Introduction Time-Frequency Analysis And Wavelet Transforms
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Lecture – 7.5
Scaling Function
Part 1/2

Hello friends, welcome to lecture 7.5 where we shall discuss the concept of Scaling Function. In the previous 4 lectures we have focused on wavelets. But, now we will introduce this complementary function called the scaling function. This will play an important role in discrete wavelet transform.

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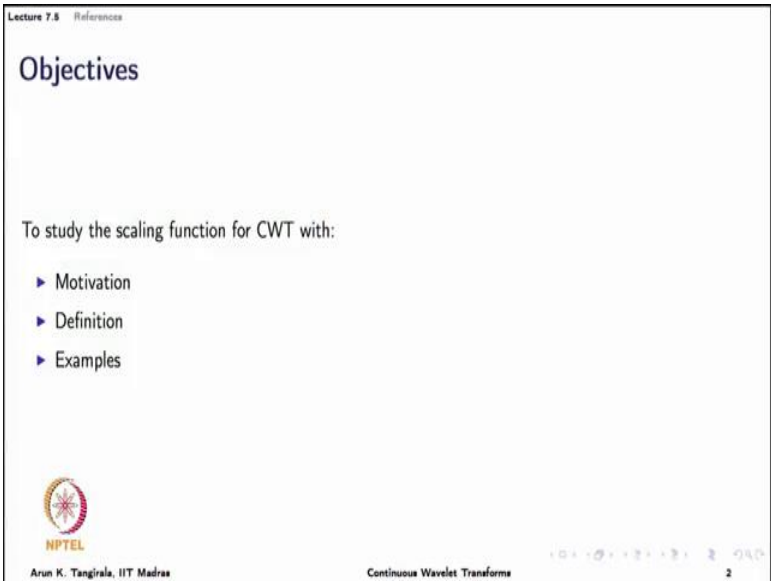

In the case of continuous wavelet transform the role is kind of limited. Nevertheless, purpose of this lecture is to lay foundations for what we shall see as multi resolution analysis in the contexts of discrete wavelet transform. So, this specific objectives of this lecture is to study the scaling function for continuous wavelet transform. As I said, there is another concept called scaling function in discrete wavelet transform as well.

Specifically in multi resolution analysis, which shares strong similarity with the one in CWT. We look at the primary motivation for introducing this scaling function. And then of course, learn the definition and look at a couple of examples. Wherever, possible will accompany and complement the theory with interpretations. So, let us ask why we would like to introduce a scaling function?

Now, we know that, the continues wavelet transform is evaluated across all scales from zero to infinity. And in that process what we are doing is, we are subjecting the signal to bank of band pass filters with different bandwidths. As we venture into the low frequency region the bandwidth of this band pass filter is narrow. And as we move to high frequency regions the bandwidth increases. Now, it is useful to partition this scale space into two regimes, one corresponding to zero to one and the other corresponding to one to infinity.

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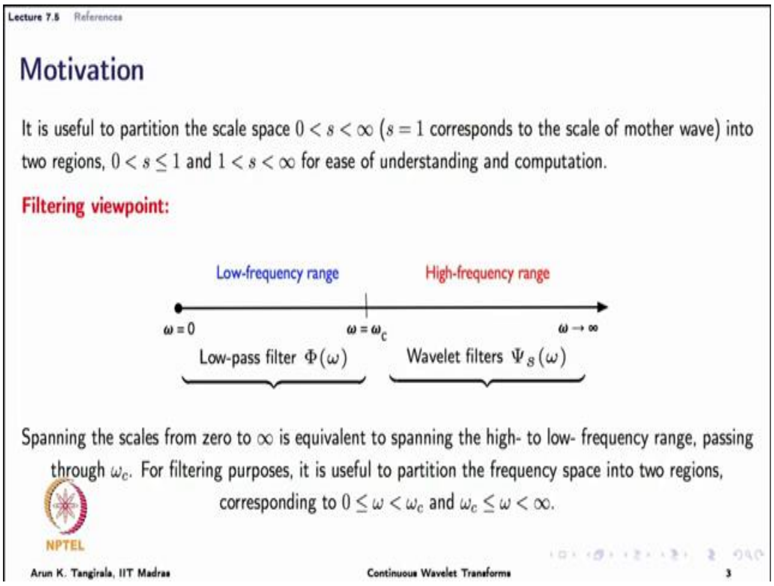

