Introduction to Time-Frequency Analysis and wavelet Transforms
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Lecture - 8.1
Discrete Wavelet Transform
Part 2/2

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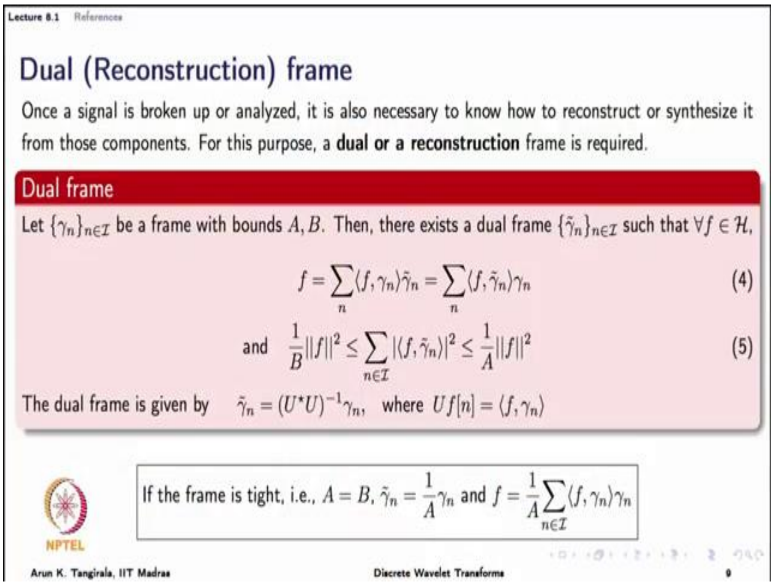

Now, obviously when I broken of the function into its components, I need a reconstruction frame as well, that is I need to know how the how to recovery the signal, and what kind of functions - synthesis functions I require to we construct my signal from the broken up components. And that is where we have what is known as a dual or a reconstruction frame. The dual frame is also a frame. So, the same frame definition applies, but it is a special frame, it is a dual; it is a dual of some other frame. So, that is why we have a reference frame here, suppose gamma is a frame that I have used to break up A and B, then there exists a dual frame gamma n tilde, again n into the finite or infinite. Such that for any function in the Hilbert space, I can write this function as a synthesis equation here, that is the inner products of f with the analyzing frame or the frame reference frame gamma are being used to the construct by f. But then I have another equation here, and will talk about that very soon.

So, let us see what is a dual frame has to satisfy, the dual frame has to satisfy one second frame condition, but because it is a dual frame - the bounds of for the dual frame should be related to be bounds of the reference frame. So, we have gamma is a reference frame, and gamma tiled as a dual frame. A dual frame is also frame, therefore a frame condition exits, but it is a dual of something. So, that something will determine what bounds I should get here, and A and B here are the bounds on the reference frame gamma. Now let us get back to equation four, what this tells me here is, I could switch the roles of gamma n gamma tilde. What this statement these results says or theorems says is that I can use gamma for analysis, and gamma n tilde for synthesis or I can use gamma tilde n for analysis, and gamma for synthesis either way I can switch the rules, such as duals of each other.

So, you do not have to use one specify necessarily one for analysis gamma for analysis and gamma delta for synthesis, you can switch the rules and that is way the duals of each other. That is what is equation means, if I use gamma n tilde for analysis, I can use for gamma for synthesis, that is what a means right. That is a very interesting thing. And again your to keep telling yourself that a frame is nothing but a collection of functions, that I am using for analyzing and recovering a signal. Now what is this dual frame? The dual frame is given by this expression here, unknown gets scale looking at this expression, you that you see here is called an operator, but you can also imagine you do be a matrix. What is this you do the essentially operates on f or it multiplies with f and produces the coefficients that you are completed.

