Introduction to Time-Frequency Analysis and Wavelet Transforms
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Lecture - 2.3
Basic definitions and concepts - Part III

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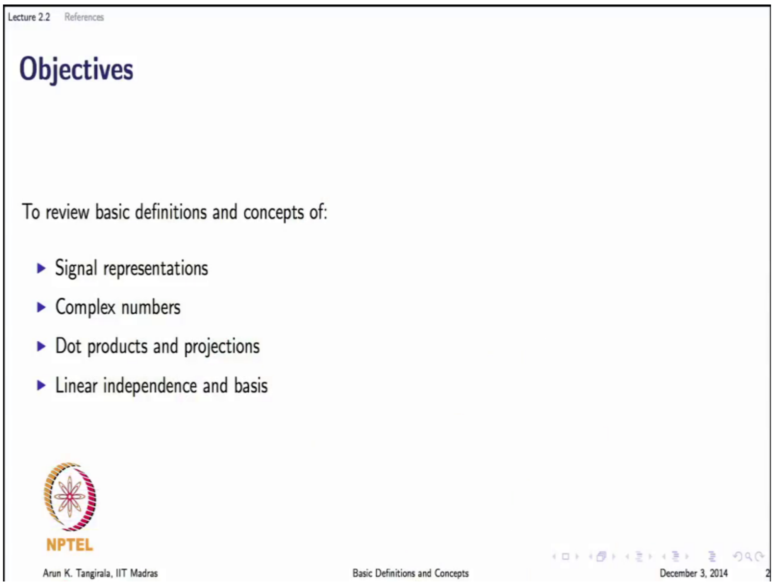

Welcome to lecture 2.3 of this course. It is the third and final module of unit 2, which is on basic definitions and concepts. So, in this module we will review concepts pertaining the signal representations, complex numbers, dot products and projections, and conclude with quick review of the linear independence and what is known as the basis. Now, these are actually essential in at various points in the course. When we apply these concepts you will realize the importance of whether it is, of these points that we are going to review today.

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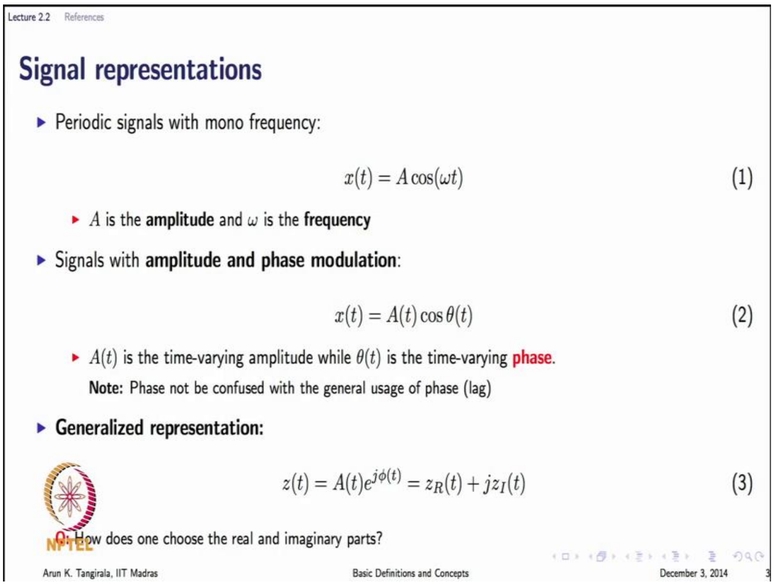

So, the first topic of interest to us in this module is the signal representation, because the entire analysis is based on how we represent the signal. That is a very important point to keep in mind. Whenever we are analyzing a signal behind the sames is signal representation that is working for you. Depending on how you represent the signal, will be your analysis. We are particularity interested in periodic signals. Of course, you may ask what about aperiodic signals? Well, aperiodic signals can be in turn expressed as a combination, weighted combination of periodic signals; that is the basic idea in Fourier transforms as well.

So, we will start with periodic signals. If I have a periodic signal with single frequency then I can write it as, that I have shown in equation 1 here, x (t) as A cosine omega t. It is a continuous time periodic signal. A is set to be the amplitude, and omega is set to be the frequency. Of course, omega is in, it is the angular frequency; the cyclic frequency would be dividing by 2 pi. Now, a quick note on this amplitude, this amplitude is not the value of the signal at any point in time, tt is rather the maximum or the minimum. Essentially, it is the, it is magnitude of the extrema of x.

