Introduction to Time-Frequency Analysis and Wavelet Transforms
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Lecture – 7.1
Continuous Wavelet Transforms
Part 1/2

Hello friends, welcome to the first lecture in the unit seven on continuous wavelet transforms. In unit six, we looked at Wigner Ville distributions quite a bit in detail and in the unit prior to that we looked at short-time Fourier transforms. Now, we move to one of the central topics of the course, which is the continuous wavelet transforms. Of course, you should keep in mind the last lecture.

In the previous unit where we said, that spectrogram, scalogram and the Wigner Ville distributions are a part of this long chain and we have studied this spectrogram and Wigner Ville distribution, now it is time to study the scalogram. In this lecture, however, we shall not study the scalogram,,, but we will build the foundations for arriving at the scalogram by studying what is known as continuous wavelet transforms. Of course, wavelet transforms are very popular,,, but what we will do is, we will slowly build theory and I will give you an introduction to the continuous wavelet transforms in a manner that is not seen in many text books.

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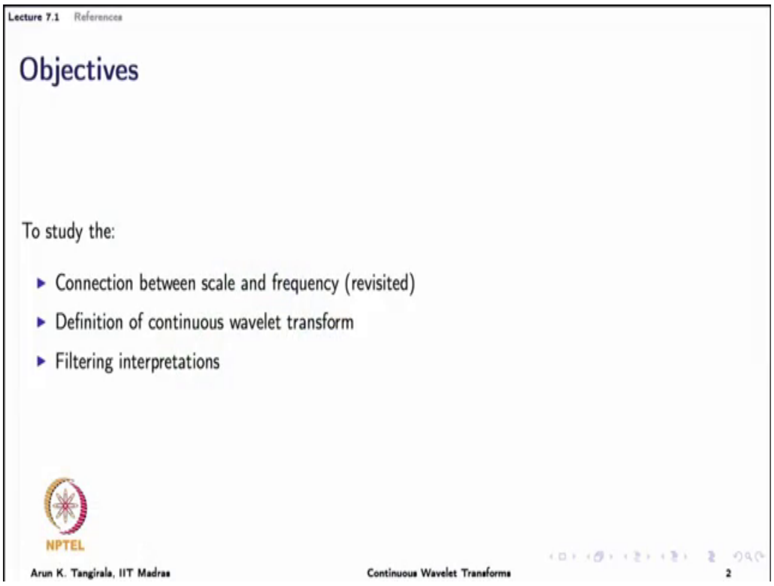

Typically, one would see the transforms equation first and then the synthesis equation or the reconstruction equation. But if you recall, when we reviewed Fourier transforms and even short-time Fourier transforms, we had mentioned, that it is important to begin any transforms with the synthesis equation first. Therefore, that is the strategy that we shall adapt and provide an intuitive way of arriving at the synthesis equation and then write the expression for the continuous wavelet transforms.

But before we do that you should keep in mind that wavelet transforms work in the so called time-scale plane unlike the short-time Fourier transform and the Wigner Ville, which operate in the time frequency plane. But we can still bring the wavelet transforms back into the time frequency plane by studying the connections between scale and frequency and that is what we shall begin with today, and then move on to studying the synthesis equation, learn the definition of continuous wavelet transforms and conclude the talk with filtering interpretation of the CWT.

The talk is going to be a bit heavy on a mathematical side. So, you may have to read and watch this lecture carefully and probably listen to it more than once.