So, essentially you rewriting the inner product as a matrix multiplication with an vector, that is all. And if you can think of U as a matrix, then U star U inverse is just an inverse of the product of U star which is a complex conjugate of U, because your gamma ends can be complex value would also. Therefore U could be complex value, and U star denotes a complex ((Refer Time: 03:57)). You take U star U inverse times gamma n, that is what we will get you gamma n into (Refer Time: 04:02)) this means is all I need to do is choose gamma, and then the dual frame gets fixed or choose gamma n tilde, the other frame gets fixed. So, I do not have to really search for another set of functions that will help me recover the signal.

Now, if the frame is tight there is a nice recovery expression from here, that is when a equals b you can show quickly that gamma n tilde is 1 over a gamma. Now this proof I avoid you can referred to Mallat's books or any literature it is a fairly easy proof very simple proof, but in interest of time I am just avoiding it. So, when the frame is tight the dual frame is simply 1 over a gamma or the dual frame function. What this means is my recovery expression simplifies to this, all I do is in my gamma n for my gamma n tilde as substitute 1 over a gamma. And then I get f equals 1 over a sigma this inner product of f with gamma, n gamma, and so on. So obviously, what does means is when the frame is orthonormal, any orthonormal frame is a tight frame, but any tight frame is not a orthonormal frame right.

So, if A equals B equals 1 further, then what happens? The reconstructing frame functions at the same as the analyzing one or if you think of gamma and tilde is the reconstructing once. So, the dual frame is the same as the other frame itself, they are identical. Therefore, the recovery expression becomes very simple as we will see later.

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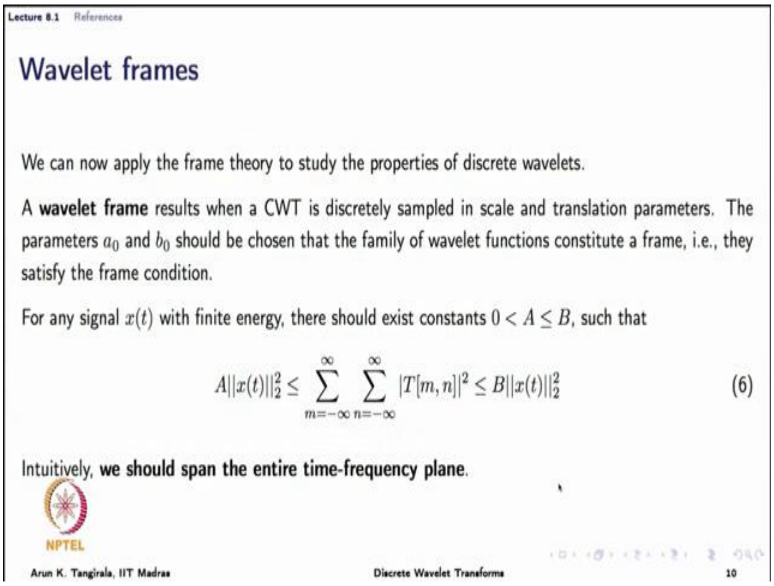

So, let us quickly move on to wavelet frames now. What we why did we study frame theory, because we discretized cwt, when we discretize cwt we are breaking up the signal f or the function f on to a family of wavelets, evaluated at specific scales a naught to the m, and translations n b naught times a naught to the m. Frame theory allows as to now comment and qualify on weather this breaking up of the signal on to this bunch of wavelets, that I am obtain my discretization yields any sensible representation or not.

Now you can think of fourier frames, you can think of any frame, your gamma could any think. Now my gammas are wavelets right. So, wavelet frames results essentially when a cwt is discretely sample in scale in translation. So, this discrete is not d i s c r e e t; that means, something else in English. So, this is d i s r e t e. So, discrete is sample in scale and translation parameters, and the a naught and b naught should be chosen such that this frame condition is satisfied for the wavelet coefficient. So, your T of m comma n is nothing, but your inner product of the signal f with the discrete wavelet at scale m or scale a naught to the m and translation location m. So, you ideally what now this frame theory tells me I should choose a naught and b naught, such that I get a frame, because only frames will allow me to produce stable representation. This energy condition essentially means that I have stability in my representation, that is nothing. If you look at it other way round or intuitively I should choose my a naught and b naught.