And there is nothing so, sacred about this reference point s equals 1. But, initially we shall use this as midpoint or as the reference point for the interval. Towards the end of this lecture, we shall generalize this reference point as well. The basic idea is to partition this scale space into two regions, one corresponding to so, called low frequency regime. And the other corresponding to the high frequency regime and the partitioning is justified because, all wavelets with reference to the mother wave, which has scale with scaling parameters greater than one have center frequencies to the left of the center frequency of the mother wave. And all wavelets which are scale, such that they are compress that is for values of s less than one has center frequencies to the right of the mother wave.

Therefore, it is reasonable to call the frequency regimes, that the dilated wavelets pane as the low frequency one. And the compressed wavelets the regime pane with the compress wavelet as high frequency range. So, there are two view points for partitioning this scale space. One is something that we are just discussed, which is from a filtering viewpoints. It is nice to partition the scale space, rather than the scale space here, we talk about the frequencies space.

And the reference point being the center frequency itself because, s equals 1 corresponds to the center frequency of the mother wave. So, here I have simple schematic for you on the slide, where I show the partitioning. The frequencies to the left of the center frequency is constitute what is known as a low frequency range. And the frequencies to the right constitute what is known as the high frequency range.

Now, what we are going to do is, we are going to replace in the process of defining the scaling function. We are going to replace this infinite set of band pass filters, that I have in the low frequency range, with a single low pass filters. That is the basic motivation. Why do I want to do that? Well it is not so, immediately obvious right now. But, when we discuss the next view point, that is from a functional analysis view point and also when we discuss DWT. This partitioning will become more obvious. That is a reason for partitioning will become more obvious. At this moment, let us say it improves our understanding and also improves a computation. So, what we are doing is, we are saying that I am subjecting the signal to through two filters, one low pass filter and the other is a bank of band pass filters. So, not necessarily two, but one low pass filter and a bank of band pass filters they are characterize by this wavelet filters.

So, that is the basic idea we are going to replace all this infinite set of band pass filters with a single low pass filter. Now, obviously, we know that signals when passed through a low pass filter produce what is known as an approximation. That is, you filter out the high frequency components the once that change very fast or relatively very fast from sample to sample. In other words, we are leaving out the details and only retaining the approximate part of the signal. So, that is the functional analysis view point that we take next.

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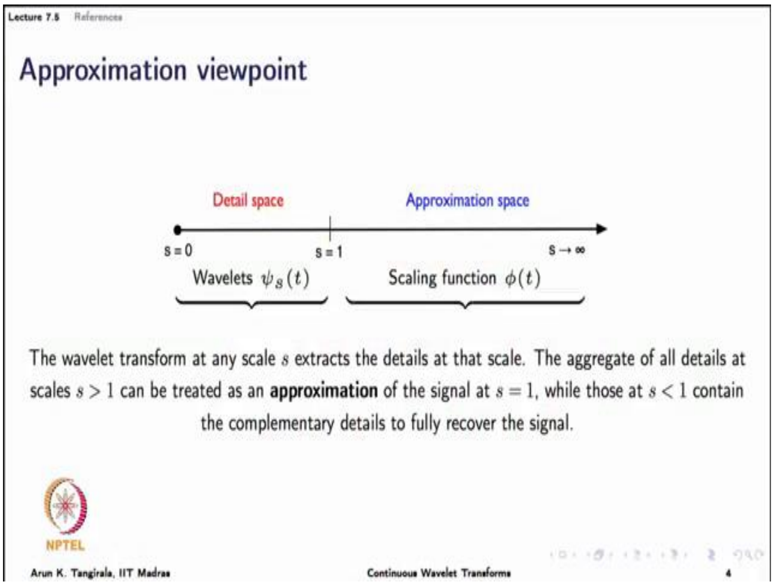

