Introduction to Time-Frequency Analysis and Wavelet Transforms
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Lecture - 2.2
Basic Definitions and Concepts- Part-II

Welcome to lecture 2.2 in the unique on basic definitions on concepts of the course on Introduction to TFA and wavelet transforms. In the introductory lecture and subsequently we have used on, such as energy density, power spectral density, energy and power and so on. There we have use a terms loosely but, it is a time to learn these terms in a more formal frame work. So, the objective of this module is essentially that, we will formally learnt, what are power and energy signals.

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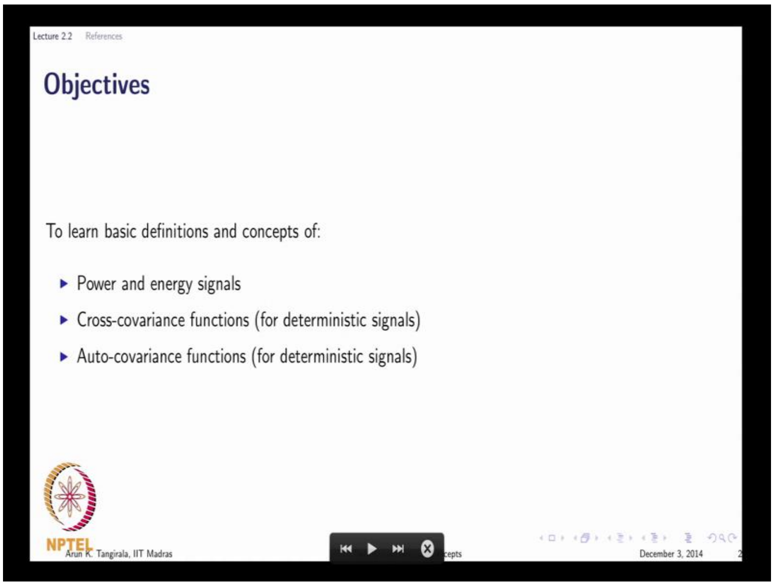

In fact, it is just the review of these concepts. These are supposed to be known, before even we well in to time frequency analysis. And these energy densities are defined in time. But, when we move to frequency domain particularly for discrete time signals. But, one can strike densities may not exist in the time domain for discrete time signals. But, densities exist in the frequency domain.

And one can show that, these densities in frequency domain are related to, what are known as covariance functions. Again, these are concepts that are taught in a typical second year or a third year signal processing course. And we will review for the benefit of everyone. And discuss two measures known as a cross covariance functions and auto covariance functions. Again for deterministic signals which is the frame work for this course.

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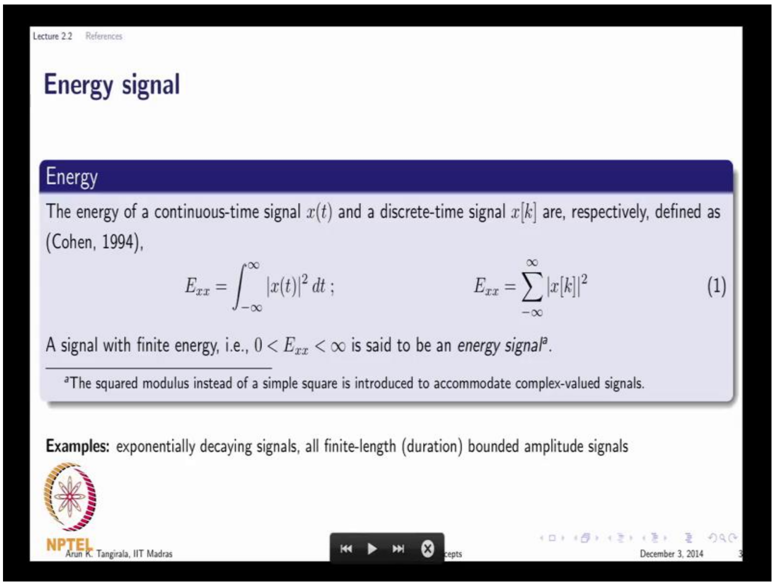

So, let us begin with the definition of an energy signal. And for that we need to define, what is energy. The definitions for the continuous time and discrete time signals are shown here in equation 1. Obviously, continuous time signals involve an integral and discrete time signals involve summation. Essentially, these definitions come from Electrical Engineering and Physics. The definition is fairly straight forward.

These definitions originally assumed that, the measurements or the signals have electrical quantities. And therefore, energies are defined in this manner. And you also notice that, there is a modules square rather than simply x square. That is essentially to accommodate complex valued representation. In fact, we will come across this complex valued representation, shortly in the next module and subsequently as well throughout the course.

So, the energy of a signal has to be understood, as the energy expanded by the process to generate this signal and we evaluated or the entire existence of time. When a signal has a finite energy, continuous or discrete we say that, it is an energy signal. And number of examples can be given. All finite duration and finite amplitude signals have finite energy. So, an example is an exponentially decaying signal.

