Introduction to Time-Frequency Analysis and Wavelet Transforms
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Lecture - 2.1
Basic definitions and concepts

So, welcome to lecture two point one of the course in unit one.We obtained an overview of the topic of time frequency analysis and wavelet transforms. We now formally get into the course. Since we are going to deal with signals, periodic signals, aperiodic signals and so on, it is important to learn a few mathematical definitions and also obtain suitable interpretations before we even review Fourier transforms and that is the objective of this unit. And this is the first lecture in this unit, therefore number 2.1.

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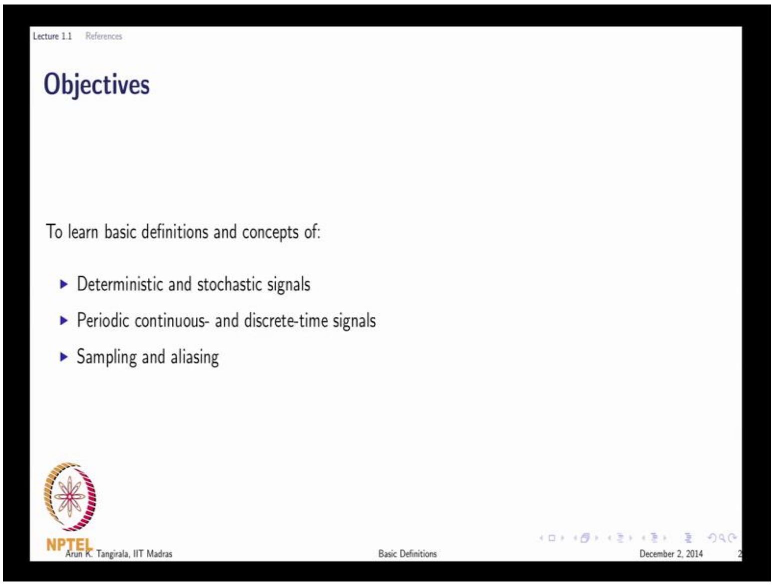

In this module we will learn the basic concepts of aperiodic signals, both continuous time and discrete time periodic signals. Importantly, we will learn the difference between deterministic and stochastic signals. It is quite important to know that and we will conclude the module with brief review of the sampling theory because we are going to primarily deal with sample data.

Theoretically, we will work with continuous time signals, but when it comes to the application of this techniques, we are going to work with sample data and therefore, it is important to know at least some basic theory of sampling and particularly the sampling theorem itself.

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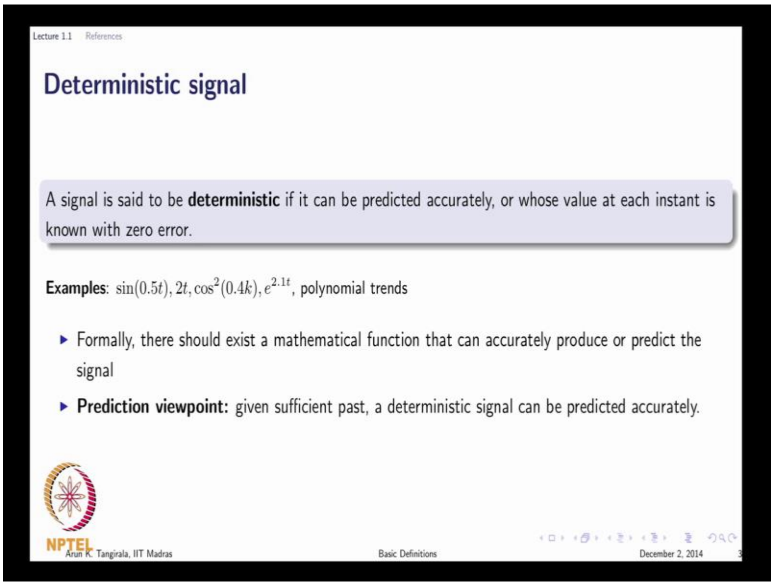

So, let us start with the concept of a deterministic signal. We all have an intuitive feeling of what a deterministic signal is. A signal is set to be deterministic if it can be predicted accurately, that is one way of looking at it; that is a prediction view point. An alternative way of looking at a deterministic signal is, that there exist an mathematical function, that will predict the entire course of its evolution, that is, over its entire existence one can give numerals examples. Deterministic signals need not be periodic, although I give a few periodic signal as examples. They can be anything, which you can predict accurately. So, the keyword is the accurate prediction.