In introducing the scaling function, the idea is to partition the scales space into two parts. One an approximations space and other detail space. So, that way we are able to extract two important features of the signal, one a global feature because, approximations are carry the global details or variations of the signal. And details carry the local variations in the signal. So, let me just draw simple schematic for some arbitrary signal for you on the board.

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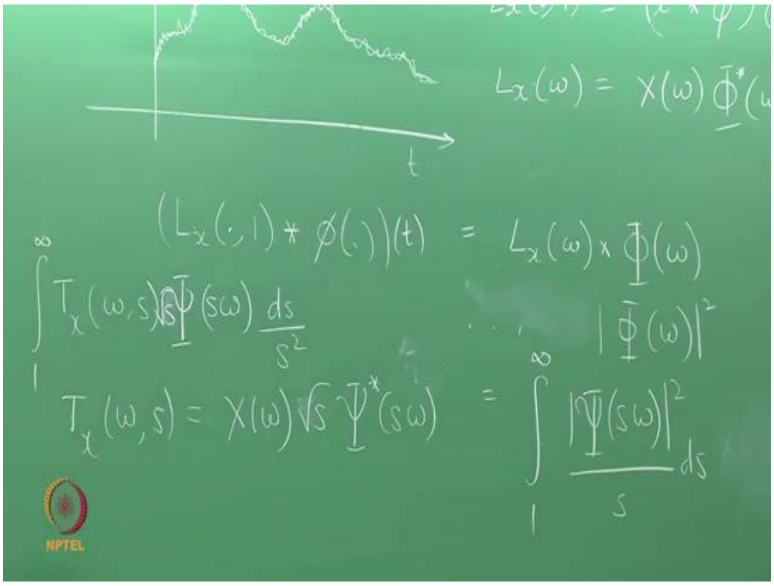

So, let say I have the signal x. And here it the signal consist of this global trend and let us say that it is consists of this minor, but minor in amplitude, but high frequency to mid frequency fluctuations. So, there are this fluctuations riding on the global trend. What we are attempting to do here is to get this global trend that is underlined this signal. And also get the fluctuations that riding on top of this global trend.

Now, classical filtering when you subject x of t through a low pass filter. What it does is it completely looses out the details and only gives you the global trend. So, these details are lost and you will never be able to recover x of t from the approximation. If you want to recover the full signal from the approximation then of course, you need this complementary information, that makes of the entire signal as well.

And that is the basic idea here in wavelet analysis. In fact, that is the basic idea also in filtering, signal estimation using DWT where, we not only obtain the global trend of x of t. But, also retain this details. But, what we are doing is we are separating them out, we have on one plate, the approximation of x of t. On the other plate we have the details of the x of t.

What I mentioned earlier from a filtering view point is, as you can see this local fluctuations are of high frequency content. Because, a change very fast with time relative to the global one. Therefore, if I want to extract the details when I should subject x of t through high pass filter are a band pass filter. And if I want to extract this smooth global trend. Then, I have to subject x of t through a low pass filter. What is special about wavelet filters is, that they constitute both the low pass, band pass and the high pass filters.

And you do not lose out anything until you manually throughout the particular components. You can always use or retain or through away parts of the sig filters signal components, depending on what information you have priory. But, you are in a position always to recover x of t from the filter components, unlike in the traditional filters.

So, the purpose of introducing the scaling function is therefore, to think of passing x of t through low pass filter and a bunch of band pass filters. Or through to construct an approximation plus details at finer scales. So, the other point that you should remember is approximations are always at a courses scale. Because, you keep throwing away every other point. For example, a very crude approximation is to through away every crude every other point or every even three points, two points and. So, on. And what you will be left with a very crude approximation of x of t.