Of course, that does not have a finite duration. But, it is exponentially decaying. You can show it has finite energy. But, otherwise all finite link signals have finite energy. And therefore, they are energy signals. On the contrary, a periodic signal exists forever and it has infinite energy. Therefore, it is not an energy signal.

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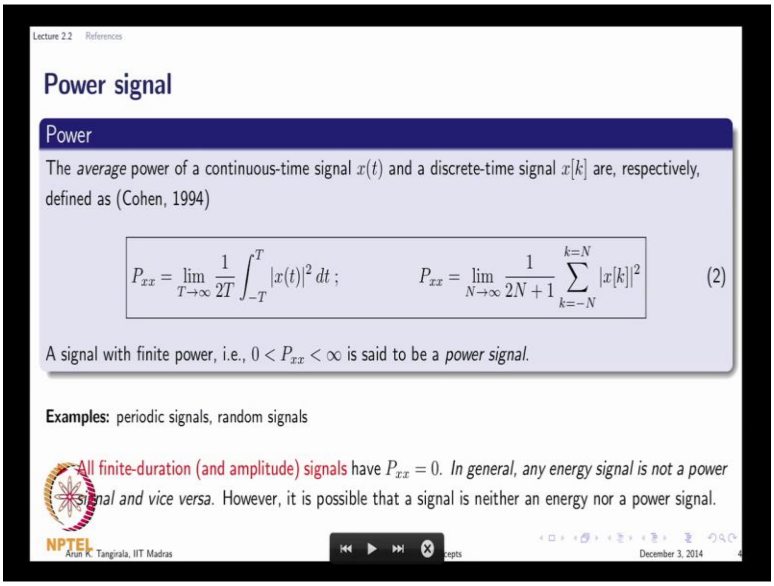

Accompanying this definition is the notion of this power signal. Again, the definition of power is inspired from Electrical Engineering and Physics. Strictly speaking, this is the average power. As you can see from the definition, it is defined in a limiting sense over the entire existence of the periodic signal. Here capital T for a continuous time signals, refers to the periodicity and capital N or you can say 2 N plus 1 is the period of the discrete time sequence signal.

Again here, these are the average definitions for both continuous time and discrete time sequence signals. If a signal has finite power, then we say that the signal is the power signal. And we just mention earlier, periodic signals have infinite energy, but have finite power. The interpretation of power is once again on the same lines, as energy. It is the rate at which, the system is expanding the energy to produce these signals.

Always remember that, no signal exist without a system. So, whatever properties we derive of a signal, actually what we are doing is, we are referring to this system that is generating this signal. The system could be a man made system or a natural system. But, it is a device or a system, that generating the signal. The random signals also are power signals. Because by definition random signals are assume to exist forever and they never decay.

And therefore, they have infinite energy. And also, they have finite power. In general, any signal can either be a power or an energy signal. But, there are also exists classes of signals, which are either finite energy or energy or power signals. They may have infinite power and infinite energy. But, we shall not talk about this in practice. You either run into power or energy signals and so on. Having said this, I should tell you that all the observed data. Because, all the observe data collected are a finite time.

And therefore, over that finite time the signal has a finite energy and the average power is zero. Why should an energy signal have zero power? Because, the moment I say signal is an energy signal, it exists for a finite time. Or it is energy is finite, whereas average power is the energy spent over the entire time of it is existence. And because, the average power is defined in a limiting sense, all energy signals will have zero power and so on.

So, you have to be comfortable with these definitions. And you have to be clear in your mind, whenever you are using the term energy spectral density and power spectral density and so on. We have to recall these definitions, until you are really comfortable with it.

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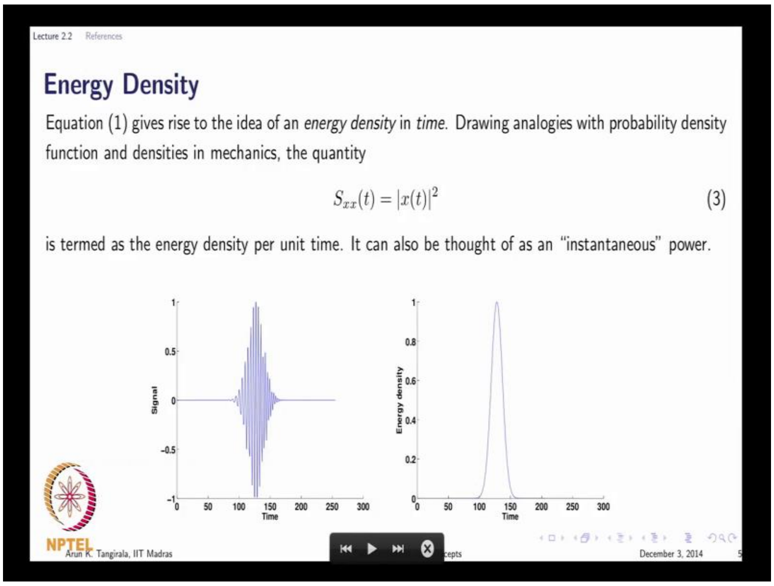