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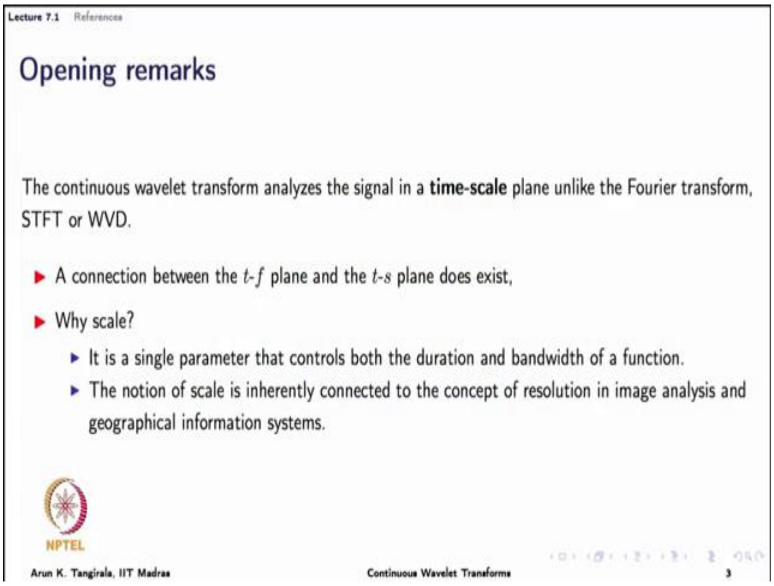

So, as we have said just now, that wavelet operates in the time-scale plane,. So, we are moving from the time frequency plan to the time scale plan when it comes to wavelet analysis,,, but fortunately we have a connection between scale and frequency. Therefore, we should be able to switch back and fourth between these two domains. However, an important question, that arises is, why should I even consider this. So-called scale, what is this scale, because we have looked at the scale briefly when we were studying the properties of Fourier transform. We know, scaling corresponds to stretching and compressing and so on,,, but why should I look at scale?

Now, when you look at signal analysis recall, for example, a short-time Fourier transform, what we are interested in is the localization of the energy in a time frequency plane. And in short-time Fourier transform, we achieve this by slicing the signal using a window function and then, we move along the frequency axis, as we do in the Fourier analysis.

Now, unfortunately what happens, as you have seen in short-time Fourier transform is, once we decide on a window length, then we are not going to change it during the entire course of analysis unless we choose another window of different length. So, once we choose a certain window length, the duration bandwidth principle tells me, that there is a limitation on my ability to resolve or localized energy frequency domain. So, if you choose a wide window, then in time, of course,,, then I will be able to localize energy nicely in frequency. And if I choose a narrow window. So, as to be able to capture short lived features in the signal, then the energy localization in frequency becomes very broad.

So, what happens in short-time Fourier transform is, a lot of book keeping is necessary. I have to try windows of different lengths and capture different features of the signals. However, in a lot of realistic situation, the short-lived features are high frequency components, that is, they have high frequency content and the long lived features, which persist throughout or typically, the low frequency components of a signal. Of course, this is not the scenario all the time,,, but a lot of time this is the situation, then it will be nice to be able to adjust the window, that is, the duration of the window and the bandwidth as I move along the time and frequency plane. The scale parameter allows me to do that.

We have seen this when we were studying the properties of Fourier transform. We have seen, when I stretch a particular signal, then its duration is increased,,, but simultaneously the bandwidth decreases. And when I compress the signal, then the duration decreases,,, but the bandwidth increases. So, if I can have a handle, a kind of a knob, that will adjust both the duration and bandwidth. So, as to suit signals, which have shortly high frequency components and long lived low frequency components, that would be really great and that knob is a scale, which is missing in short-time Fourier transform.

In short-time Fourier transform, you will really have to be play around with two knobs where you adjust the window length and then, you go ahead and determine the frequency content and again try another window length and determine the frequency content and so on. So, you have to play around with two knobs, whereas, here we have a single knob, which is a scale parameter that will allow me to adjust the duration and bandwidth simultaneously.

In the language of signal processing, particularly filtering, we have seen, that the short-time Fourier transforms for a fixed window length acts as a band pass filter with fixed bandwidth throughout the time frequency plane. We shall realize shortly, that the introduction of scaling parameter I will be able to generate a series of band-pass filter, which will span the entire time frequency plane,,, but with a very nice feature. As I move in the high frequency regime, the bandwidth increases, allowing me to use narrow windows. And as I move to low frequency regimes, the bandwidth decreases allowing me to use wide windows and so on. So, this is probably the core message or one of the core messages of continuous wavelet transforms, that is, as far as signal analysis is concerned.