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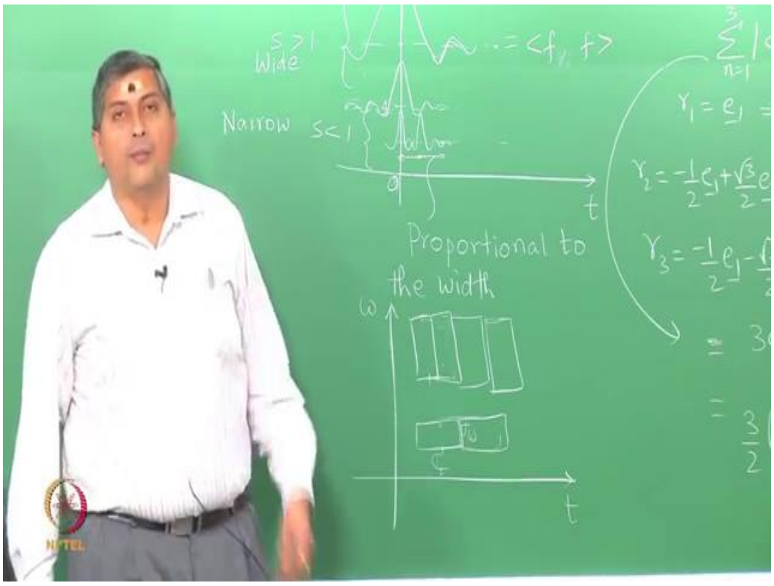

So, if you look at the figure that we had the a naught and b naught will determine the grids on scales and translations, I should choose my a naught and b naught in such a way that the entire time frequency plane. So, now I have time and omega whenever I choose a particular wavelet, what I am trying to do is I am trying to capture this signal in some box here; these are as you know Heisenberg boxes with sigma t and sigma omega for that wavelet, and I say move into the high frequency region I have these boxes. So, as I am choosing different a naughts and b naughts, what I am doing is I am tiling the time frequency plane. Now I should choose my a naught and b naught from such a way that the entire time frequency plane is covered. I want to make sure that there is no gap in this tiling, that's is you are a mason, you are laying down tiles on the floor, you should choose your tile widths in the two dimensions, in such a way that there are no gaps. Not only tile widths that the spacing between tiles, your allowed to have overlapping tiles, but not tiles with gaps, and that is what essentially frame theory allowing to you to mathematically determine whether there are gaps, that is a key point about frames.

Now I would like to choose I can have tiling's like this, and likewise I can have tiling's like this, but when I choose a naught and b naught, such that there are no gaps between this tiles CWT does this, in dwt now I have the opportunities of a choosing a naught and b naught, such that there should be no gap. So, when I do not have any gap then this box is do not overlap. Then what I would have is a tight and orthonormal frame also; orthonormal means tile, that is the key. I can choose overlapping tiles, I can choose non overlapping tiles; these provisions are both. So, I hope you understand what we meant by spanning the time frequency plane.

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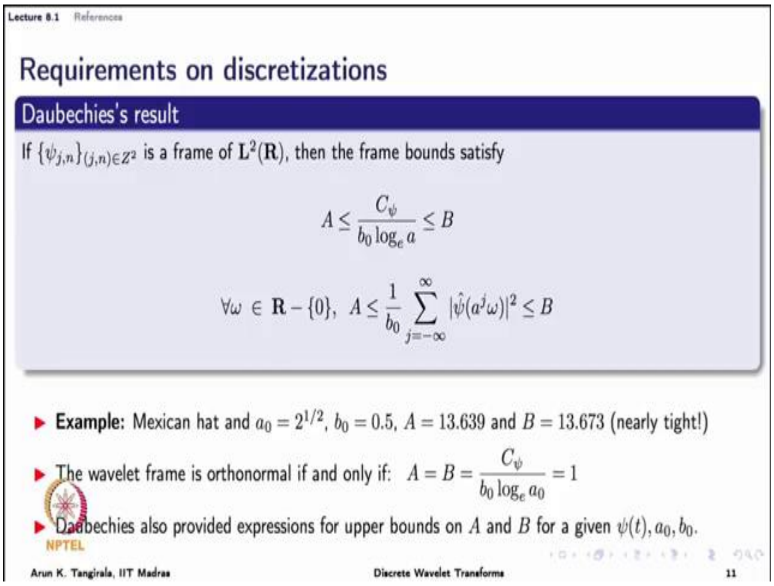

