Introduction to Time-Frequency Analysis and wavelet transforms
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Lecture - 7.1
Continuous Wavelet Transform
Part 2/2

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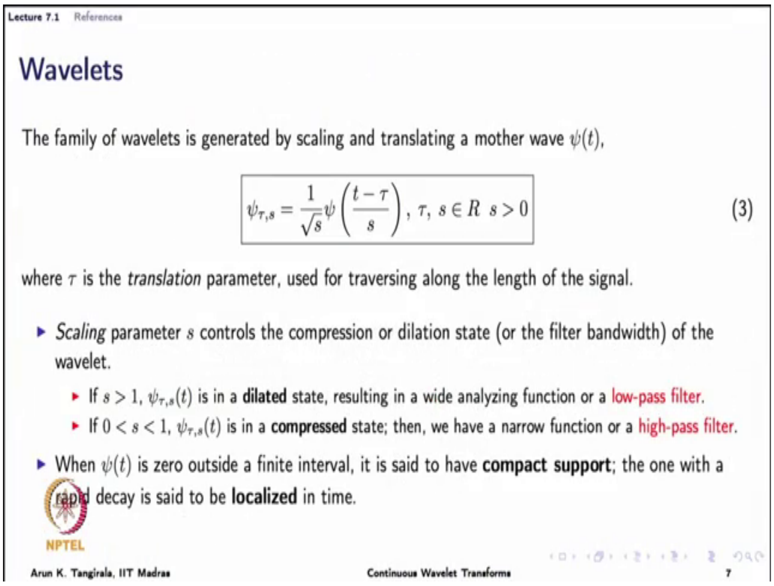

So, with this we have the family of wavelets in general, where I have dropped a subscript naught, psi naught. In general we say psi of t is a mother wave, and psi subscript tau comma s, given by this expression, is the generic child of the mother wave, and the entire family that we obtain, is called a family of the wavelets. Once again tau is a translation parameter s is a scaling parameter. Just to recap, the scaling parameter controls the compression or dilation state. If you are looking at values of s greater than one, then the child is in dilated state, or in a dilated shape, and that acts as a low pass filter, as we have seen earlier and something that we will again learn in this lecture. And when you choose values of scaling, parameter between 0 and 1, then the child is in a compressed state, which essentially allows us to, extract the high frequency components of the signal; that is it access a high pass filter. Some terminology when the mother wave is zero or the child is also zero outside a finite interval, then this mother wave or the wave is said to have a compact support. And when a function has a rapid decay, we say it is a localized in time. So, does not have a compact support, that is it does not exactly go to zero in finite term.

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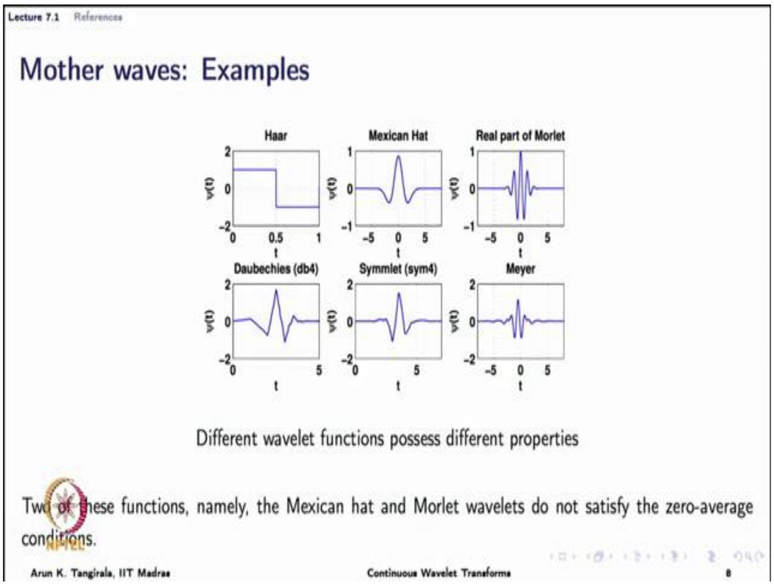

Some examples of mother waves, as you can see here, we have the Haar wavelet on the left top, which is also historically the first noon wavelet proposed by Haar. It has of course, discontinuous nature. And we have Mexican hat in the real part of Morlet, because Morlet waves are in general complex wavelets, it is showing in a real part. Both belong to the Gaussian family; Mexican hat and Morlet wavelet. Then you have Daubechies wavelet which has some nice properties, that will discuss in the contest of the discrete wavelet transforms. And then you have what are known as simulates, which are similar to the Daubechies, but different phase characteristics, then you have Meyer wavelet transforms.