So, associated with the energy definition, if again take you back to the definition of energy, you can observe in this integral here, in the equitation 1 that, the energy is the area under a curve, which is modules x of t square. So, drawing analogies with mechanics and probability, density functions and so on, we can treat modules x t square as an energy density. So, that the area under this density curve gives me the total energy. So, that is the idea. And we extend these notions somewhat to the discrete time signal. Although, it is not an area, but in a discrete senses it is an area. So, it is a lose interpretation there.

And here, we have now, what is known as the energy density, in time. Slowly, one should get comfortable in time frequency analysis with energy densities in different domains. This is energy density in time. Shortly, in the sense in a few modules from now, we will also speak of energy densities in frequency domain. Every time, we talk of densities we should ask of if they exist.

For a continuous time case, if it is an energy signal, the energy density exists. And I am showing you, an example here of a Gaussian modulated signal, which of course if you can see from the plot, it exist for a finite time T case, but in T case to 0. And what you see on the right is the energy density. I just give you a feel. These energy densities are later on used in deriving, what are known as duration, defining duration and band width and so on.

The energy density in frequency will allows to define the band width. Loosely speaking can interpret this energy density as an instantaneous power. But, I do not really advocate the definition for long period of time. Just a loose interpretation.

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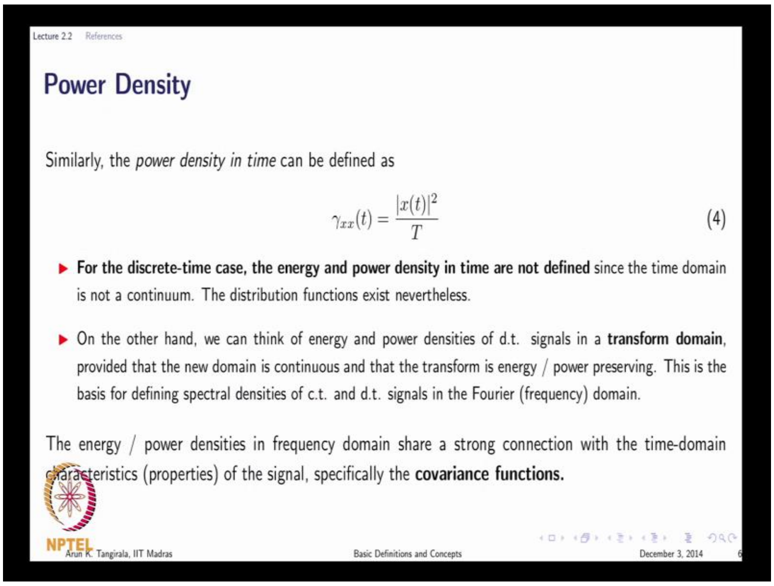

Along the lines of the energy density, one can define power density. Again, I will take you back to the definition of the power density. If you look at this equation 2 here, what you can see is, the modules x t square. Strictly speaking should be by 2 T, but you can always define by T. The area under that actually gives you the power. And therefore, I can treat modules x t square by T, where T is the period of the signal as the power density in time.

Again, this is an average power density in time. And for both the discrete time case, both the energy and power densities in time do not exist. And the reason is, here if you look at yet, the densities are only applicable to continue. This is what exactly, I said earlier. You can only think of modules x t square in a loose term as a density. But, strictly speaking it is not a density. However, we can think of distributions.

And here, if you have some random variables of probability theory, recall that. The probability density function exists only for continuous valued random variables. And they does not exist for a discrete valued random variables. Whereas, in both cases the notion of probability distributions exists, that is valid.

Likewise here, for discrete time signals, you cannot talk of densities, simply because in discrete time, time is not a continual. You only have discrete points. However, you can have densities of energy and power, in a transform domain. Here, we are hinting a frequency domain, Fourier domain and so on. And we will learn shortly that, these energy and power density do exist for discrete time signals, in frequency domain. That is, what we call as energy spectral density and power spectral density and so on.

Assuming of course that, the energy and power are defined accordingly. Now, I said mentioned earlier on this module, these energy and power spectral densities are very closely related to what are known as covariance functions. So, now we are moving from the univariant ((Refer Time: 11:08)) momentarily to bi-variant. What I mean by bi variant is two signals.