So, what is a requirement on discretization? This was a fundamental result by daubechie, you showed what conditions are required to generate a wavelet frame, in other words, you should be no gaps in the tiles. In terms out that any family of wavelets that you obtain by discretizing, they should not be j comma n, they should the n comma n. Any family of wavelets that you obtain by discretizing the scale and translation will give rise to a stable representation or a frame in the two dimensional dual space, if and only if the frame bounds satisfy this. So, ((Refer Time: 10:22)) is the admissibility constant you know that b naught should be a naught clear, b naught and a naught should ((Refer Time: 10:31)) of the user define parameters. Put together there should be two constants, a and b that you can always fine. Now you know for example, morlet wavelet does not have finite C psi strictly. So, morlet wavelet whatever you do discretize, you will never we able to find a and b, therefore you can never be used for dwt; that is why earlier in cwt, we learn that morlet wavelet cannot be used for dwt.

Now, Mexican hat wavelet with these values of a naught and b naught results in this A and B very tight frame is in fact, this calculations are shown in mallat's book, and many other sources of literature. So, we have nearly tight frame, but the redundancy is extremely high, because if you do not have redundancy then A and B should be equal to 1, it is very high, it is still dwt, but not dwt with a minimal representation. So, I can perform the DWT with the mexican hat wavelet, when normally DWT in literature means orthonormal DWT. Therefore, you will see even mat lab tool box, and in many places in the literature saying mexican hat is not suited for DWT; that is not completely right, it is suited, but not for orthonormal DWT. Now when is a wavelet frame orthonormal, well if and only if; these bounds are equal to 1. A remember I can choose my a naught and b naught and also my wavelet such that this is satisfied, and that is what is entire theory of designing wavelets for DWT - orthonormal DWT, and that is what daubechie work extensively on, and many others to produce orthonormal wavelets, such that this orthonormal frame condition is satisfied. Of course, daubechie also provided expressions are upper bounds and A and B, but will not go into that. So, the entire design of a orthonormal wavelet is about finding a wavelet psi designing a wavelet psi, and choosing a naught and b naught such that this orthonormal frame condition is satisfied. Now, what happens in realities that when practice is we choose a naught and b naught typically s 2 and 1, and will talk about very quickly.

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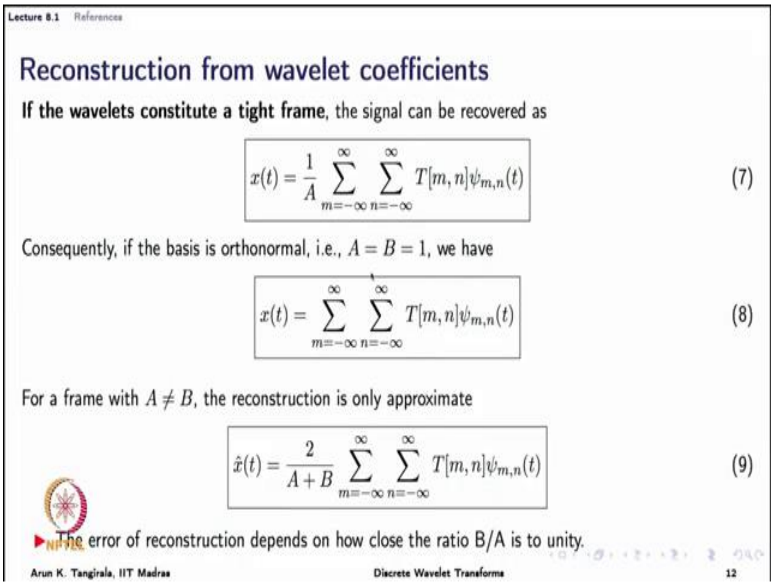

So, the moment I have a wavelet frame; that means, I have chosen my psi and a naught and b naught, such that the frame condition is satisfy, it does not have to be orthonormal. But if it is tight frame, then I can recover it this wave, this again expression comes from this result that we have here. If I have a tight frame gamma n tilde is this, and the recovery expression is this. Now my gammas are wavelets, and the inner products are nothing but T of m comma n. So, I have here this expression 7.