So, you can imagine the periodic signal as a constant ridding on a cosine or a sine wave. It could be sine as well. So, that is the basic signal representation. Of course, not all periodic signals can be represented with the single expression, but that is the idea in Fourier series. If I have another periodic signal I could express that another periodic signal in terms of these cosines and sines and so on. When the amplitude itself, that is the extrema itself is changing with time, then we say that there is an amplitude modulation.

Further we can bring about a phase modulation. Now, what is phase here? Loosely speaking phase is the product of omega times t here. This phase is different from the phase lag that we normally use. So, you should not get confused between these 2. So, whenever I have an amplitude modulation and phase modulation, I have the expression here in equation 2; it is natural generalization of the representation in 1.

We can further generalize this to obtain what is known as the complex representation of a signal. So, equations 1 and 2 of a real valued signals, what if I choose to have a generalized representation such as normally the equation 3. The question that comes to mind is why do I need a complex representation when the signals that I am dealing in real life, actually real value? But, we will answer this question at a later time when we speak of so called analytic signals.

Remember we are talking of signal representation. Whenever we talk of signal representation all that we are looking is how to represent the bunch of numbers that I have in the signal. And there are numerous ways of doing it. The complex representation is particularly useful in time frequency analysis, particularly in the evaluation of so called instantaneous frequency, as we will learn later on. So, we will not really dwell into this so called generalized or complex representation of a real valued signal.

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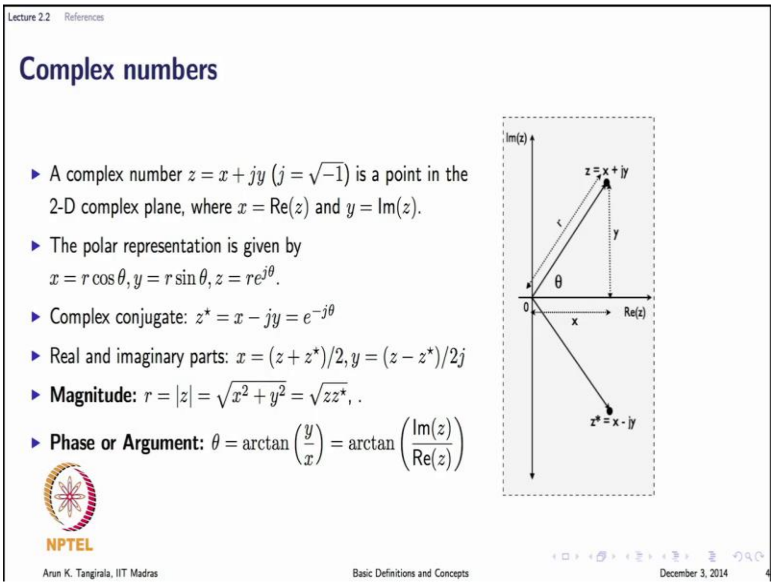

In the signal representation, now we have come across this complex numbers and so on.

And we will encounter this complex numbers very frequently in Fourier analysis. So, a quick roundup of complex number theory. Complex number is nothing but a point in the 2 dimensional complex plane where the horizontal axis is conventional in real part of the complex number, and the vertical one is the imaginary one. So, fairly straight forward.

The, of particular importance is a polar representation of the complex number where we rewrite the complex number in terms of what is known as a magnitude and a phase or an argument or an angle and so on. So, there is a relation, 1 on 1 relation between the magnitude and the phase, and the real and imaginary portions, as I have shown in this schematic here; r is the amplitude; it is the magnitude of the complex number. And theta is the phase or the argument of the complex number.

In complex number, in the analysis of complex numbers we often encounter what is known as a complex conjugate that is fairly straight forward to understand. The sign of the imaginary part is reversed. And therefore, I can write the z star, there is a small mistake here; it should be r e to the minus j theta. I will correct that for you. And the real and imaginary parts of the complex number can be rewritten in terms of the number and its conjugate itself.