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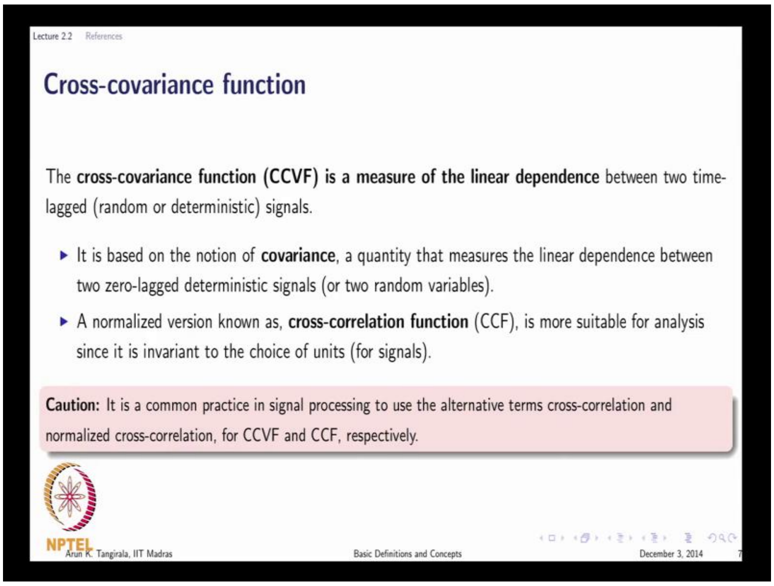

So, let us begin with this very familiar measure called cross covariance function. This cross covariance function is actually a measure of linear dependents between two signals that are shifted in time. And the sense what you do is, you take signal x and signal y. These are the two signals. I shift one of them with respect to the other, by a certain amount l. And measure the covariance between m at that light l.

When I evaluate this covariance between this x and shifted y or y and shifted x at different values of l, I get what is known as a cross covariance function. I will give you the definition, shortly. What is very important to remember is that, the cross covariance function is a measure of linear dependence. And this is used widely in testing linearity and so on.

If you actually do a search for covariance literature search or if you read any books on covariance sense form, typically you will see this covariance is define in a statistical sense that is, in the context of random variables. And also cross covariance functions are defined for random signals. But, here we are going to review the definitions for deterministic signals. Cross covariance is a concept. It is a concept that asks how two signals are varying together. That is why, it is called the cross covariance.

You can apply this to deterministic as well as random signals. And here, what we are going to do is, we are going to look at the deterministic case. And normally, this cross covariance is normalized to get cross co relation function, so as to make it independent of the choice of units. So, if I am relating temperature and pressure of a gas, all from relating voltage and current in a circuit or in a wire, then the units will make a big difference to value of the cross covariance, that you are evaluating.

You want to measure that, invariant to the choice of units and that is one of the reasons, why we choose to work with cross co relation functions. And I want to caution the viewer here that, these terms cross covariance and cross co relation are used slightly differently in statistics and in a signal processing. The term the time using here is from a statistical prospective, cross covariance.

If you turn to the signal processing text, the cross covariance function is replaced by a term called cross co relation, which is actually a normalized cross covariance for statisticians. So, without confusing you further, cross covariance and cross co relation in statistics mean, cross co relation and normalized cross co relation in signal process. So, you have to be careful with the terminology that you are using.

And a lot of times, if you turn to statistics I will talk about this with respect to this definition here. So, this is the definition of the cross covariance for periodic signals.

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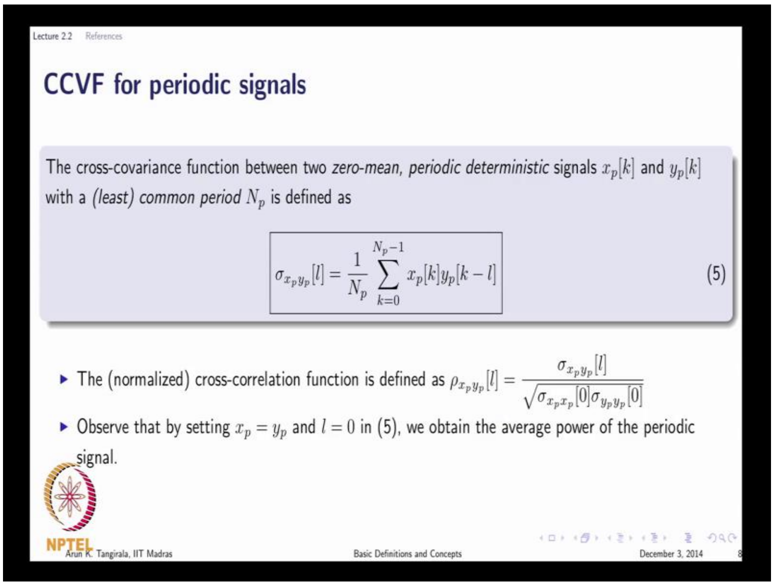

You look at this definition given in this equation. You can see that, we are evaluating this cross covariance over the period of the signal, because this definition is for periodic signals. Notice, that the subscript is sigma x y. This is the notation that we are going to use. And as I said earlier, the notation is such that for discrete time sequences or signals, we uses square bracket to keep track of the index.