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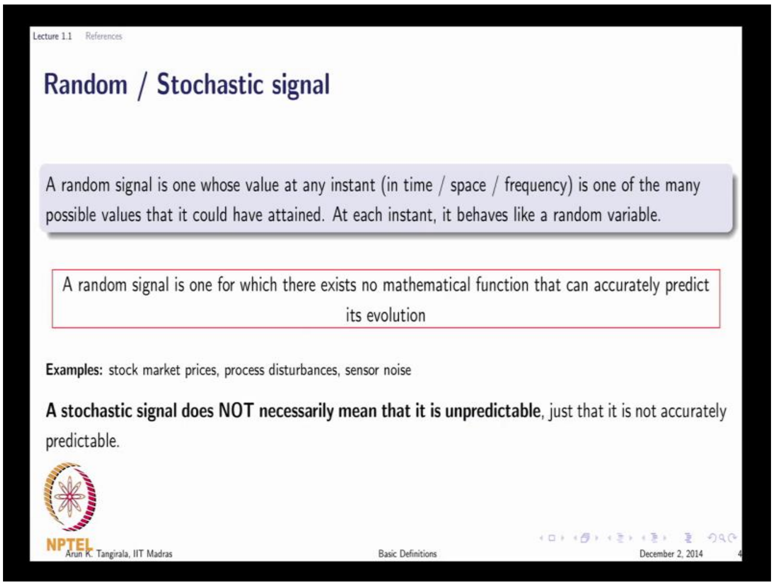

And when it comes to stochastic signals, obviously, one should expect the prime difference being, that the stochastic signal or a random signal is that signal, which cannot be predicted accurately. It does not mean however, that you cannot predict it. So, you should note the point that I make towards the end of the slide here.

The stochastic signal is typically misunderstood as being unpredictable, that is not true. You can predict a stochastic signal, but not accurately. You may be able to predict with 99.99 percent accuracy, but definitely not 100 percent accuracy. So, the prediction accuracy is anywhere between 0 to 99.99 percent. When you cannot predict a stochastic signal, that is, when the prediction accuracy is 0, then we call that as a white noise signal and so on. So, that is a prediction view point.

Again, the other view point that we have is, that exists no mathematical function that can actually predict or that can describe the behavior of a stochastic signal over its existence. You may take small part of the random signal, may be able to fit a mathematical function, but this mathematical function will not be useful in extrapolating. You may be able to fit. For example, if I generate 100 samples of a random signal, I may be able to explain it accurately using a 99th order or degree polynomial, but it will miserably fail outside that interval when it comes to prediction.

Now, there are several examples in real life, that we cannot, that that fall into this category of random signals and you can find engineering examples economics. These are very prevalent in economics, in engineering, of course, where you have disturbances and so on. In economics, we looks at stock market prices and so on. In reality, there exist no signal that it is accurately predictable, which means all signals that we encounter in reality are random. Then, why do we even deal with the world of deterministic signals is something that we should ask.

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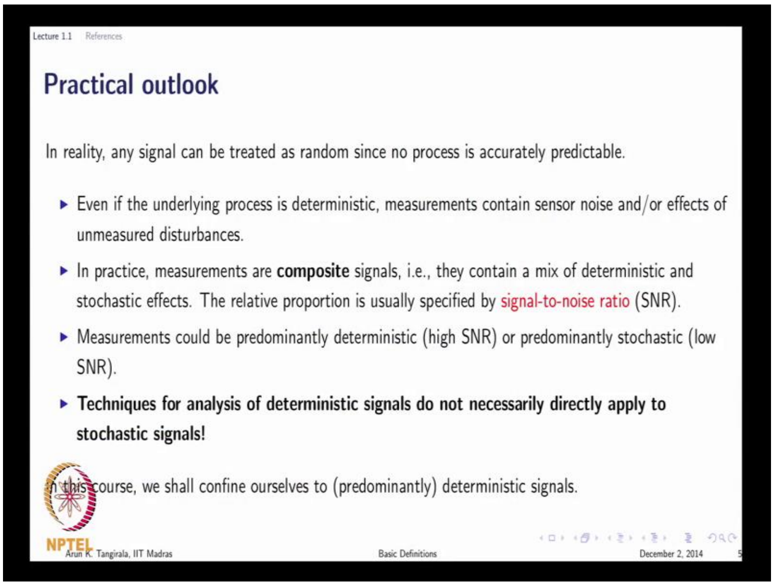