So, each wave although I say wavelet, these are mother waves; each wave as you can see has different characteristics, and depending on what you want to analyze in the signal you choose a particular wavelet. As an example if I want to deduct discontinuous, very short, discontinuities like one is see in Haar, then Haar wavelet is a ideal choice for that. but if I want to deduct regularities, then I choose wavelet that is more regular; that means, most move then so on. And I should mention in passing that two of this function; that is Mexican and Morlet wavelet, do not do not satisfy the so called zero average condition, that is generally required of a mother wave. We have not yet imposed that conditions until now, but we will see why that condition becomes necessary in the next couple of slides.

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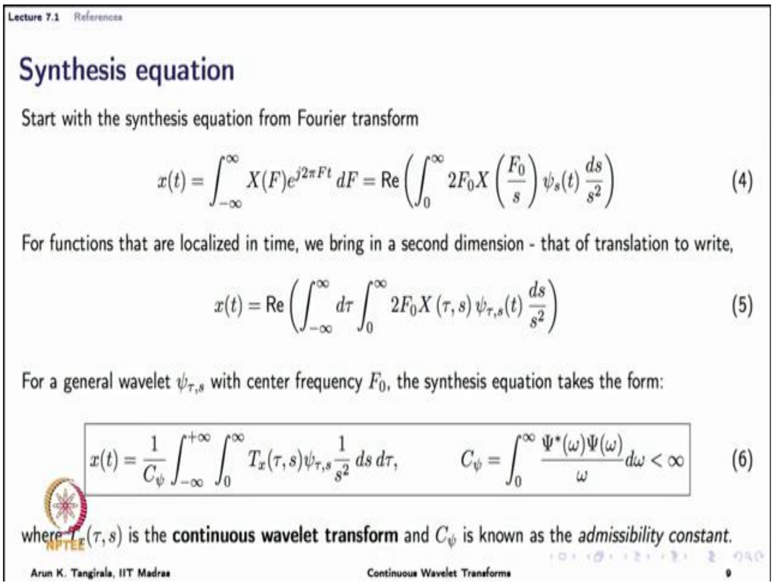

So, having understood how to intuitively arrive at the wavelet starting with the sinusoid function. We will also now understand, how intuitively one can drive the synthesis equation in the case of wavelet analysis, starting with the Fourier synthesis equation. So at the top in equation 4 you have the Fourier synthesis equation, where F is a cyclic frequency. The upper case x as usual Fourier transform. What I have done is, re written this expression, using the relation between the scaling parameter on the frequency. So, I have written s equal f naught by f or f s f naught by s, and also replace this d f incremental variable in frequency with increment able variable in scaling parameter. And I replace e to the j with the scaled wave. This is for the case when this mother wave as psi of t is not localized in time, but for functions at a localized in time, we need to bring a second dimension, and that is why the synthesis equation, now involves in equation 5, a second dimension which is the tau. That is difference between the equation 4 and 5 is in equation 4 psi of t and psi s of t are spread infinitely in time, but the movement I require, or I bring in localization in time nature to psi of t.

Then I have to travels along time also to recover x of t, in addition to travels in along the frequency access. Notice that I am taking the real part, whereas, in equation 4 and 5, where as in the first expression which is the Fourier synthesis equation there is no real part, because frequencies are allowed to take parameters, values from minus infinity to infinity. And since we have restricted ourselves to positive values of scaling parameters, we pick the real values of this, and there is a two introduced, so as to be able to take care of the negative side of it. So, its all intuitive one, but fairly consistent the math. Now the other difference between equation 4 and 5 is; in equation 4 this x big x is only a function of this scaling parameter x, but now it is in equation 5, it is a function of the local parameter in time tau, because now this size are also function of tau. To a general wavelet like we did not deriving the wavelet atom. Now the general mother wave, which is not necessarily sin wave. When we move to the general case that is when we looking at the general reference function on the mother wave, and of course, the wavelets, which have general frequency f naught, then the synthesis equation in 5 takes the form given in equation 6. Of course, I have skipped all the derivation and I do not think it really necessary at this movement. Of course, we can always give you the suitable references. What is more important here, is the string of developments that we have gone through, starting with the Fourier synthesis equation, then introducing the scaling parameter there, where replace frequency with scales, and then restricted the width of the sinusoid in time and then we have generalizing that to function with center frequency f naught; that means, now their also not necessarily localized in frequency as a sin waves. That is more important then actually worrying about how to derive this equation.