And further If I have an orthonormal basis, then the recovery expression from my wavelet coefficients is the same as 7, but with a equals 1; it is very simple. So, I use a same wavelets for orthonormal DWT for synthesis as well, same as analysis and synthesis. And that is the beauty about orthonormal DWT besides other advantages. What if I use the same wavelet for synthesis when a is not equal to b, that is when it is not a tight frame. Then you get only an approximate recovery, and that is what happens in cwt recovery. That you get some you do recover the signal, but up to a certain approximation error. And that depends on how you have discretize your scale and translations which will in turn depend on determine a and b. So, this reconstruction error will vanish or will start diminishing as the ratio b over a approaches unity. So, that is the thing.

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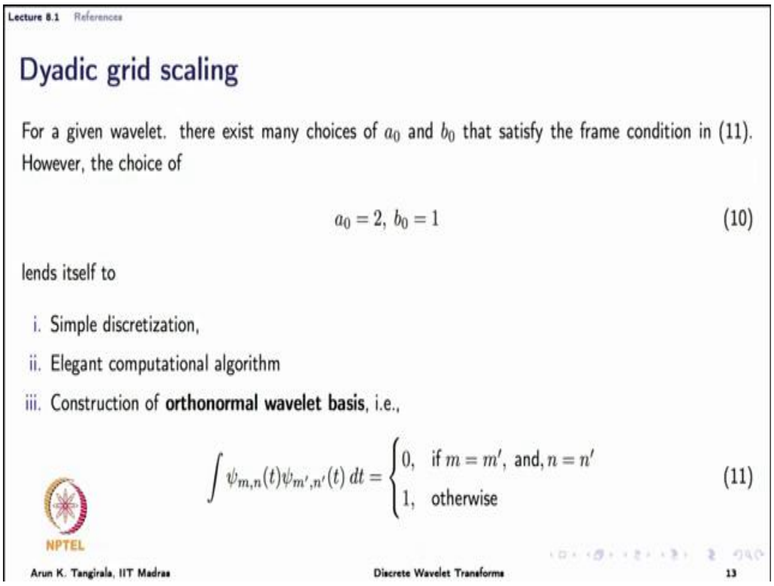

So, finally will conclude with orthonormal DWT, which is what will be the center piece of the entire unit, as we have just learnt I have considerable freedom in choosing and a naught and b naught, but I should make sure that these choices for a given wavelet should yield a frame, in other words should yield a sensible are stable representation. So, there are many choices in principle for a naught and b naught that satisfy the frame condition, but the choice of a naught equals 2, and b naught equals 1 has found to be have been found to be ideal in many respects, first of all it yields a very simple discretization known as the dyadic discretization.

So, my scales are going to be powers of 2, and the computational algorithm as usual learn later is also going to be very elegant, it involves just down sampling by factor of 2, and filtering and then down sampling a factor of 2, and so on. So, it is a very beautiful algorithm, and you can it does not mean that when I choose a naught equals 2, and b naught equals 1, I will always get orthonormal DWT, you have to be really careful. This only means I get a frame, if I want an orthonormal frame, then I have to impose this condition further in my design of wavelets. That is what is also important to know, because all these means is I will be able to satisfy the frame condition, but if I choose my wavelet such that this is also satisfy, that is a equals b equals one. Then I have orthonormal DWT, what is orthonormal dwt? Essentially the inner products again here size or real wavelets. So, the inner products, therefore we throw away the star the inner products between the wavelet at a scale and translation, and at another wavelet - the same wavelet at another scale and translation should be 0. So, long as the scale and translations are different from each other. In other words, you are generating a non overlapping tiling in the time frequency plane, if you just understand that that is more than enough.