Well, a lot of measurements that we obtain a lot of processes that we encounter have a deterministic nature as well. For example, even if the true process is predictable, there is some predictability for sure. But when we observe this process, then measurement errors and disturbances are going to corrupt your observation. As a result, the measurements, which you deal with in reality are going to contain mix of deterministic and random effects.

Now, the question is how much of deterministic and how much of random randomness is present in the signal? It is a very qualitative question, but usually it is quantified by what is known as signal to noise ratio. It essentially gives you an idea of what is the extent of determinism present in a signal to the extent of uncertainty. When we have high signal to noise ratio we say, that the measurement is predominantly deterministic and when we have low signal to noise ratio, we treat it as predominantly stochastic. So, you, you are going to encounter typically, what we known, what we call as composite signals.

In this course particularly, we are going to deal with primarily deterministic or predominantly deterministic signals. You are going to switch off the random component, but with a big word of caution. The techniques, that we learn for deterministic signals are not necessarily applicable to the class of random signals. There are number of examples, that can be given, but I am not going to, going to back, but that is a fact that one should remember. You cannot straightaway apply this the techniques that you learn for deterministic signals to the random signals as well as and classic example is Fourier transform. Fourier, a transform of deterministic signals exist, a class of deterministic signals, but Fourier transforms of random signals do not exist.

And another point that should be noted is a random signal by definition is assumed to exist for infinite time. There is, there is no randomness about it existence. What is random about random signal is the uncertainty associated with the value of the signal at each instant. So, a prime difference between deterministic and stochastic signal or a random signal is, at each instant in time, a deterministic signal can only assume one and only one value. There is no uncertainty about it, but when it comes to a random signal at each instant, there exist many possibilities out of which you end up observing one. And philosophical point that you should find, you could find useful is, that no process is truly random or truly deterministic. It essentially depends on the knowledge and that is why the prediction view point is helpful.

A signal becomes random if I do not have the complete knowledge of the process that is generating it. On the other hand, if I have complete knowledge of the process that is generating, then the signal is deterministic to me. So, as I obtain more and more knowledge of the process that is generating the signal you can say, that the determinism is increasing, it is just in loose terms, but the uncertainty is shrinking and so on. Anyway, since we are going to deal with deterministic signals, primarily we will not.

Well, further on random signals I would recommend, that you read any other book, that is that deal with this stochastic signal and so on. Occasionally, we may talk about it, but not too much in detail.

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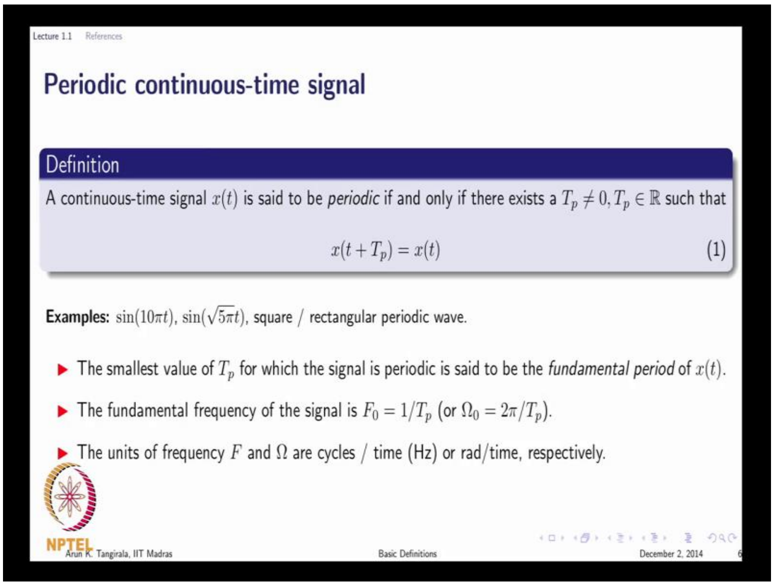

So, let us move on to discussing what is a periodic continuous time signal. These are elementary concepts that we learn in even in our first year of under graduation and so on. The definition of a periodic continuous time signal is straight forward. If you can find a finite time after which the signal repeat itself, then you say, you can say that the signal is periodic. This finite time can be continuous, that is the important fact to remember because when we move in to discrete time signals, there is going to be huge difference.

