is this series here, will not necessarily converge to the signal x as is, but it will converge to the signal x in what is known as a mean square error sense. That means, the means the squared integral squared error will actually go to 0 rather than the error itself as m goes to infinity.

What we mean by m here is, as I include more and more building blocks, there are other sets of conditions necessary and sufficient conditions that have been studied long ago. For example, it says one of the conditions says, the series converges to x of t if it is continuous and of bounded variation, which means, as I said earlier, you cannot have really unbounded amplitudes in this as signal. When I have finite extrema and finite number of discontinuities, the series converges to the average value of the left and right limits and so on. So, these are nicely discussed in books by Priestley and Bloomfield and so on. I strongly recommend you to read the particular sections of these books and of course, there are other books as well.

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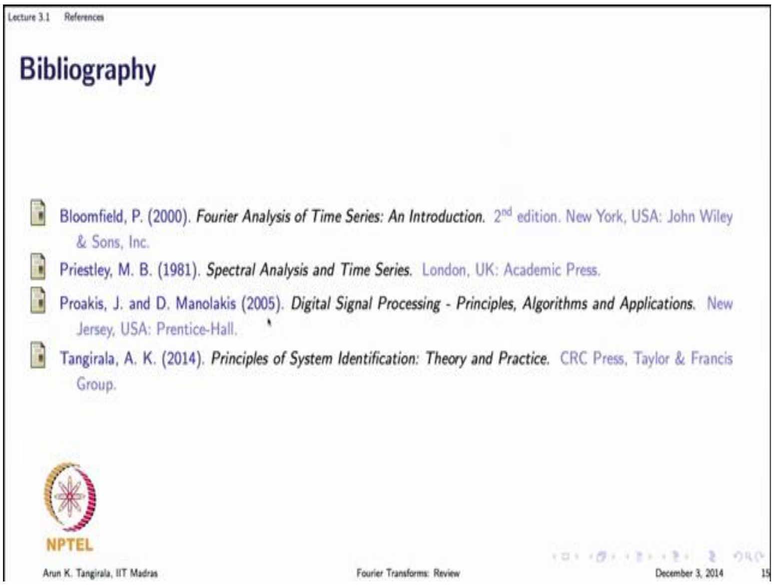

So, with this we come to a close of the continuous-time Fourier series. Essentially, we gave you a very quick review. Truly speaking, you are supposed to have a prerequisite, but we wanted to actually make sure that the foundations are clear, and therefore went through interpretations, the math of particular interest is the power spectrum, and the message that you should take with you is periodic continuous time signal has only a line spectrum and not a spectral density. So, and when the period goes to infinity, then you can think of a spectral density, which is what is a subject of the next module in this unit.

Thank you.