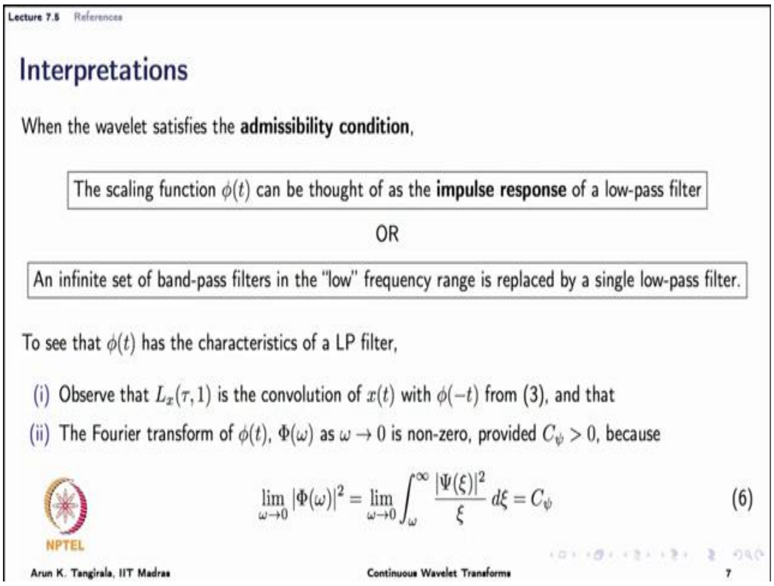
Introduction to Time-Frequency Analysis and Wavelet Transforms
Prof. Arun K. Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 7.5
Scaling Function
Part 2/2

(Refer Slide Time: 00:15)



Now, we just now said that only when the wavelet satisfy, that there is some similarity here between the integral in expression 5, the final integral here, and the one for the admissibility constant; and also, that the admissibility, existence of the admissibility constant is a requirement for the scaling function, and that is exactly what we are saying here.

When the wavelet satisfies the admissibility function, the scaling function can be thought of as a impulse response of a low pass filter; pretty much like how we have been treating the wavelet as a impulse response of a band pass filter. Or, in other words, an infinite set of band pass filters in the low frequency range is being replaced by a single low pass filter. Either of these interpretations is ok.

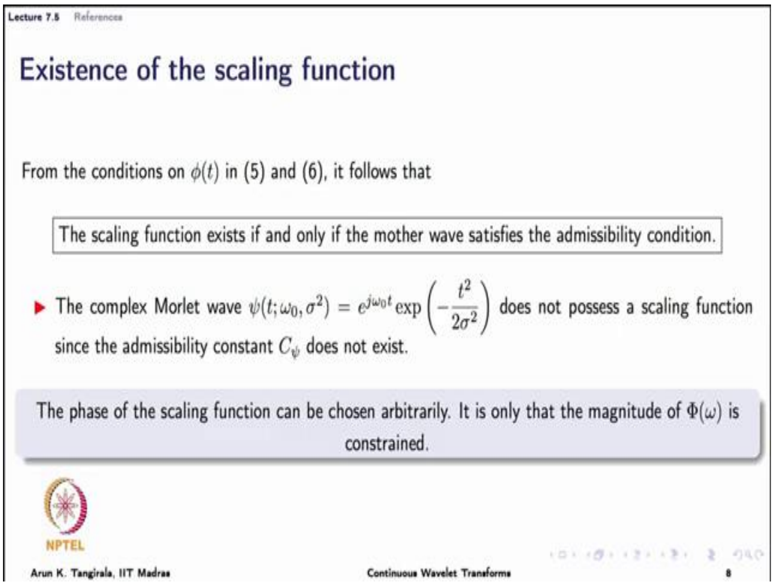
Now, why is this true? That is, how can I think of phi t as the impulse response of a low pass filter? Note this, to see this, observe that is a first point, that something that you have used earlier, that the approximation question L is a convolution of x with phi minus t, l whether its phi minus t or phi t, it is almost the same. If one is a low pass filter, other is also pretty much access a low pass filter.