So, equation 4 is the Fourier synthesis equation. Equation 5, is where we have introduce scaling parameter and the translation parameter. And then equation 6 is a general version, that we normally see in wavelet analysis, which has offend presented as a recovery equation, or reconstructed equation. notice that in this equation I have one over c subscript c, and the expression for this constant, so called admissibility constant, is given again here in equation 6 on the right hand side, where this c of omega is a Fourier transform of the wavelet. Although I have written c star of omega time c of omega, is essentially magnitude square of the Fourier transform of the wavelet. Rather than simply worrying what is this expression here ,you should ask when does this c subscript c exit, And if you look at this expression it is fairly clear, that as omega approaches zero, you may have an issue, unless the c of omega, the magnitude square of c of omega approaches zero, more rapidly than omega itself. And that translate to what is known as a admissibility condition for a wavelet, or for the mother wave, which in fact translate to the so called zero average condition. the reason being I want the c of omega that is

Fourier transform of the wavelet, to go to zero as omega approaches zero; that means, I want the Fourier transform of the wavelet to be zero at omega equal to zero to be very clear.

Now whenever the Fourier transform of a function is zero at zero frequency, then its average should be zero, because we know from the expression for the Fourier relation, that the value of the Fourier transform at zero frequency is nothing, but the average value of the function itself, and for continuous functions average is simply integral minus infinity to infinity c of t d t. So, that is the famous zero average condition that is presented. On the zero average condition can be looked at in different ways, and we will talk about it soon. So, now, just to summarized in this line what we have done, we started from the Fourier synthesis equation, and arrived at the synthesis equation for the signal in terms of this wavelet transform of the signal, which is denoted by T subscript x and it is a function of tau, s. Compare this with the short term Fourier transform, which was the function of tau comma omega or z; a frequency variable. Now, it is a function of tau, scaling parameters.

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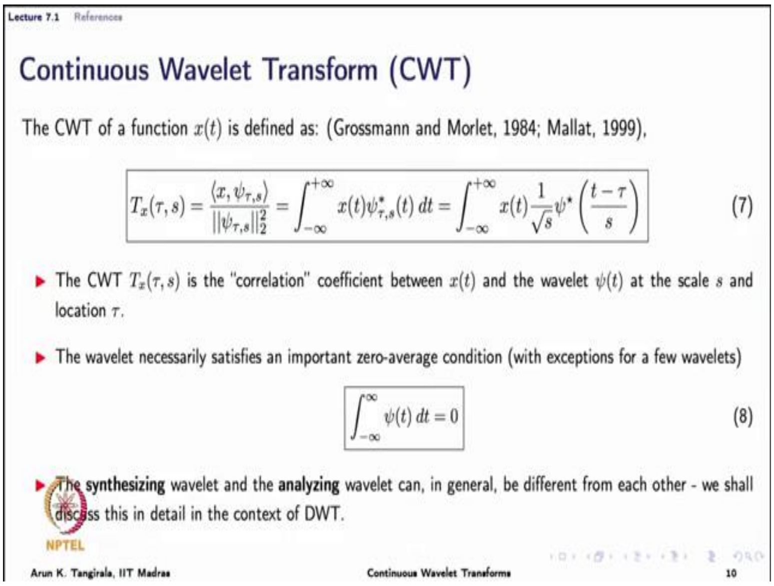

With this background, and with this prelude, we now present the continuous wavelet transform. Of course, as a said earlier much of what we have done, that is starting from the Fourier atom arriving at the wavelet atoms, starting from the Fourier synthesis arriving at the synthesis equation, is not found in many text. So, this is somewhat new to you probably. If you find it bit uncomfortable, you can straight away look at the continuous wavelet transform as yet another transform, but those derivation have really helpless understand. A very key point which is the transition from frequency to scale, and another key point connection between scale and frequency. This the wavelet transform, that we have just seen in equation 6. It is again here has a same shape as a regular transform equation, I have integral x of t c star of tau comma s of t d t; that means, we do anticipate c to be complex value.