The first time after which you, you find the repetition of the signal is known as a fundamental time period and the inverse of this fundamental time period is call the fundamental frequency of that periodic signal. There are again several examples that one can give as you can see here. There, there are sign signals and there are other square rectangular signals and so on. And also note, that the frequency can be expressed either in cyclic frequency as cycles for unit time or in angular as angular frequency as radians per unit time. The relations being very straight forward omega is 2 pi f.

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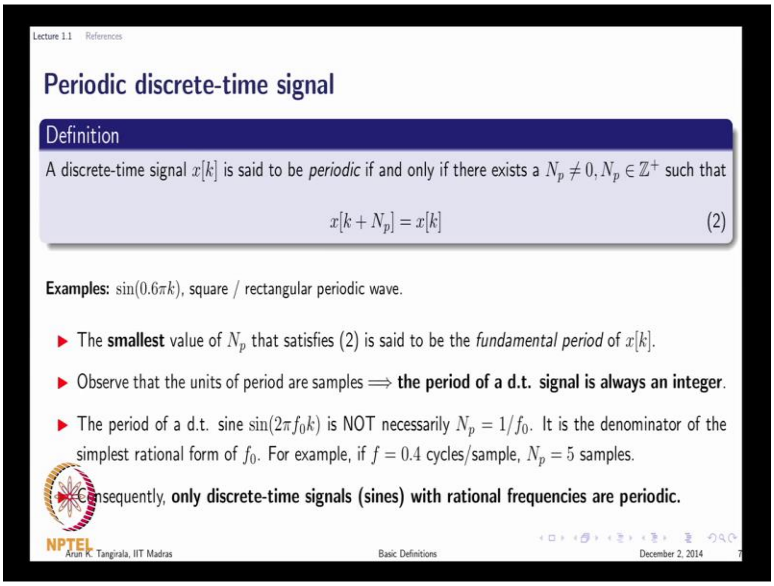

When it comes to periodic discrete time signals, there are some striking differences with respect to the continuous time counterparts. The first fact being, that a signal is periodic, discrete time signal is periodic if and only if you can find an integer number of samples under which you can notice a reputation. So, this is in contrast to the continuous time signal where the period was continuous value. Now, the period is integer value. In fact, it is an, it is a positive integer. Therefore, the period is always expressed in terms of samples for discrete time signal.

Now, as a consequence of this you cannot really say, that if I have a sine wave, let us say I have sine 2 pi f naught k, I cannot really straightaway say, that the periodic of this discrete time signal is 1 over f. For example, if f was, let us say, 0.3, I cannot say, that the period is 1 over 0.3 because 1 over 0.3 is not an integer. So, what do I do? I have to, even 0.4 for example, I cannot say 1 over 0.4 is 2.5. There is nothing like a 2 and half samples. Then, I will have to express a frequency in the rational form and its simplest rational form. So, 0.4 can be written as 2 over 5 in its simplest rational form. And, and if you look at the units of the frequency of a discrete time signal, it is cycles per sample.

So, when I have f equals 0.4 it is completing 0.4 cycles in a sample. In other words, it is completing, it takes 2 and half samples should complete one cycle, but I will not be able to observe that. So, the first time after which I will observe the reputation is 5 samples, therefore 5 is a period of the signal.

Now, what this also means, that discrete time signals are periodic if and only if you can express f as a rational number. So, if the frequency of the discrete time signal is irrational, then automatically it means, that it is not periodic, which means, you will be never be able to find an integer number of samples after which you will see it is repetition. This is probably the most important difference between discrete time periodic signal and continuous time periodic signal. This is something that we shall remember.