Likewise, the magnitude can be expressed as a product of, square root of the product of z and z star. And the phase of the argument is nothing but the tan inverse of the imaginary over real part of the complex numbers. So, these are the basic concepts that one should be familiar with in dealing with the transforms.

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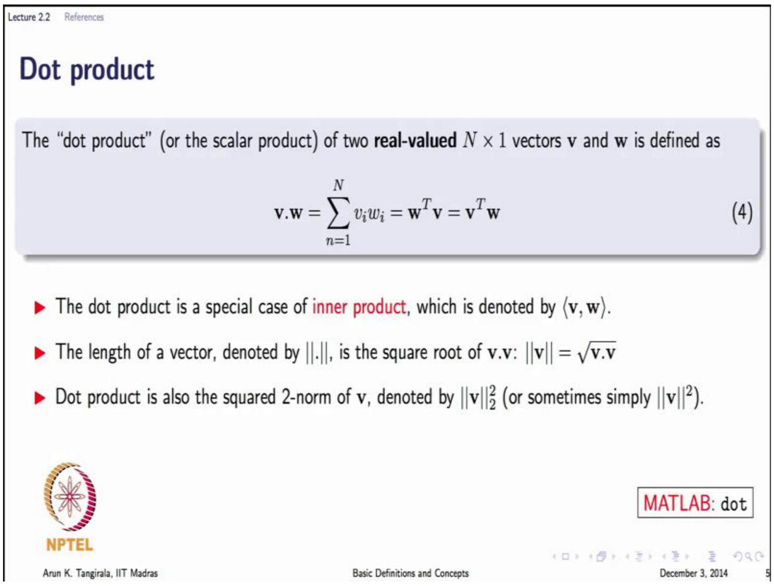

Now, moving on, one of the key linear algebra operations that we would involve in signal analysis; remember, we are performing operations on signals when we are analyzing them. So, one of the key operations that we will be performing is what is known as a dot product. And this is a fundamental operation that one would encounter in all signal analysis.

We learn this in high school mathematics, the dot products, dot product of 2 real valued vectors, v and w. We say that they are in the n dimensional utility and space, is defined as shown in equation 4 here. It is nothing but the element wise product, and summation of those element wise products of the 2 vectors. Obviously, they have to be both of the same dimension. You could interchange the order of the inner product. So, it is not sensitive to the order. However, if v and w are complex valued, the ordering matters, as we will see shortly.

Now, the dot product is a special case of what is known as an inner product in vector algebra, linear algebra which is denoted by this left and right angular brackets. And the length of a vector denoted by this norm here, this double dash lines is nothing but the square root of the dot product of vector with itself. And the dot product is also the squared 2-norm of the vector itself. So, these are some of the useful relations. And in MATLAB, you can compute the dot product between 2 vectors using the dot comma.

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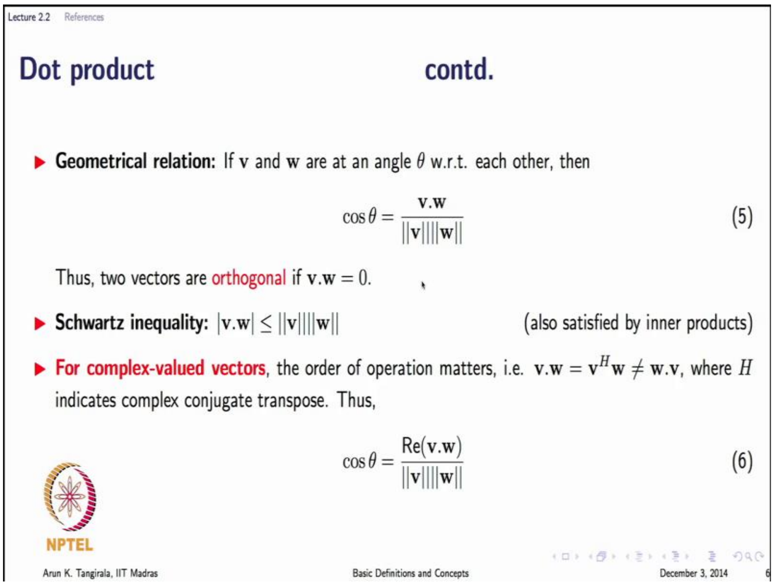

Now, further insights into the dot product. The earlier we saw linear algebral perspective;

now the geometrical definition of the dot product can also be obtained. You can show that the dot product is nothing but the magnitude of v times magnitude of w that is the length of v times length of w times cosine theta. That gives us also definition of the angle between 2 vectors in terms of the dot product. Clearly from this result, if they, if these 2 vectors are orthogonal then cosine theta is 0, naturally then the inner product is 0. So, that is a quick check for orthogonality of 2 vectors; and in fact it also constitutes the definition of orthogonality.