In fact, that is true we can have wavelet set or complex value, why not. In fact, when we start when we will go back to the derivation, we said denote c of t as a complex exponential. So, it is possible that c is complex value. Of course, will talk about under what situations we will use complex wavelets, and in what other situations we will use real wavelets and so on in a different lecture. All I have done here is, I will just substituted to arrive at the last integral here in equations and just expanded the wavelet expression. So, it is an inner product between the signal x and the wavelet itself, normalized in a product. The normalization factor being the energy of the wavelet itself. So, the energy of wavelet is normalized one, it is simply the inner product itself.

So, the first useful interpretation of this continuous wavelet transform; like in Fourier transform in short term Fourier transform, is that it is a correlation coefficient, between the signal and the wavelet at that location tau and that is scale s. the scaling s remember controls, how why the wavelets is, or how narrow it is, and the choice of mother wave will determine the shape itself. So, the basic shape is decided that a choice of mother wave, and scaling parameter will then control whether you are looking at a, why window or a narrow window. Of course, translation parameter will determine at what portion of the signal you are situated. The second point is, as we have mentioned earlier the wavelet necessarily satisfies an important zero average condition. Now as I have even mentioned earlier, there are a few exceptions to this, where the integral does not turn out to be zero, but nearly zero. An examples are; Morlet wavelet and Maxican hat wavelet.

In such cases perfect recovery of the signal it is not possible, and I show this to you in a next lecture when you look at some mat lab based examples. Whenever a wavelet satisfied this zero average condition, perfect recovery is possible; that is exactly what this synthesis equation also says. And the third point is, that we have use a same wavelet in synthesis equation and the analyzing equation. What we mean by analyzing equation is, equation seven, where we are computing the transform. They need not be the same, I could use different functions; of course, which are related, for the purpose of recovery and for the purpose of analysis, and we look at this more in details when we discuss discrete wavelet transform. For the movement we have chosen this synthesizing function to be the same as see analyzing function itself that is we use a same function for analyzing and recovery.

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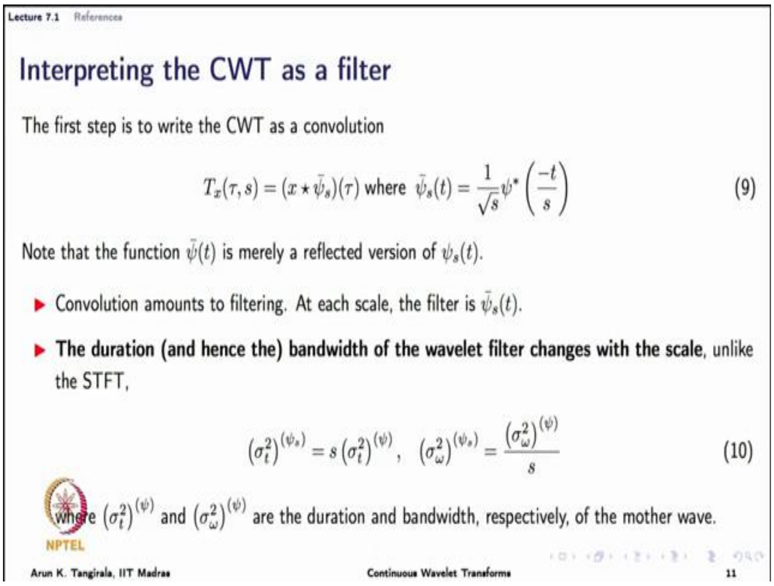

The last point that I want to make in this lecture, is that the continuous wavelet transform can be interpreted as a filter, and this is very important, because this as consequences in a understanding of d w t, the implementations of c w t and so on. Of course, you can always view c w t at a time frequency analysis tool, but this interpretation helps as nicely switch over to d w t and this signal processing interpretation. To see the filtering nature of c w t, all we have to do is rewrite the wavelet transform is a convolution of x with of function c bar which is nothing, but a reflected version of the complex conjugated of a wavelet. So, does not much difference between c bar and c. Once I write this a convolution it is now obvious, that the wavelet transform act as a filter, because I know convolution, all convolution operations are filtering operation from leaner system scale.