What is happening here is, convolution of x with phi t. And, we know from linear systems theory that convolutions are nothing but filtering operations. What kind of filter is phi? How do I know? Well, simply evaluate the phi omega at omega equals 0, like the way we did for the wavelet. We said that wavelet is a band pass filter, primarily because its magnitude that is a Fourier transform of the wavelet at omega equals 0 is 0. Because wavelet is a 0 average function, and we know, therefore, that psi omega at omega equals 0 is 0.

Now, what do we want for a low pass filter? The Fourier transform of this low pass filter or the frequency response function, what we say, Fourier, should be non 0 at 0 frequency. That is fairly obvious because low pass filters have necessarily a non 0 magnitude in the f r f's. So, to see whether the phi omega has a non 0 value as omega goes to 0, all we do is we evaluate the integral that we had in equation 5 here, in the limit as omega goes to 0.

If the magnitude is 0 or non 0 then phi omega also will be non 0. Therefore, it is sufficient to evaluate the limit of the magnitude, of the squared magnitude. And, what we have here is nothing but the admissibility constant itself. In other words, the value of the squared magnitude of the low pass filter at omega equals 0, or the f r f at omega equals 0, is nothing but the admissibility constant. And, we want this to be bounded and non 0, and that also tells us that only wavelets that are admissible will have scaling functions.

(Refer Slide Time: 03:42)

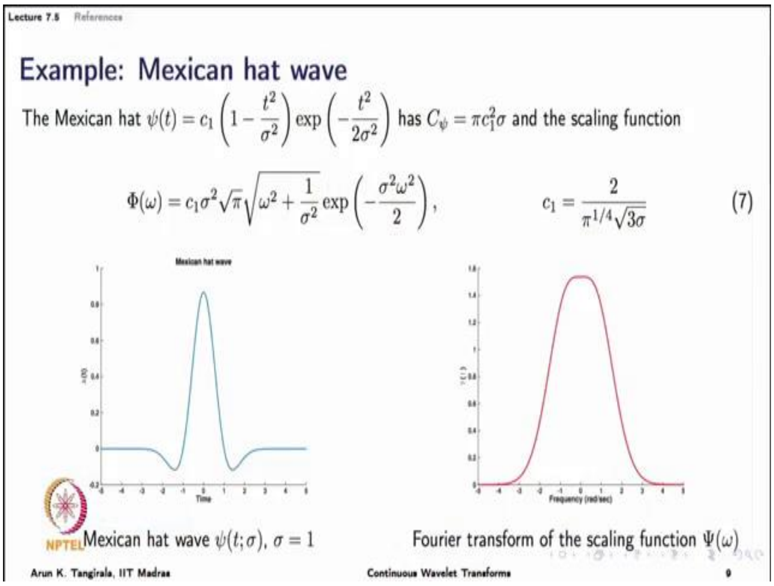


And therefore, what we have here that the scaling function exists if and only if the mother wave satisfies the admissibility condition. Consequently, the complex Morlet wave that we have does not possess a scaling function, why? Because, the admissibility constant for this particular wave does not exists. In other words, it is unbounded. The other way of saying it is, strictly this Morlet wave is not of 0 average.

However, as when you choose omega 0 in the range 5 to 6 roughly, then the average of this wavelet is of the order of 10 power minus 5. But, we, but, and of course, C psi is going to be bounded, but very large value. And, we are not going to really consider that as something that satisfies a requirement here. Strictly we want the admissibility constant to be finite for all possible values of the parameter.

Now, as I mentioned earlier, the phase of the scaling function can be chosen arbitrarily; the only requirement is on the magnitude, squared magnitude which depends on the admissibility constant. So, that is how the wavelets and scaling functions are tied together.

(Refer Slide Time: 05:09)



As a simple example, let us look at the Mexican hat wave. Here, there is some confusion in the literature. For example, if you look at the documentation on wave fun, the routine, that we have seen earlier in MATLAB's wavelet toolbox of released 2014 be, the documentation would say, there exists no scaling function for the Mexican hat wave. Unfortunately, that is not entirely true.

It is true only when you are using the Mexican hat wave for multi resolution analysis or DWT. But, if you are using Mexican hat wave for CWT, the scaling function exists as given in this equation 7. In fact, this expression is given in Mallat's book. So, the Mexican hat is a second order derivative of the Gaussian, equally minus t square by 2 sigma square. And, the c 1, here is a constant that ensures unit energy for the wavelet; in other words, integral mod psi t square d t, should be 1.