And the standard Schwartz inequality on the dot product, we will use this at some point in time. In fact we will use this in for deriving the duration bandwidth principle at a later time. And for complex valued vectors, the ordering matters, as I mentioned earlier; but the rest of the operation is the same. A key difference is, instead of taking a transpose we take what is known as the conjugate transpose, also known as the Hermitian of the vector. And scalar number that comes out of this dot product operation can be complex valued. Therefore, in evaluating the angles between this complex vectors we only use the real portion of it. So, that is the prime difference now, between the dot product of real numbers and complex number, complex valued vectors and real valued vectors.

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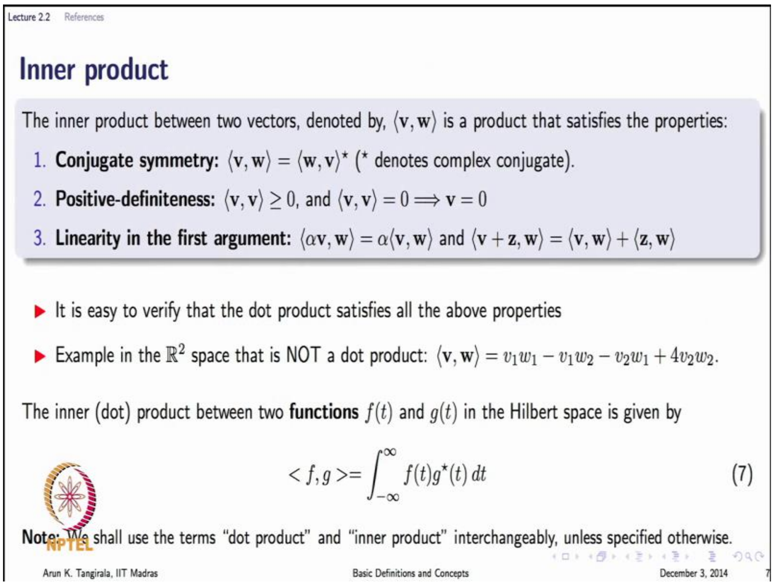

So, as I said, the dot product is a generalization of an inner product, and very often we will work with inner products. Whenever I am looking at a signal transform, whether it is Fourier transform or wavelet transform and so on, I work with I express the signal as a linear combination of some basis functions or analyzing functions and so on. The coefficients of these expansions it turns out are evaluated by first evaluating the inner product between the signal and the analyzing function itself; you will see this in module 3.1.

So, the inner product between 2 vectors is a product that has to satisfy these properties known as conjugate symmetry; conjugate because we want to take into account complex value vectors, positive definiteness which means the inner product of a vector with itself can mean non negative; and if it is 0 then it can only occur under trivial conditions, and it has to satisfy linearity. So, although I say, linearity in the first argument, it has to actually be linear in both arguments, but if you bring in the conjugates symmetry into picture you can show that it will also be linear in the second argument.

Now, it is easy to verify that the dot product satisfies all the above properties, and therefore, is a, is one of the possible inner products. You can actually define an inner product that is not a dot product. And I have given an example here. I leave it to you to verify that this indeed satisfies all the 3 properties here. The inner product between 2 functions, remember there is a difference between vectors and functions; functions are defined; here functions are defined over a continuum, and therefore, you can think of this functions as infinite dimensional vectors.

If these functions are in Hilbert's space which means they are there is an inner product that is defined, and there is a finite norm that is induced by the inner product, that also exists, and certain other properties satisfied. I recommend you to look up a book on linear algebra that talks about Hilbert's spaces. If these 2 functions live in the Hilbert's space then the inner product is defined in this fashion, and we will keep using this inner product many times.