Now, there is a condition here, if you go back which also tells you how the frequency access is being discretize. We are being talking about discretization of the scale access, but the frequency access also being discretize in propositional to be A here, it is not being discretize linearly; obviously when I discretize s in an exponential manner, I do not necessarily get a linear discretization omega; the omegas are also being discretize in some exponential fashion. And you go from narrow band a small bandwidth in the low frequency to large bandwidth in the high frequency; that is something that will revisit later on.

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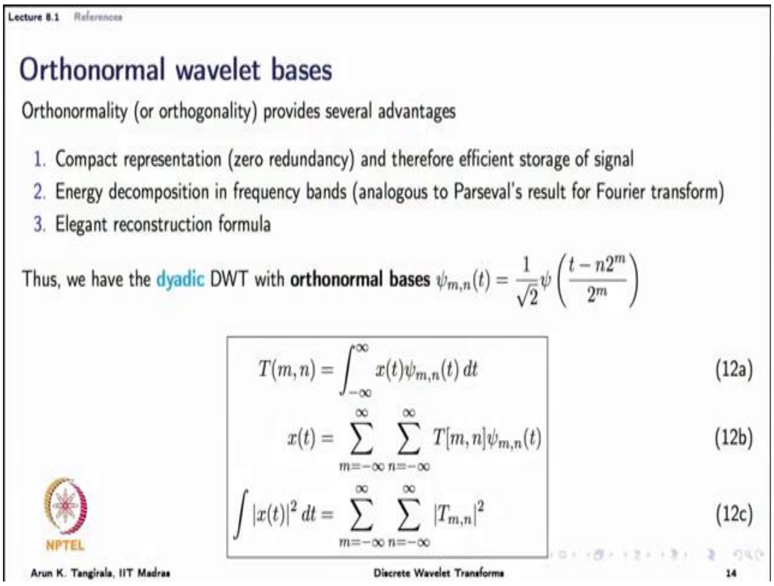

We will conclude therefore this lecture with a summary of orthonormal wavelet basis, the orthonormality provides several advantages, compact representation minimum representation or 0 redundancy. And therefore, efficient storage of the signal that is useful and signal compression. And also it provides a nice energy decomposition of the energy of the function, that is decomposition of the function into scales are frequency bands, which is analog as to what we saw as an energy decomposition for fourier transform provide, that is useful in characterizing, and feature extraction in many many applications. You can determine for example, in which frequency band how much energy is presents, and you can then say you can do a monitoring, for example how the energy shifting based on how the energy shifts from one scale to another scale are among frequency band to other frequency band.

And then you have an elegant reconstruction formula as we seen earlier. So, when we choose a naught equals 2, and b equals 1. We have dyadic dwt; dwt in general does not mean dyadic, but in literature dwt almost always means dyadic. So, we shall drop this dyadic term unless otherwise necessary will not use this in referring to this dyadic DWT, we will simply use DWT. And further will look at DWT with orthonormal basis, and these are the expressions for the orthonormal basis. Of course, equation 12 a is a general wavelet coefficient calculation expression, 12 b is recovery, if it is not of the orthonormal and if you it straight then you have a 1 over a. And the last one 12 c is the energy decomposition, that I was referring to earlier. The energy of the signal is being decompose this way, straight away now we have the energy not the density, because the right hand side is not summation, but we have an energy spectrum in the time frequency plane. This we call as the discrete wavelet energy spectrum, which is the function of this m and n pretty much like the line spectrum that you saw in the fourier case. So, that is about it for this lecture. Now hopefully you would all clear as to what is DWT, dyadic dwt, orthonormal dwt, what is the frame, why did we look at frame theory, and so on. In the next lecture, we shall look at how this orthonormal DWT naturally leads to multi resolution analysis. So, these are some references for your ((Refer Time: 19:46)), and we will see on next lecture.

Thanks.