If you look at image analysis, there is no mention of the word frequency,,, but rather you hear the term resolution. We talk about image resolutions and finer resolutions and coarser resolutions and so on. Likewise, we can talk of coarser scale and finer scale and so on and a classic example is that of geographical map. Even if you were to recall the use of Google maps and so on, or any other map from any other software provider, you will find, that as you zoom in, you get more details. So, you are actually going at, going at, looking at a map at finer scales and when you zoom out, you get a coarser picture. So, then we say, that the map has been drawn at a lower resolution or a coarser resolution. So, there we are talking about the connections between scale and resolution.

Again, we have seen this earlier scale and frequency have an inverse relation and so, do scale and resolution, right. If I, lower scale would mean very fine resolution and higher scales would mean poorer resolution; that means, coarser images.

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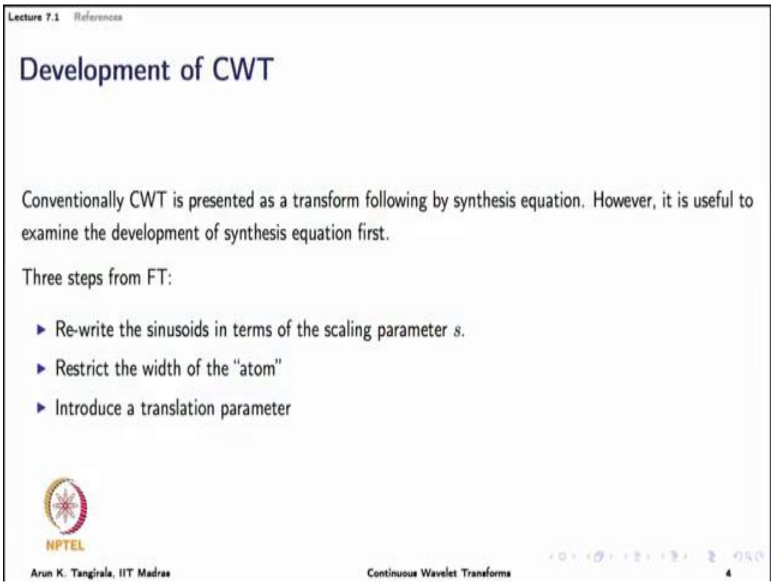

Primarily, in this course we have been focusing on signal analysis. So, therefore, , we will talk about the relation between scale and frequency. Before we look at that let me tell you also again a point that I made earlier. Typically, when you look up texts on continuous wavelet transforms, you will see the wavelet transform expression provided upfront,,, but what I would like to do is present this synthesis equation first and part of this material is being inspired by, is inspired by the material, by the book or the presentation in the book by the ((Refer Time: 09:39)) on time frequency analysis and therefore, you can also look up the particular chapter in the text to get more details.

So, what we shall do is we will begin with the Fourier transform and rewrite the sinusoid symptoms of the scaling parameter. So, that I see the connection between scale and frequency straightaway and then, we will restrict the width of the atom. We call this as atom now because it need not be a basis function necessarily. Atom essentially means, the analyzing function. And then, introduce a translation parameter because I am going to restrict the width of the analyzing atom, I should be able to traverse along the length of the signal. So, we will develop CWT in a stepwise,,, but intuitive fashion.

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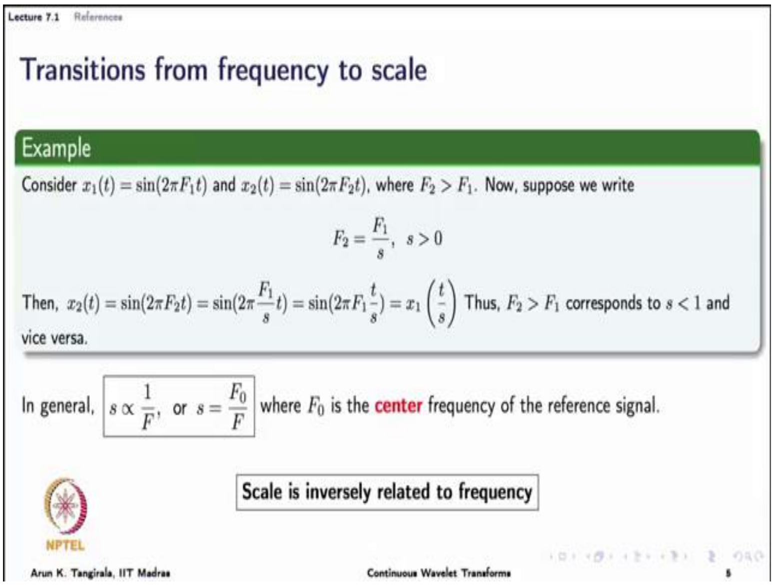