So, sigma x y at l for a periodic signal is given as a time average, summation time average product of x y. But, not just plain x y, but x y k minus l. So, that is a shift that I was earlier referring. In statistics, if you turn to this definition of cross covariance functions, you would see that, the means are subtracted from the respective signals. Whereas for deterministic signals, we do not and that is one of the prime differences that, you should keep in mind.

In mat lab, there are two different commands called x c o v and x c o r r. X c o v subtracts mean, means from the respective signals before computing the cross covariance, whereas x c o r r does not. So, in a sense x c o v is to appeal to the statistical community and x c o r r is meant for signal processing. But, if you know what you are doing, you can use either of these functions. It is not a big problem.

So, this cross covariance function for periodic signals is evaluated only over the period, that is something that is usually, you should observe. And what you see here is the cross correlation, as I mentioned earlier. It is a normalized cross covariance. Their normalization factor as you seen the denominator, is a square root of a product of two terms, which is sigma x x at lack 0 and sigma y y at lack 0 and x p x p. But, I am ignoring the subscript.

So, what is the sigma x x of 0 and sigma y y of 0? These are auto co variances evaluated at lack zero, which I will talk about shortly. Now, it is easy to see by setting x equals y and evaluating this cross covariance at lack zero, gives me the average power of the periodicity, if we go back to the definition. Now, the cross covariance for a periodic signals is naturally defined over the entire existence of the signal.

Now, the summation runs from minus infinity to infinity, as you can see. But, the form of the expression does not change significantly. There is no averaging in the sense here, because the existence in principle over the infinite time, whereas for a periodic signals, you average it over the period. The rest of the things carry forward. The cross correlation, again here is a normalized cross covariance.

And when you said, x equals y and evaluated at lack 0, you recover the energy of that particular periodic signals. Again, this definition of cross covariance only holds, if this signal is a periodic and has finite energy. You can have a periodic signal that is it has no period, but has infinite energy. A classic example of that is, a sin square root 2 times k.

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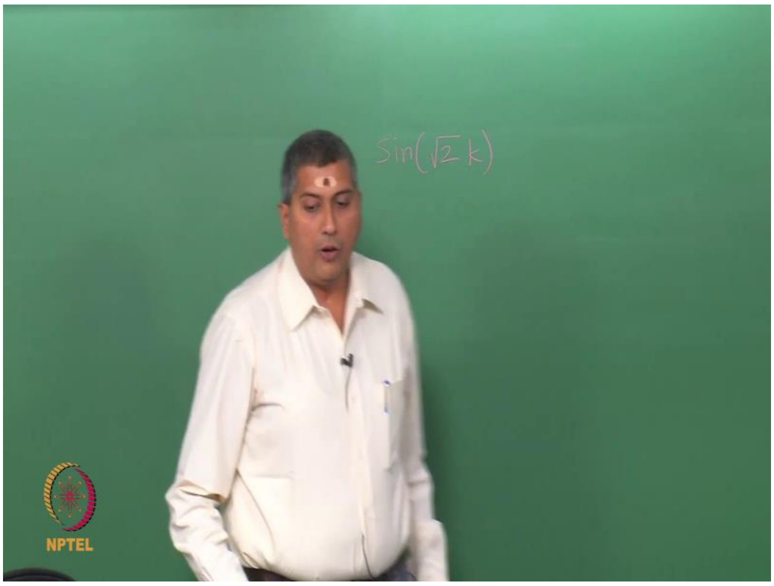

So, if I look at a signal sin root 2 k. This is a discrete time sin wave, but it is not periodic. The reason is, the frequency is irrational. We recall the, that we had in the previous model. This has infinite energy, but it is not periodic. And therefore, you cannot apply this definition of cross covariance ((Refer Time: 18:08)).

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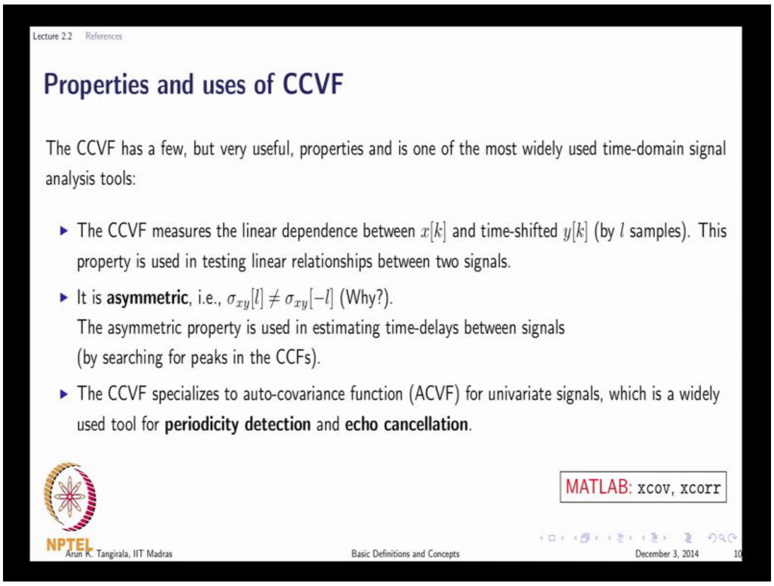