If you recall, I had defined the continuous time wavelet transforms for, the continuous time wavelet transform for you in the introductory lecture. And you can relate, f is taken by the signal, the role of f is taken by the signal, and the g is replaced by the wavelet itself. So, the continuous wavelet transform can be viewed as an inner product between the signal and the wavelet itself. Although dot products are special cases of inner products we will use these 2 terms interchangeably. So, whenever we say inner product, unless otherwise specified we mean dot product and so on.

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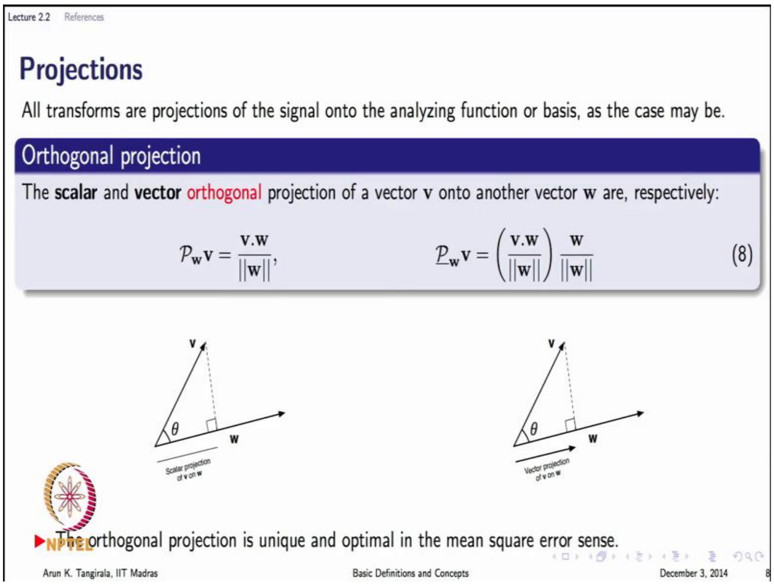

So, moving on to projections. So, we have defined what is the dot product and what is an inner product. And now we want to move on to projections. Now, the projection is a very important concept in signal analysis because projections are nothing but approximations, as I will explain to you shortly. In fact, all transforms can be viewed as projections of the signal onto the analyzing function of the basis function.

So, there are different kinds of projections that are possible. You can have orthogonal projection, you have an oblique projection. Very often we work with orthogonal projection, but there are of course, many situations where oblique projections are involved. So, let us understand what is the orthogonal projection? Let us take a look at the schematic that I show here on the left. There is a vector v, and I am projecting this vector v onto the vector w. And I have indicated here, this, distance here, this length as being the orthogonal projection of v onto w.

Now, what we are doing here is we are dropping a perpendicular from the, from v, from where the arrow ends, and onto w, so that it is perpendicular to w; and therefore, it is called orthogonal projection. The other term for projection is shadow. When we walk on the road in daylight or sometime even in any light, we see our shadow on the road, that is the projection of the 3 dimensional body onto the 2 dimensional plane which is the road, right.

So, here I am projecting v onto w. I can also project w onto v; nothing prevents me from doing that. Now, this is called the orthogonal projection because simply I am dropping a perpendicular here; I might as well draw any line from the tail, the arrow of v, the head of v onto the w, then it becomes an oblique projection. I will show that to you on the board.

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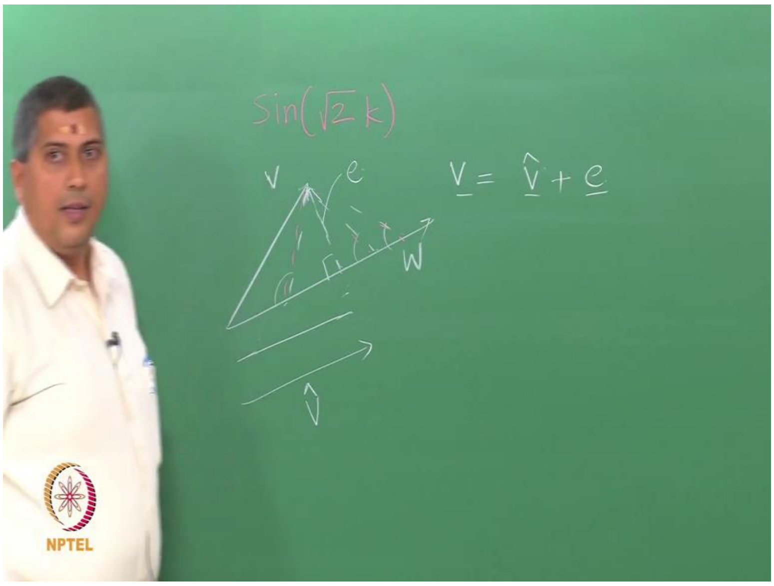