So, the first step, as we have just noted, is to observe the connection between scale and frequency. So, consider these two basis functions or the complex sinusoid that we used in Fourier analysis, which are sines of two different frequencies, all are chosen sines, the discussion pretty much applies to cosines as well. So, there is no loss of generality here.

Assume, that F 2 is greater than F 1. And now, what we shall do is, we shall rewrite F 2 as F 1 over s. See, there are two ways of looking at F 2 being greater than F 1, I can say, F 2 is F 1 plus delta F. So, that is an additive relation. Here, I am introducing a multiplicative relation between F 2 and F 1, that is the prime difference between what we have been seeing earlier until now and what we have seen at this moment. So, we write F 2 as F 1 over s, which is kind of a multiplicative relation,. So, that if s is less than 1, then F 2 is greater than F 1. And since in this example F 2 is greater than F 1, s is indeed less than 1. We shall assume for the purpose of discussion, that s is greater than 0.

Then, what happens is, I can rewrite the second signal, which is sine 2 pi F 2 t as sine 2 pi F 1 over s times t. And the trick here is to shift this 1 over s on F 1 onto the time variable. So, that I rewrite this as sine 2 pi F 1 t over s as a result of which I immediately see this, nothing,, but x 1 evaluated at t over s.

So, very easily I am able to see, that two sine waves of different frequencies are nothing,, but scale versions of each other that is the prime message that you should take from this example. Until now, we have been now looking at these two sines as sines of two different frequencies separated by additive relation.

Now, we are looking at a scaling relation. Clearly, when s is less than 1, F 2 is greater than F 1, which means, whenever the scaling we call says, the scaling parameter, whenever the scaling parameter takes on values less than 1, I am generating sinusoids of higher frequencies with respect to F 1.

So, now we need a reference point that is another second key point that you should keep in mind. One, we have introduced a multiplicative relation; two, we are looking at a reference signal now. If I keep the reference signal as F 1, then F 2 is a scale that is a sinusoid of frequency. F 2 is a scaled version of x 1 with this scaling parameter taking on values less than 1. Of course, it follows, that if I choose the scaling parameter greater than 1, then I generate scaled version of this x 1, which have frequencies lower than F 1. So, F 1 acts a reference. In general, if I have any reference signal as F naught, then any other sinusoid of a frequency different from F naught will be a scaled version of this sinusoid with frequency f naught, the scaling parameter, a parameter being F naught over F.

Now, we can say, that the scale and frequency share an inverse relation. So, the simple example really brings out the connection between the scale and frequency in a nice way. Now, if you further generalize this idea where we are not dealing with sinusoids, I am dealing with some arbitrary function and I am looking at the scaled version of that function, then this scale version of this function will be related to the frequency. Now,,, but we are talking about center frequency. So, the reference function has a center frequency f naught and now you can say, that the scaled version has the center frequency f. So, you can see, that the center frequency of the scale version is center frequency of the original function, that is, we call as a mother wave in a wavelet analysis divided by s.

So, the key is now, that there is a center frequency scale and frequency share an inverse relation,,, but this proportionality constant depends on the reference signal that you choose, that is the main point, ok.