Therefore, approximations are always at courses scale. And details are relatively at finer scale. That is relative to the approximation. And that is what you see in the figure as well.

So, the approximation space is span by scales greater than one. That is a course scales and details space or span is span by wavelets at finer scales, relative to approximation of course,,. So, that is the basic idea and the basic purpose of introducing the scaling function.

Now, the question is how do we do this mathematically? So, we hopefully understood why we want this scaling function. Now, before we move to the math in both the viewpoints ((Refer Time: 11:19)), whether it is a approximation on the functional analysis view point or the filtering view point, observe what we are doing. In the case of from the filtering prospective what we are doing is, we are replacing a bunch of band pass filters with the single low pass filter.

From a functional analysis view point, we are replacing the entire set of wavelets that other scales greater than one with a single scaling function. At this moments it may seem a bit intriguing as to how it is possible to do that. And other what condition really it possible to do that. So, let us now turn to math for help on that.

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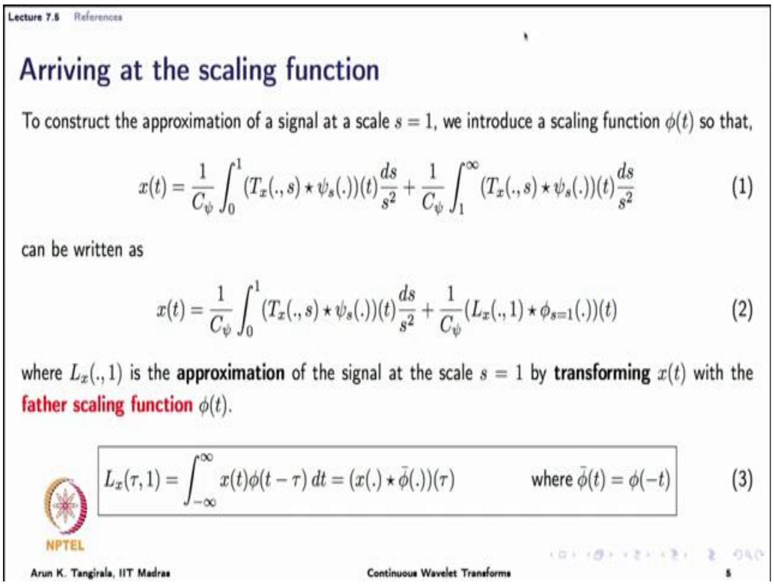

For this what we do is, we start with the recovery expression of the inverse continuous wavelet transform expression. Assuming that we are using the wavelet, that is admissible. In other words the C psi exists, it is of finite value on non zero. So, what I have done here in equation 1 is I have separated the inverse continuous wavelet transform expression into two parts. One corresponding to scales between zero and one and the other corresponding to all scales greater than one.

As I mentioned earlier this partitioning with reference to one is only as of now you can generalize this idea and you can partition with respect to any reference point, it need not be one. So, when I do this of course,, the expressions are fairly straight forward, the integrant look identical here. Then, what do we want to introduces scaling function phi of t. So, that I can rewrite equation 1 as equation 2.

So, what is the difference between equation 1 and 2. Of course, first turn looks identical. That is I am not touch anything, as for as the scales between zero and one or consent. But, what I have done here is, I have replace this integral in the second term with a simple convolution. The integral in the second term is the integral of a convolution. But, here we are thrown away the integral and instead introduce the scaling function. And what is the role of this scaling function?

First a construct an approximation, pretty much like how wavelet construct the detail. If according to our notation wavelets produce the coefficients T at each scale. And here the scaling function which is introduced at scale one. So, we say now are scaling function is at scale one. Because, our partitioning point is one. It introduces an approximation denoted by l. L standing for low pass filtering.