So, when I have a vector v like I have on the schematic, and this is the w here, right. Then, right now I have a perpendicular dropping from here and this is called the orthogonal, this length is called the orthogonal projection, but I could draw many lines this way, and each at a certain angle, right. So, there is certain angle to all of this. We call all these other projections as oblique projections. And these oblique projections are defined with respect to another vector that is running parallel to this.

So, the oblique projections will also involve what is known as a third vector, but we do not define that here. Now, the scalar projection can also be given a vector interpretation by saying that this is nothing but another vector in the direction of w and that becomes a vector projection. So, the only difference between the scalar projection and the vector projection is that there is no direction to this number; it is a number. But, here there is a direction to this number; clearly indicating it is in the direction of w, alright.

And that is the reason you see in the vector projection, you, there is multiplication with w by its own magnitude. So, you are essentially multiplying this number with the unit number vector in the direction of w. So, that is why it is called a scalar and vector projection. So, otherwise these 2 schematics look alike. Now, very interesting property of this orthogonal projection is that it is the best approximation of v using w. So, if I want express v using w then the scalar, the orthogonal projection offers a best projection in a minimum mean square error sense.

In other words, by if I look at this vector projection here, this is the error. So, vectorially if I call this as e; and this as the orthogonal projection, let us call this as v hat. I can write therefore vectorially as v being v hat plus e. So, this is the error that I am making in approximating. If I approximate v with v hat, this is the error that I am making. So, all other projections will give me the larger magnitude of e, and that should be a failure obvious if I look at this.

And this is also the so called d squares solution or minimum mean square error approximation of v using w. So, that is the most important property of the orthogonal projection. And now you can understand why projections and approximations are very closely related. We will use this concept of projections time and again in Fourier transforms and wavelet transforms and so on.

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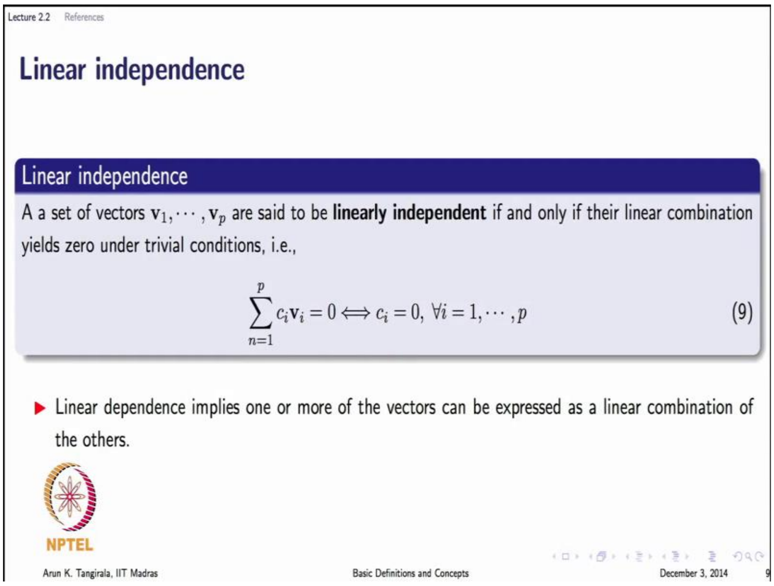

So, we will conclude this module with the concepts of linear independence and basis because we will need this when we talk of DWTs and so on. The concept is fairly straight forward. A set of vectors is said to be linearly independent, if and only if their linear combination is 0 under trivial conditions. So, what is the linear combination? This is the linear combination. If this is 0 only under trivial conditions then we say it is linearly independent; which means, if it is 0 under non trivial conditions, which means I can find non zero constants that will result in a 0 linear, 0 valued linear combination, then we say that the vectors are linearly dependent.