Now, the cross covariance function has a number of useful properties, few but very useful properties. And it is perhaps, one of the most widely used tools and signal analysis. It is used in delay estimation and it is used in impulse response estimation and so on. So, I list some of the important properties here. As I mention earlier, the cross covariance measures the linear dependence between two signals, with one of them shifted with respect to the other.

And therefore, it is used in testing, linear dependence between two signals. If you think of x as input and y as output, then you can use this cross covariance function to see, you can develop a linear model between the input and output of a given system. And it is an asymmetric measure. That is a very important property of the cross covariance function, which means sigma x y at lack l is not the same as sigma x y at minus l.

And this should be expected, because you cannot expect x to affect y k minus l. That is assume, that l is greater than 0. Then, sigma x y at lack l is examining the dependence or influence of y k minus l. The past value of y on the current value of x. That need not be the same, as how a future value of y is related to the present value of x. Particularly, if you think of x and y as input and output of a causal system.

So, naturally it is an asymmetric function. However, auto covariance as we learn later on, is a symmetric measure. Now, there is another word of caution that I want to mention here, make here.

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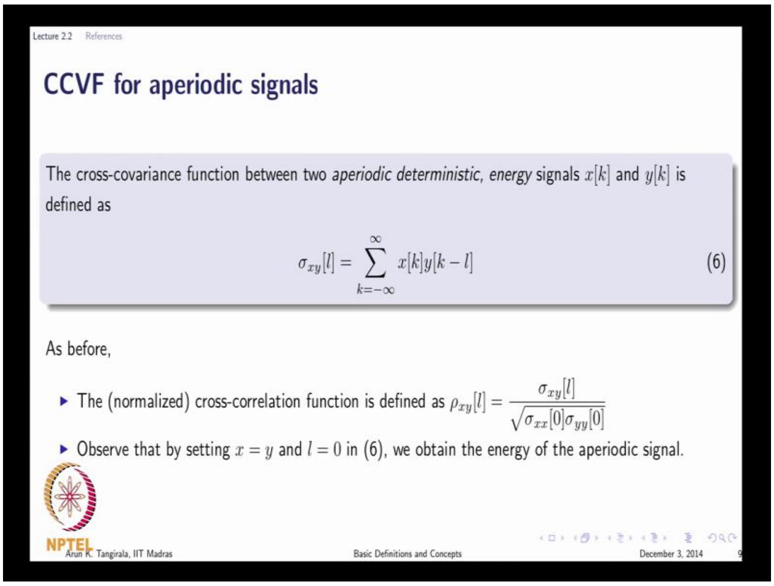

We have used the definitions as x k, I mean sigma x k y k minus l and likewise, for a periodic signal. In many text, you may see x k y k plus l. So, you have to be careful. There is nothing wrong with the definition ether, but then you have to be consistent to the rest of the development and the way you entered the cross covariance and so.

And particularly, when you are using a software package, you should make sure that you refer to the documentation, to see what definition on he is using. And thirdly, the cross covariance functions specialize to the auto covariance function for univariant signals, as I mentioned earlier. And this auto covariance function has a number of useful properties and so on. That is a last item that, we are going to discuss in this module.

And I mentioned earlier, there in mat lab there are two commands x c o v and x c o r r. And I would like you to go and look up the help of the documentation on these commands. So, you will see that x c o v computes a cross covariance, by mean centering. That is first, by first removing means from the respective signals. Whereas x c o r r does not remove the means. But, if you know what you are doing, you can actually produce the same results.

And there are other options in x c o v and x c o r r. For example, you can get normalized co efficient, which means you can get cross correlation or un normalized, which is a cross covariance. So, check out the help on these two routines. So, let us quickly now move on to auto covariance function. This is nothing, but the cross covariance function evaluated for the same signal. So, this is not a new definition in itself.

It is actually obtained by setting x equals y in the respective definition for periodic and aperiodic signals. And as I mentioned earlier, unlike the cross covariance function, the ACVF is a symmetric function. Auto covariance function is a symmetric function. Therefore, you will see in all the text, the auto covariance functions plotted only for the non negative set of logs. And the normalized versions, again give rise to what are known as auto correlation functions and so on.