Both l and T are functions of translation parameter and scale. But, because we have replaced this entire set of scales greater than one. That is the courses scale with the single function. Evaluated existing at scale one. L here is being evaluated at scale one, this dot here essentially means, it is a dummy variable it is the place are the potion for the dot is translation parameter.

So, what we are saying here is, this integral 1 to infinity convolution of T with the scale wavelet d s by s square is going to be now l convolve with phi at that scale, which is at scale one and that is about it. And then of course,, 1 over c psi remains. Now, at what we want to ask is, when is this possible can I replace this entire set of wavelet coefficients that I have which have generated with the use of wavelets at different scales with a single approximation. Can I do this? Well, it possible.

Because, what is approximation after all, it is an aggregate of all details. When I keep adding up many, many details, what I end up with is an approximation. So, that is the message that we are trying to convey here from in equation do the same. I am going to replace all the details at courses scales with a single approximation. That is obtain by the scaling function. And the definition for l is given in three here.

The definition for the approximation coefficient is strikingly similar to the wavelet. Expect that, now the replace wavelet function with the scaling function. And of course,, we are evaluating this at scale one. Because, our scaling function right now is at scale one. And once again here, like in the wavelet continuous wavelet transform. The approximation co efficient at scale one can be written as an convolution of the signal with this scaling function reflected version.

So, phi bar like this psi bar that we introduced earlier is the reflected version of the scaling function. I can also scale this scaling function, which will talk about a bit later. Now, the question remains when can I do this? When can I replace equation 1 with the equation 2 particularly the second term. Mathematically under what conditions can I do this. And obviously, from this equation and discussion that we just had. There is a very strong connection between phi and that is a scaling function and the wavelet I cannot choose my phi arbitrarily. So, let us look at now how to answer these two questions?

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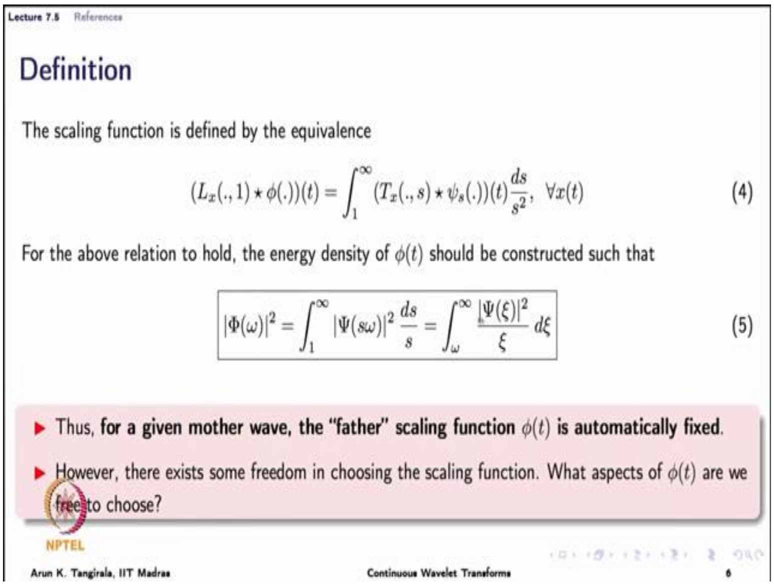

Now, to answer these two questions what we do is, we equate the second term in equations 1 and 2 ((Refer Time: 17:17)). In other words, this integral here is should be identical to this convolution that exactly what we are saying here. So, L at scale one. Once again the dummy variable is the translation parameter convolve with this scaling function should yield this integral here for all signals, when is this possible?

Now, in many text you will see the result, that is given in equation 5 given straight away. But, let me explain to you how to arrive at this condition in equation 5 starting from equation 4. It is fairly straight forward, we will make use of the Fourier transform here to arrive at the expression or the condition in five ((Refer Time: 18:06)). So, what we have here is L x at scale 1 being convolve with phi at time t on the left hand side.