Therefore, we have this constant c 1, and I have given the expression, the value for c 1, here in equation 7. Now, this has an admissibility constant, pi times c 1 square sigma. And, the scaling function can be derived starting with this equation 5. And, letting the scaling, phi t, or the phase to be not exist; in the sense, we do not really consider the phase, we let phi omega to be real value; and, that is exactly what we have here, phi omega is real value.

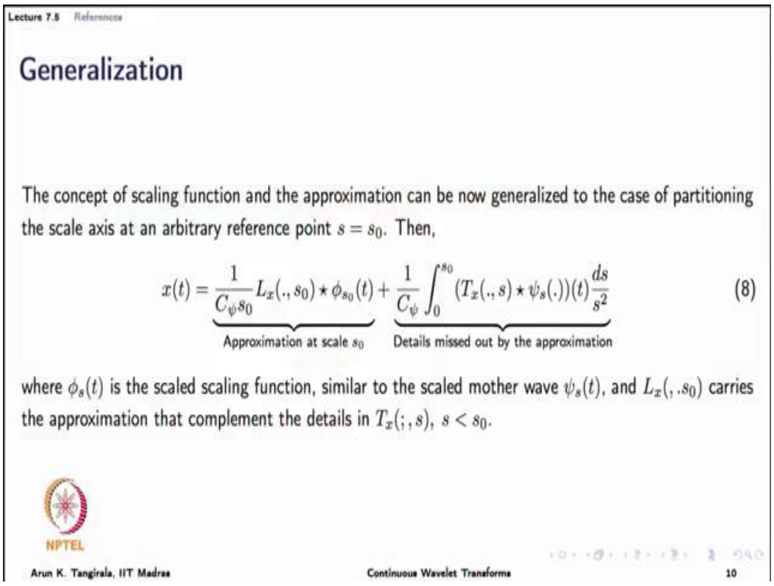
So, you, from equation 5, you get an expression for mod phi omega square, and you take the square root and you let phi omega be real value. So, you do not, you throw away the phase because that is free to choose. I have plotted the Mexican hat wave for you. The name, obviously, arises from the nature that is a shape of the wave that you see here. It looks like a Mexican hat. For those of you who have not seen how Mexican hats look like, then I think you can go to the net and see how Mexican hats look like. It is just a fancy name for this wave.

And, what you see on the right is a scaling function. Observe that this Mexican hat wave on the average will have 0 value because it exists both to the above 0 and the bottom of the 0 line in the y axis. Whereas, the scaling function should not have a 0 average because that is a requirement for a function to act as impulse response of a low pass filter.

And, in fact, if you workout the area, it should actually, sorry, workout the magnitudes square of phi omega at omega equals 0, you will get the c psi. And, what I have shown you here is, pi omega, not phi t, I am sorry about the earlier statement, about the average not being non 0. But, you can obtain phi t by taking the inverse Fourier transform of phi omega, and plotted for yourself.

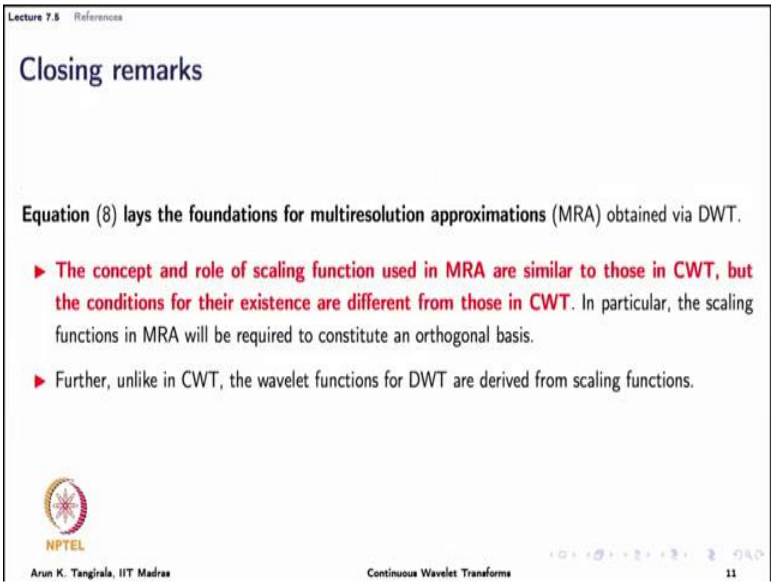
So, one way is, if you cannot do this analytically, evaluate phi omega at different values of omega. And then, use the inverse Fourier transform algorithm in MATLAB or any other software, and compute phi t; you should be able to see the phi t. So, it should be complex value because you have phi omega to be real value. And, both this plots have been drawn assuming sigma to be 1. If you change, sigma things are not going to change much, in the sense, in the terms of shape, but what you will see is the widths being different, with different values of sigma.