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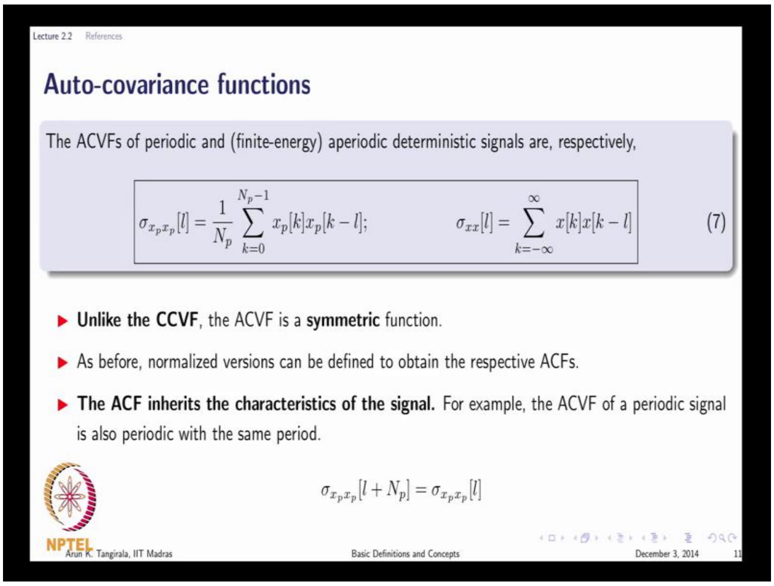

The main property of this auto correlation function is that, it inherits the characteristics of the signal. What it means is, suppose the signal is periodic, then you will actually see periodicity in the auto correlation or the auto covariance function as well.

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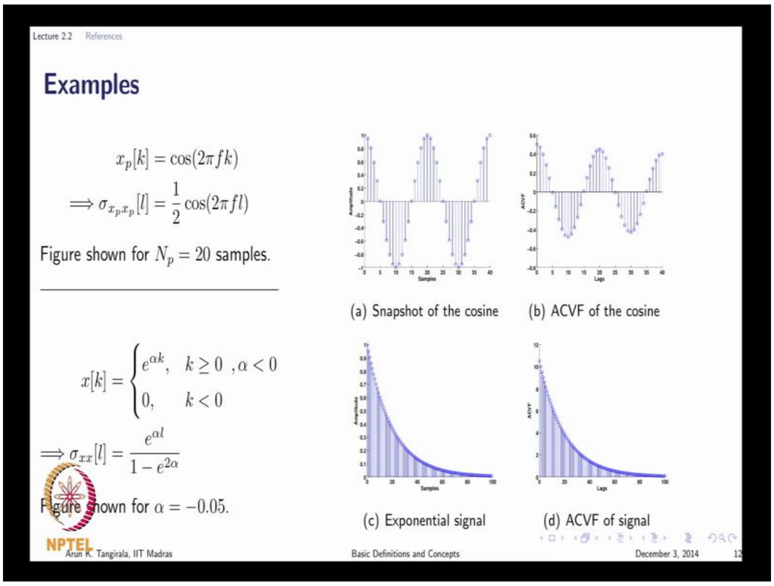

Let me try home this point, by means of these two examples. The first example on the top that you see is, for the case of a periodic signal. On the left, here you see the mathematical result. This is easy to show. I leave that as a simple exercise to you. If x is a cosine or a sin, then sigma x x which is the auto covariance at lack l is also a periodic signal with the same period. And what you see on the right is an illustration of this result.

When, whereas chosen the period to be 20 samples. And I am plotting the auto covariance, not an auto correlation. The way to find out, if it is an auto correlation or not, is to check the value at lack 0. If at the lack 0 the value is 1, then most likely it is an auto correlation. That is a thing, because at lack 0 you are measuring the similarity of the signal with itself. And that obviously has to be 1. And that is, how the auto correlation is defined.

For the a periodic case, again here I show for the case of an exponentially decaying signal. And I have derived the expression for the auto covariance function. Again, I leave that as an exercise to you, to derive the expression. And I am showing the results for alpha equals minus 0.5. And once again you can see that, the auto covariance has inherited the properties of the signal. Now, you may ask what is the big deal about this.

I mean, whatever this signal has, the auto covariance showing that. So, I can extract the features. The same features from the signal, why do I need to work with auto co variances? Well, the answer to that is… If I have a signal with noise, I do not show that here. But, at an appropriate moment I will show you that example. If let us take the top example, the periodic signal here.