So, what we shall do is, we shall take Fourier transforms of the terms on the left and right hand side. When it take Fourier transform of this left hand side expression. I obtain L x at omega times. Now, this is the product times phi of omega. Where, phi of omega is a Fourier transform of the scaling function and L x of omega is a Fourier transform of L x with respect to this translation parameter.

Now, we also have seen earlier that L x itself is a convolution at any dots. For example, I can take time t as a convolution of x with phi dot evaluated at time tau. Now, here I have the Fourier transform of L x. Which means, I am looking at the Fourier transform of this convolution. And once again I use the property of Fourier transform and right the convolution as the product. So, the Fourier transform of L is x of omega times phi star of omega.

Because, phi bar here is reflected version of the scaling function. And we know reflections in time produce complex conjugates in the Fourier domain, that is it. So, I plug in this expression in the Fourier transform of the left hand side. So, that this boils turn to x of omega times mod of phi omega square. Now, when I look at the Fourier transform of the right hand side. Then, what happens is I have Fourier transform of the integral.

Let us a interchange the operation in the Fourier transform and the integral because Fourier transform is also an integral. So, we are going to change the order of the integrations, we write with respect to the right hand side I have 1 to infinity Fourier transform of the integrand. Now, the integrand is the convolution. Once again I have by virtue of the property of Fourier transforms. I have T x of omega at a certain scale s times the phi bar of the Fourier transform of the psi there which is a wavelet. And then I have here as usual d s over s square.

But, since the wavelet is at scale s I cannot have simply psi of omega, I should have psi of s of omega times root s hopefully it is clear now. So, we taken the Fourier transform of the integral, it is a convolution. Therefore, the Fourier transform convolution is a product of T x of omega comma s. Remember, we are taking Fourier transform with respect to tau. So, it is a product of this times of Fourier transform of this scaled wave of the wavelet at scale s.

And therefore, we have root s times psi of s omega. Once again, recall that this continuous wavelet transform is nothing. But, a convolution of the signal with the reflected version of the wavelet. Therefore, I have here let me write that for you. So, T x here or T x of omega comma s is nothing but, x of omega times root s times psi star of s omega. Because, it is a convolution of x of the t x of tau comma s is a convolution of x with the wavelet at that scale, for we have the situations here.

Because of the reflection I have a complex conjugate and when I multiply both this I end up with this integral here and I let me right that here. Therefore, this x of omega times phi of omega mod square is integral 1 to infinity psi of s omega mod square by s times d s. And that is exactly times x of omega. Now, what happens is this x of omega is independents of s it falls out of the integral. And therefore, what we are left with is a relation, that you see in equation 5. That is how you arrive at this expression, it is not out of the blue that it terms about.

Now, with the change of variable, you can arrive at the final integral that we have here, as omega to infinity. All you do is you replace s omega with the corresponding variable here xi. And then, you should be a able to come up with this integral that is a fairly straight forward expression. So, hope fully now you understand, the conditions under which the identity exist. That is, under which we can replace the second integral in equation one with the one in the equation two more importantly.

The conditions under which I can find a single scaling function, that takes a role of infinite band pass filters or the infinite wavelets, number of wavelets at scales greater than one. Now, what we can see clearly here is that the condition is not really imposing anything on the face of the phi. It is only imposing a conditions on the magnitude or the square magnitude of the Fourier transform of the scaling function.

Now, because this scaling function here is playing a complementary role to that of the wavelet function is scaling function is often called the father scale or the father wave. And the mother wave of course,, is the psi of t. If I know for a particular wavelet I can always derive the scaling function. Of course, this integral here that you see, that is omega to infinity psi of xi mod square by xi d xi. This integral is familiar to us in the form of admissibility consistence.

When recall that the admissibility constant has a similar looking integral, except that the limits run from zero to infinity. And will take about that soon. And in fact, that places a requirement on when this scaling function can exists. So, the scaling function is decided automatically by the wavelet. But, there exists some freedom and that freedom is in the choice of face for the scaling function.