(Refer Slide Time: 09:14)



Now, we can extent or generalize this idea of scaling functions, as I said earlier, to an arbitrary reference point s 0. So, I am instead of partitioning this scale access at s equals 1, or partitioning the frequency access with the reference point being the center frequency, I can choose an arbitrary reference point, and I can say, that the now my reference point is s 0, and the scaling function produces, at that scale s 0, produces the approximations of x t at scale s 0, that complement all the details that are obtained by the wavelet transform at all scales less than s 0, right. You should always remember, lower scales you have details and higher scales you have approximations.

(Refer Slide Time: 10:13)



Now, this idea, in fact, equation 8, is the foundational equation for multi resolution approximations, that we shall learn in discrete wavelet transform where we divide x t, or partition x t, or break up x t into 2 parts, an approximation and a detail. And then, we further break up the approximation into another approximation and detail, and that is how we proceed to a desired scale. So, in the end what you would have is an approximation set of coefficients of x t at a very course scale; and then, all the details starting from that course scale to the finer scales; that will become more obvious when we move into the DWT arena.

As I mentioned at the beginning of the stock, the concept and the role of scaling function in DWT or multi resolution analysis is pretty much similar to what we have learn in CWT. It is only that the existence conditions are different. In multi resolution analysis, there is an additional requirement for the scaling function which is that, its translates should constitute an orthonormal basis. Here, there is no question of basis; anyway we have redundancies and so on.

Therefore, the scaling functions and there translates, in the case of continuous wavelet transform, do not constitute any basis; they are highly dependent on each other. But, whereas, in DWT, remember, we have been saying this, DWT is obtained by discretizing the scale and translation parameters. Therefore, there is a possibility for a certain choice of discretization that I can generate a basis.

And, for multi resolution analysis, it is important that the scaling function constitute an orthogonal basis, or orthonormal basis if you on normalizations. And, that is why the scaling functions that exist for CWT may not exist for DWT; and, that is exactly what wave fun probably tells you. So, you should not get carried away when wave funs say is that Mexican hat wave does not have scaling function, or any other text book that says.

You should verify if that statement is being made in the context of CWT or DWT. Otherwise, this scaling functions here, whether it is CWT or DWT, both take the role of low pass filters. In fact, when we move to DWT, we shall learn that the scaling functions are specified in terms of the low pass filter coefficients.

Recall that one particular example where we tried to connect scale to frequency for the Daubechies wave. And, we said Daubechies waves, wavelets and the scaling functions do not have closed form expressions, rather they are specified by the low pass filter and the high pass filter coefficient. So, hopefully, now that you are able to put that puzzle together a bit more better. Of course, when we discuss DWT, everything will be clear to you.

There is a another important distinction between this scaling functions used in DWT and DWT which is that, in CWT, as we have seen in this lecture, we have derived this scaling function starting with the wavelet function; whereas, in DWT, specifically the multi resolution analysis, we derive the wavelets from the scaling function. So, the starting point is a scaling function. We start with approximation spaces, and then ask what basis functions exist for generating the complementary details. So, you want to keep this in mind, when particularly we talk of MRA, you can compare notes.

So, with these closing remarks we will conclude this lecture. In the next lecture we will get back to wavelets and discuss how to choose wavelets in CWT. When we talk of DWT, we will also talk about wavelets that are specifically used for DWT, and how does one go about choosing there; the conditions of pretty much similar. And, in the last lecture on this topic we will look at a few CWT applications.

Till then, see you, bye.