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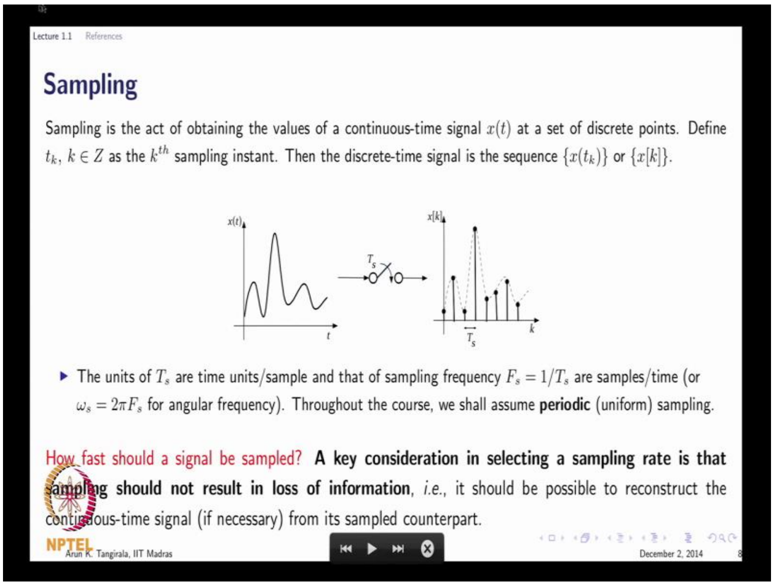

Now, as I said earlier, since you are going to primarily deal with sample data, it is important to understand the sampling theory, the consequence of sampling and so on. The process of sampling itself is fairly straight forward, as you see in the schematic here. The act of sampling is nothing but essentially, obtaining values of continuous time signal at specific instance in time. And sampling does not necessarily mean regular sampling, does not mean periodic sampling necessarily. When we observe this signal at regular intervals of time, then we call this as uniform or periodic sampling, which is what we are going to assume in this course.

So, as you can seen in the schematic, there is a sampler that is involved in producing the samples for you and the sampling interval is denoted by T s and the inverse of this is call. The sampling frequency, typically expressed in hertz, but you should also note the units of F s. It is actually the number of samples obtained in unit time.

Question that we want to ask is, what is a rate at which I want to sample a signal? How fast should I sample? How fast should I observe? Intuitively, we know that it depends on the rate at which it changes, right. If the signal is changing slowly, then I would, I can ((Refer Time: 13:41)) to observe at a slower sampling rate. And if the signal is changing rapidly in, in, in, amplitude, then I need to observe it fast. Now, this question was studied at least more than 60 years ago by many among them a prominent being ((Refer Time: 13:58)) and so on.

And the result is the sampling theorem, that we learn today in all signal processing and communication text and so on. So, what is the sampling theorem, tell me? It essentially tells me how fast I should sample a signal given the knowledge of the frequency content of the signal. So, once again you see why frequency domain analysis is useful.

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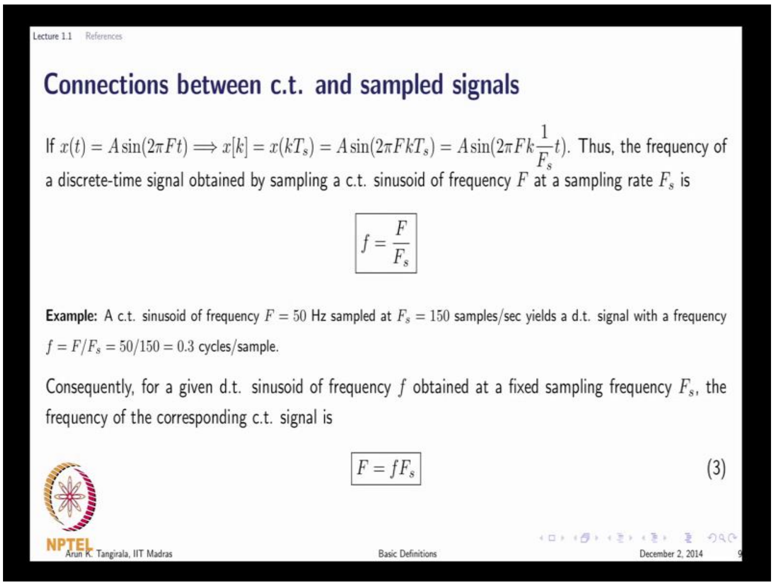

So, let us understand this sampling theorem in an intuitive and practical manner rather than going through rigorous proofs. To understand is, first we will establish the connection between a continuous time and a sample signal. Not all discrete time signals are necessarily coming out of a sample continuous time signal. They are not a consequence of sampling. But if the discrete time signal is being generated by sampling a continuous time signal, then they, you can establish a connection between the frequency of the discrete time signal and frequency of the continuous time signal.