And what linear dependence naturally means is that 1 vector can be expressed as the combination of other. So, let me give you an example of set of linearly independent vectors. So, if I look at these vectors here, 2 and 5. So, this is v 1 and this is v 2, then these 2 vectors are linearly independent because simply I cannot express 1 as a scalar multiple of the other. The other way of saying it is, I cannot write, cannot find a non trivial solution to this equation. On the other hand, if this was 1 1 and 2 2, then it is obvious that v 2 and v 1 are linearly dependent. So, that is the basic idea.

Now, a quick note on the connection between linear independence and orthogonality. So, earlier we had this vector and we showed, we realized that these 2 vectors are linearly independent, but that does not mean that they are orthogonal; very often it is confused. What is the way to check orthogonality? I compute the dot product, and if it turns out to be 0 then I say that they are actually orthogonal. So, the dot product between this 2 vectors is 1 times 2 plus 1 times 5, and that comes out to be 7 which is not 0. And therefore, it is not orthogonal. On the other hand, all orthogonal vectors are linearly independent, and that I will leave it to you to check, alright. That is a special feature of this orthogonality. And orthogonality is desirable in compact representation, signal comparison, and so on.

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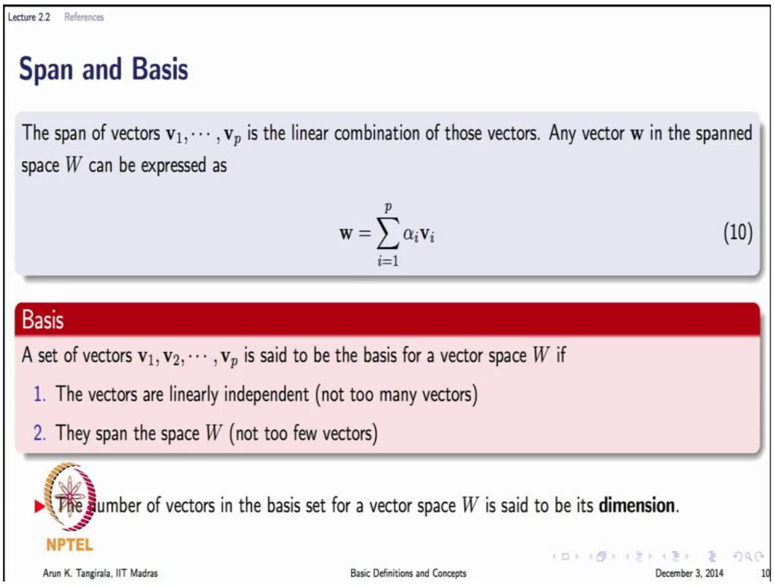

So, finally we talk of span and basis. Span of a set of vectors is nothing but the linear combination. So, already we saw the linear combinations in the context of linear independence. Now, we are talking of basis, a set of vectors is said to be the basis for a vector space W. Whatever that vector space is, it is just a matter of notation. If 2 conditions are satisfied, the vectors are linearly independent which means in as Gilbert's Strang puts it, do not have too many vectors.

What you mean by basis is, how many basic elements you require to express all the vectors in the vector space. So, if you have too many, more than required, then that means, you will bring about a linear dependence between them. So, you do not want that. And you do not want too few, which means if you choose fewer than required then you will not be able to span the entire space. What you mean by span is, to be able to generate all possible combinations, right.

So, the classic, if you talk the 2 dimensional real space, the classic set of basis are 1 0 and 0 1. So, those are the 2 different basis vectors that we normally are given as examples – 1 0 and 0 1. But I might as well choose these 2 as the basis for the 2 dimensional real number space, because this also can span the entire 2 dimensional space. They are linearly independent.

The only difference between these 2 is that this is orthogonal and this is linearly independent. So, since orthogonal vectors are also linearly independent, basis admits are orthogonal basis. And in fact, orthogonal basis is a highly desirable feature, as I said earlier in signal combination and so on. And finally, the number of vectors in the basis set for a vector space is said to be its dimension.

So, these are the basic concepts that we will require before we move on to the world of Fourier transforms, wavelet transforms and so on. I would suggest that you may get good reading of the book particularly by Gilbert Strang, because it gives you very nice ideas on, very nice insights into linear algebra. It is an excellent book, very renowned book. Please go through some additional reading before you jump to the next unit.

Thank you.