The filter here being c bar, but it is not different from the wavelet, it is only a reflected version. So, for all practical purposes you can treat a wavelet itself as a filter. In fact, it is an impulse response of the filter so to speak, because if you recall leaner system theory, the output of the filter is the input convolves with the impulses response of the system. And the most important feature of this filter, is that the duration and bandwidth change with this scale, unlike in the short time Fourier transform. That is if I look at the duration of the wavelet well square duration, then it is related to the duration of the mother wave through the scaling parameter s. And the bandwidth of the wavelet its one over s times the bandwidth of the mother wave; that means, something that we have seen already, as a stretch the bandwidth becomes narrower, but the duration becomes longer, and as compress the vice versa happens .

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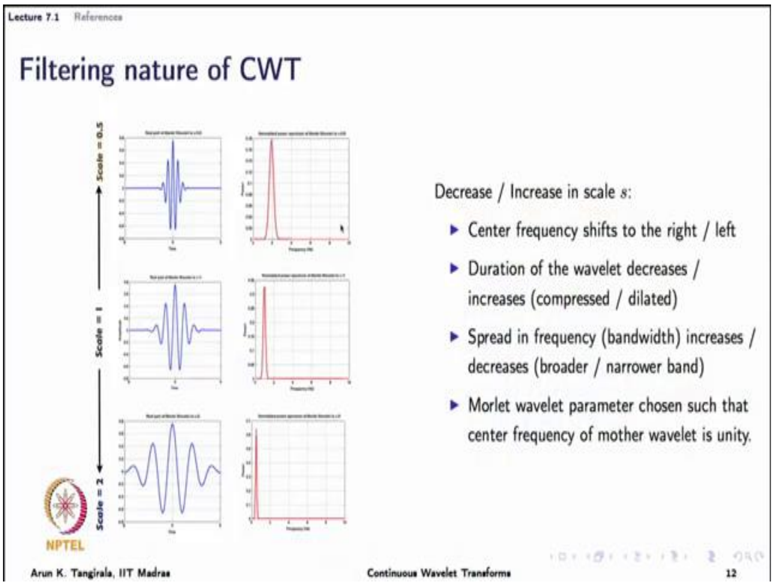

This is something that you have seen before. Again this is an illustration of the point that figures mention. At the center we have this reference function that we have been speaking off mother wave, whose magnitude of the Fourier transform is shown here. This is a representative of the energy, spectral density of the mother wave in a frequency domain, and it has some finite bandwidth. When I stretch that is when I look at scaling, when I use the scaling values greater than one. in particularly I use scale value two here, then it the mother wave; obviously, becomes wider. So, the child is wider, but the bandwidth is narrower. What this means is, if I analyze the signal with this child with this dilated way, then I will miss out on the local features in time, but I will get very nice localization of the energy in the frequency. In which frequency band, in exactly this frequency band to the left of the reference line that I have mother wave; that is the center frequency of this dilated one, is to the left of the center frequency of the mother wave.

Obviously, because center frequency of the child is f naught by s; that is the center frequency of the mother wave by s, and same argument now applies to the compressed wavelet as well. When I use a compressed wavelet, then I am able to capture the local features in time, but then I will have more smearing of the energy in frequency, relative to that of the mother wave. So, everywhere there is a reference function that we are looking at. So, essentially what we are doing is, if you think of mother standing at this f naught in frequency domain, then to the right it has children, which act like high pass filters, and the left it has children which act like low pass filters. And this partitioning of the frequency into the high and low frequency regime, understanding of this is very important, later on when we talk of, as concept of a scaling function; of course, in a different lecture.

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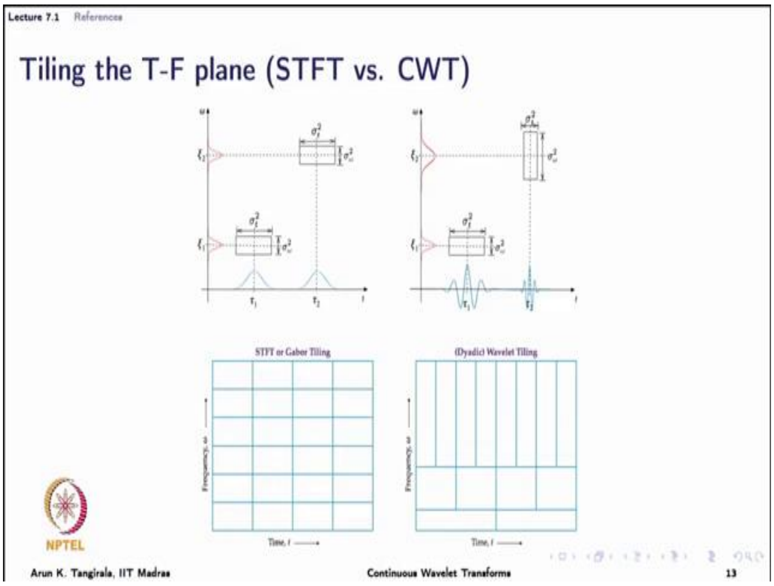