So, if you look at the example, that we use here, we consider a continuous time sine wave of frequency big F and when I sample it uniformly at a sampling interval of a T s, then I obtain a discrete time sine wave. As you can see, the math is fairly straight forward. Now, when you re-express this discrete time sine wave in terms of f, that is the small f, then it is clear, that the small f is F over F s. You can see, the small f is big F over the big F s. F s is the sampling frequency, which means, that I can always calculate what is the frequency of the discrete time signal if I know the frequency of the continuous time signal and sampling frequency. Now, there is a, let us look at this example.

So, suppose I have a continuous time sine wave of frequency 50 hertz being sampled at 150 samples per second, then I obtain a discrete time sine wave with a frequency of 0.3 cycles per sample. Now, this also means, that if I am given the frequency of the discrete time signal and the sampling frequency, I can back calculate the frequency of the continuous time signal and the formula is, relation is fairly straight forward. So, that is just a consequence. So, the equation that you see here is a consequence of the relation that we see at the top. This is something that we will use in understanding the sampling theorem.

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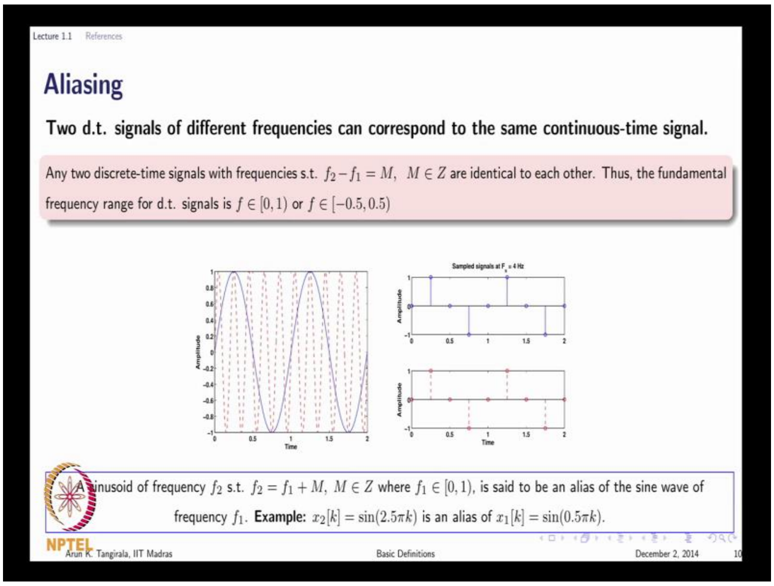

So, let us move on and learn an important concept called aliasing, which is useful in understanding the sampling theorem. So, we understood the connections between the frequency of a discrete time sample signal and, and frequency of the continuous time signal. As a consequence of that relation, that we studied earlier and as a consequence of the property of the discrete time signal, we will understand, that how the sampling theorem emanates.

In aliasing, the most important fact to remember is the property of discrete time sine waves, which is, that two discrete time signals or sine waves of different frequencies can correspond to the same continuous time signal, that is something, that is something, that is not so intuitive straight away, but it becomes clear when we understand the nature of this discrete time sine waves.

So, let us assume, that I have two discrete time sine waves of frequency f 2 and f 1 and if these two frequencies are separated by an integer, we are talking of cyclic frequencies. So, if I have f 2 being f 1 plus sum M, where M is an integer, then you can rewrite this sine wave of frequency f 2 as a sine wave of a frequency f 1. They are going to be identical, which means, if I generate a sine wave of frequency f 2 and I generate a sine wave of frequency f 1 and they are only deferring by an integer. So, let us say f 1 is 0.4 and f 2 is 1.4 and I plot this signals, I look at the signals, even when I look at a values they are going to be indistinguishable.