If this periodic signal was corrupted with some mile amount of noise, then it would not be easy for you to observe the periodicity, by a direct visual inspection of the time domain signal. Whereas, if you plot the auto covariance function of this corrupted sin wave, then it retains its periodicity because of the properties of the noise. And I am particularly referring to a kind of noise here. If you add an uncorrelated noise with right noise, then what happens is…

The property of the right noise is such that, its auto correlation is 0 at all non zero logs and 1 at lack 0 by definition, which means the noise affects the signal at every point in time, but in lack domain the noise affects the ACVF only at lack 0. And theoretically, it does not affect the auto covariance of the measurement at any other log. So therefore, the periodicity is reveal in a more significant way in the auto covariance domain or the lack domain. That is one of the key properties of ACVF that exploit in periodicity detection.

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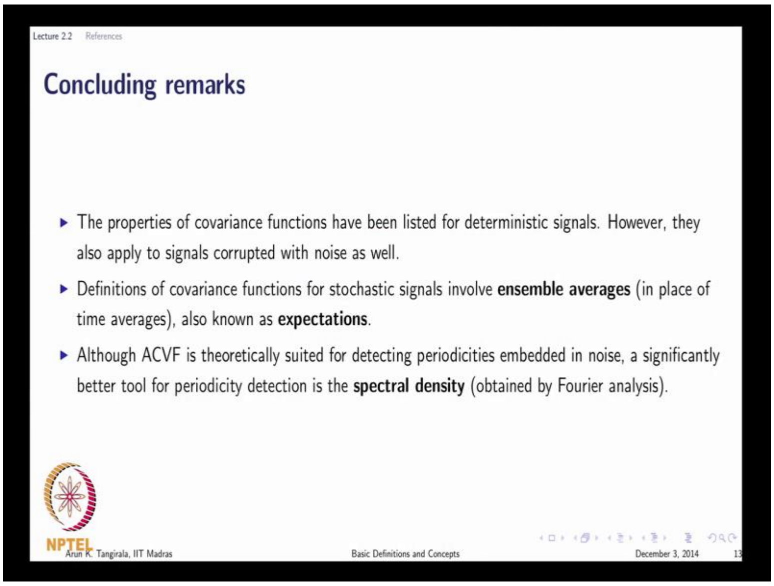

So, with this I want to conclude this module. And of course, I want to also make a few remarks. We have actually defined covariance function here for deterministic signals. But, as I mention you can apply these two signals which are corrupted with noise. Without much abuse of these definitions, more importantly the definitions of covariance functions also exist for stochastic signals.

In fact, mostly you will find the definitions introduce stochastic functions everywhere or stochastic signals everywhere. And for, you will find very relatively few definitions for deterministic signals begin ((Refer Time: 26:07)). But, the stochastic signals definitions involve ensemble averages. That is averages across all realizations. Instead of time averages, as we have done for deterministic signals. And these ensemble averages are evaluated, what are known as expectations.

And finally, what I want to make a point is, with regards to the use of ACVF for periodicity detection and I mentioned earlier, that is a very useful tool for the periodicity detection. But, a much better tool is a spectral density, particularly when you have more than one oscillatory component in the signal. I showed you earlier example, where it is a single oscillatory component.

Theoretically, ACVF is periodic. It allows you to detect periodicities, even if you have multiple components. But, the practical utility of lack becomes low, whereas the spectral density is excellent and doing that. Of course, when we move on to the third unit on the review of Fourier transforms, we will know the definitions of spectral density.

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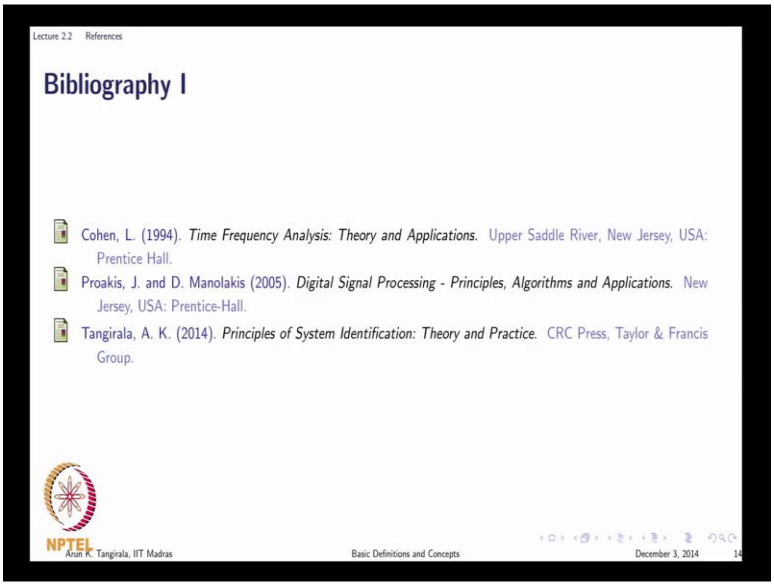

So, here are few references for the reviews. In the next module, that is 2.3 we will learn a few more basic definitions and concepts on signal representation and dot products and projections and so on, which will dense at the platform for the review of Fourier transforms.

Thank you.