And I would like to conclude this lecture with the comparison of the short time Fourier transform versus the continuous wavelet transform. So, we have seen this figure earlier, the left figure in a contest of short term Fourier transform is to how, the bandwidth remains same across all frequencies, for the short term Fourier transform. Now for the continuous wavelet transform in case what is happening is, by design; that is a beauty here, by design when I choose a wide wavelet, its bandwidth as you have seen. Here is restricted to the low frequencies and it has a very narrow bandwidth. So, come back here and see, that the wide wavelet always concern itself to the low frequency components. And a compress wavelet is always worried about the high frequency components.

Whereas, in short term Fourier transform, the window width fixed, and I travels along the entire frequency access using the same window in other words what is happening is, that I search for all frequency components, with this window in short term Fourier transform. Whereas, with the continuous wavelet transform the movement I choose a particular wavelet at this scale, it automatically figures out what frequency components you should search for in the signal, and that again is enable by this single scaling parameter s.

So, in effect what I am doing is, I am controlling, or I am travelling along the entire time frequency plain, using this single scaling parameter. Of course, a translation parameter to travels along the length of the signal. That time together, the filtering and the time localization. Filtering in frequency, frequency localization and time localization is missing in short term Fourier transform. The tiling here shown at the bottom, explains the same point that have been making until now. The short term Fourier transform, is equivalent to passing the signal through a bunch of band pass filter which have same bandwidth, regardless of where you are standing in the frequency access. Whereas, with wavelets, here I am showing for what is known as a dyadic wavelet tiling, you will understand this when we talk about the d w t, but for all dyadic means powers of two, but I am only showing you how as you change scales in powers of two, the frequency and the time localization is vary. When you are looking at low frequencies, you are using wide windows, and when you want time localization, nice time localization when you are working in the high frequency regime and vice versa. So, as you move up the high frequencies the nature of the tiles changes, so it rearranges itself according to the duration bandwidth principle. Both short term Fourier transform and wavelet transform respective duration bandwidth principle, but it is the knob or the handle that you are using which makes the big difference.

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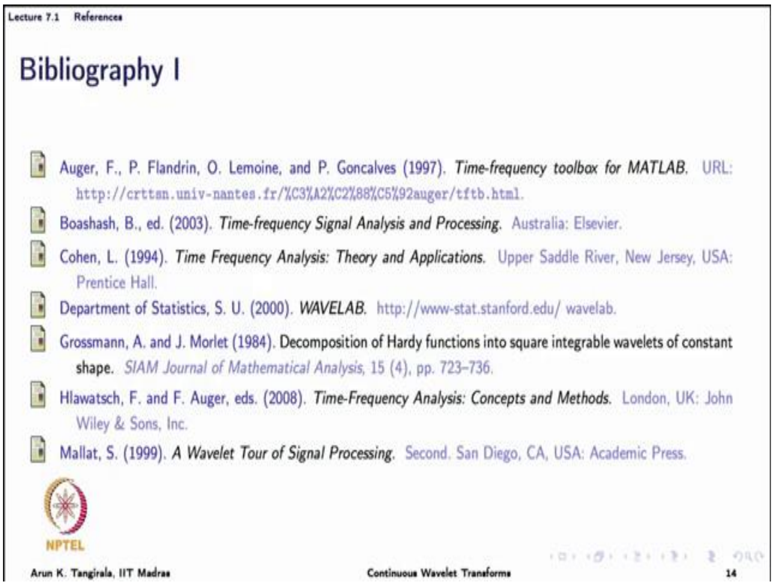

With this we come to a close. Of course, this section has been fairly theoretical, but it is very important to understand, the points that we have made in this lecture, so as to be able to move forward. In the next lecture we will learn what is known as a Scalogram, and also look at the few mat labs based examples on, how to generate wavelets, and how to compute the continuous wavelet transform, and how to compute the Scalogram, and will; of course, look at this with the few examples. So, hope you have enjoyed the lecture. See you in the next lecture.

Thanks.