Why is that happening? Because this is a property of the trigonometric function, which is a sine or cosine. They have a period of 2 pi in angular frequency or a period of 1 cycle in in terms of cyclic frequency. So, what this means is, that not all discrete times sine signals or sinusoids are unique. Only those signals in the frequency range 0 and 1, either you include 0 and exclude 1 or you exclude 0 and include 1. Typically, we include 0 because that corresponds to easy understanding of a DC component. So, 1 is excluded and that is why you see, that I have a parentheses for 1 and square bracket for 0 or you could say, minus 0.5, 2.5, any frequency range of interval 1 with 1 will contain sine waves that are unique. That means, no two sine waves in this interval are going to be identical, but outside this interval you can always, you can always find a sine wave, but you can always find a counterpart of that sin wave within this fundamental interval.

So, just to illustrate the point here, I take two continuous time sine signals although aliasing per say is not necessarily relative to sampling. Aliasing is purely a property of discrete time signals. Since we are studying sampling I am giving you this example.

So, let us consider two continuous time signals of two different frequencies. One has a frequency of 1 hertz and other has a frequency of 5 hertz. Now, I am going to sample these two signals at the sampling frequency of 4 hertz. So, what happens is, what you see here on the right, the blue signal on the top and their corresponds to the sample version of the blue or the solid continuous time signal that you see, which is 1 hertz. And the red one at the bottom corresponds to the sample version of the discrete continuous time signal, which has a high frequency.

Now, if you, if I did not give you these continuous time signals to you, these two discrete time signals we look perfectly alike, you will not be able to distinguish, which means, if you were to sit and reconstruct the corresponding continuous time signal, that is, if I were to back calculate the frequency of the continuous time signal, that generated these two, the answer will come out to be the same. You will not be able to distinguish essentially between these two signals. This is what is called aliasing in sampling and this is what has to be avoided, which means, I have not sampled the high frequency signal fast enough.

And you see this phenomenon in many films and movies. The car wheels, if you see the motion of a vehicle and the car tyre is actually rotating very fast, but the frame rate is not fast enough to capture the rotation of the wheel, as a result you will see the car, the tyres being going very slowly as if they are not moving. In fact, sometimes in an opposite direction. So, this is basic idea in sampling, deriving the sampling theorem.

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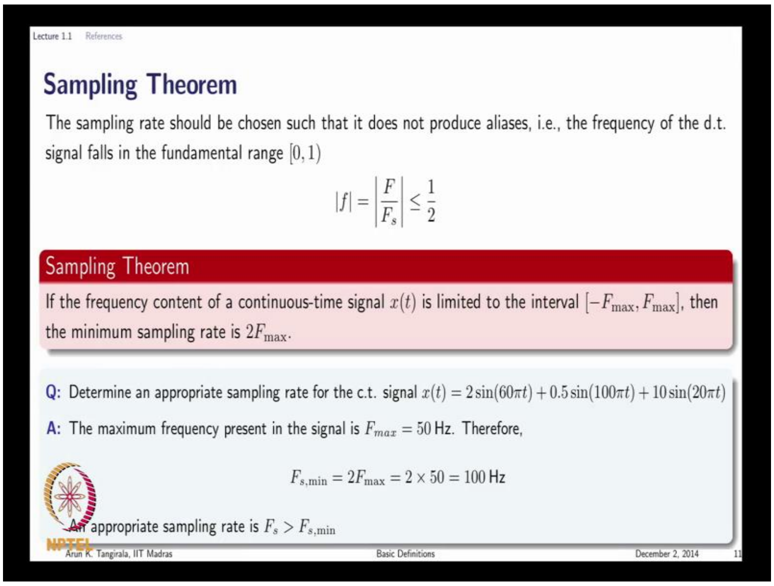

So, what we want to make sure is, I should choose my sampling frequency such that the frequency of the resulting sample signal is always less than half in magnitude. Because we have this restriction, that the fundamental frequency for, for uniqueness of discrete time signals. The fundamental frequency should be in interval minus 0.5 point to 0.5. In other words, F s has to be greater than 2 times F that is the essence of sampling theorem.

Of course, if we look at the formal proofs for sampling theorem, the formal proofs do not rely on this kind of adhoc derivation. But this adhoc derivation is also theoretically sound, but not regressive enough to prove this sampling theorem. Nevertheless, we got the point. The point is, that I should choose a sampling frequency that is at least as twice as the frequency of the continuous time signal. In practice, what is going to happen is, signal is going to contain mixed frequencies and therefore, I need to look at the maximum frequency. And sampling theorem says, base your sampling frequency with respect to the maximum frequency.

So, there is an example that I give here, which you can easily go through. The maximum frequency in a continuous time signal is 50 hertz and therefore, at least I should choose 100 hertz. In practice, I choose much greater than this. So, quick note on what is the situation reality. Situation reality is, that I do not know the maximum frequency a priory. So, what is done is, there are going to be antialiasing filters that are going to be placed in your sampling line, which will clip the maximum frequency based on the prior knowledge, other process and then the sampling is performed.

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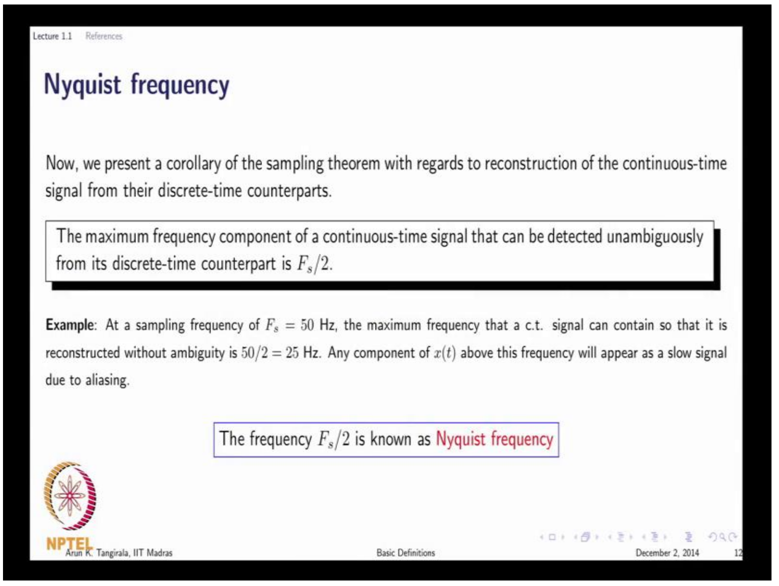

Now, very quick note. We will conclude with the concept of nyquist frequency, which is nothing but a corollary of the sampling theorem. If I am given the sampling frequency and I am asked what is the maximum frequency that I can recover unambiguously, then that is half the sampling frequency that is fairly obvious from the sampling theorem. This half the sampling frequency is known as nyquist frequency.

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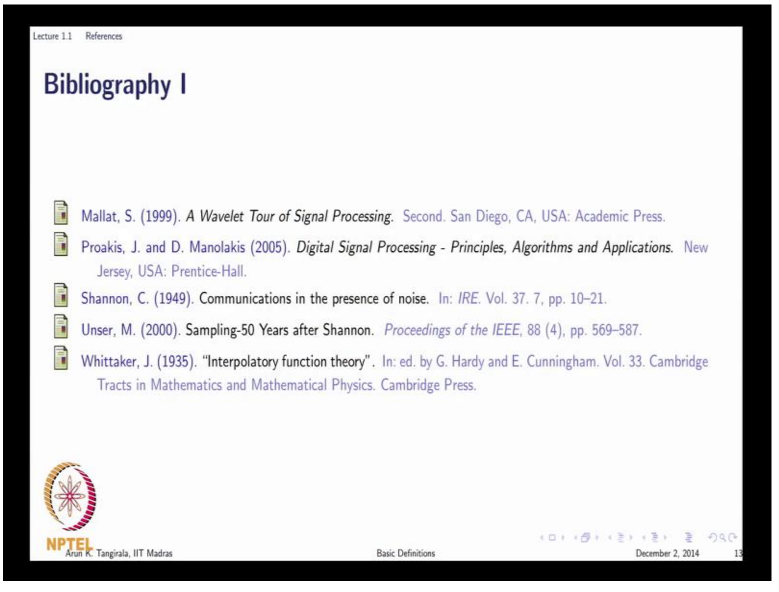

So, with this we come to the conclusion of this module where we have learnt the concepts of deterministic stochastic and stochastic signals, periodic continuous time and discrete time signals and also we have looked at the sampling theorem bit more in detail. In the next module we are going to learn concept such as autocovariance function for deterministic signals and energy and power densities as well as ((Refer Time: 24:24)) on power signals.

Thank you.