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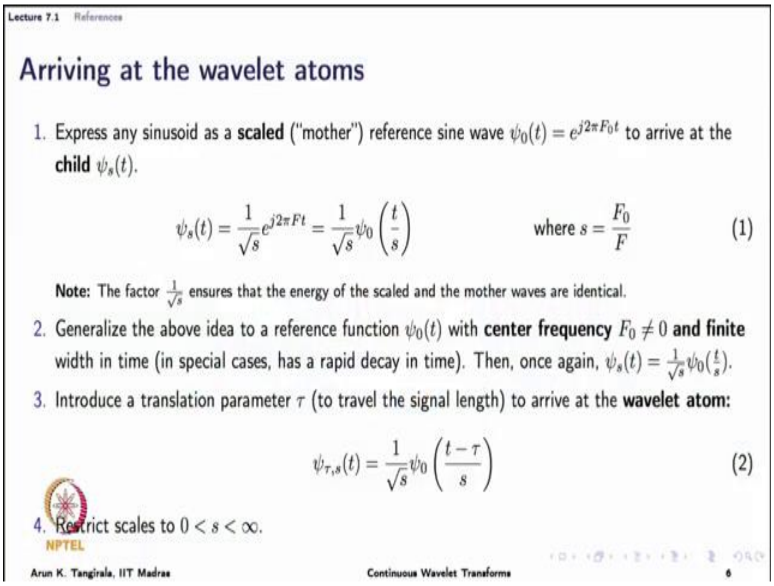

Now, from the previous example we have some idea of how to construct the wavelet atom. We will start with the sinusoid. So, let us say, I have a sine wave of frequency F naught. I am going to scale it first and this, we call this scaled function as a child and the reference signal as a mother sine, we denote, as is a convention, we will denote the scale with one subscript s.

Now, we have 1 over root s coming in a, appearing in front of this e to the j 2 pi F t. This 1 over root s is essentially to preserve the energy of the signal, that is, the energy of the child signal or the scaled signal should be same as the energy of the reference or the mother signal. Why does this 1 over root s appear, I will just show that to you on the board.

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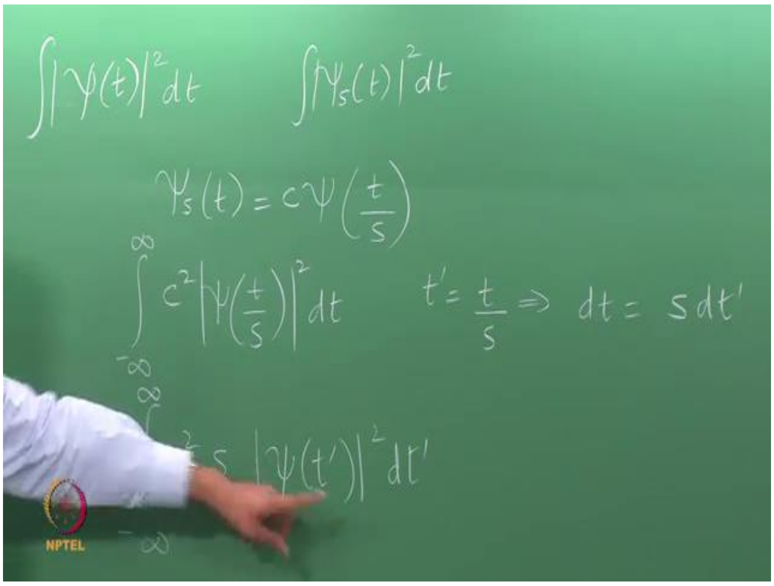

Suppose I look at psi t, which is the reference signal, then its energy by definition, if you recall unit 2, we had given definition for energy that is the energy of the mother wave or mother function. And if I look at this scaled signal, its energy is similarly given by this expression. We want both to be identical, right. We know, by definition, this scaled function of the child is simply related to the mother function through this relation.

Now, suppose I assume, that there is a constant C that I have to introduce. So, that the energy of the child function should be the same as the energy of the mother function. Then, let us substitute this in expression for the energy of the child there and I have C square psi t of s mod square d t. And of course,,, here the limits run from minus infinity to infinity because we are doing a theoretical analysis.

Now, I can introduce a dummy variable t prime, which is t over s. So, that I can see clearly, that d t is s times d t prime. As a result, I can write this integral as minus infinity to infinity c square. I have s square, sorry, s time psi of t prime square d t prime. Now, I want, this t prime is a dummy variable,. So, you should not get carried away with this dummy variable. We want this expression to look exactly like this to work out to this expression. Therefore, I choose C square as 1 over s, C is 1 over root s. Since we are working with we assume, that we work with positive value of s, it suffices to have 1 over root s otherwise you would have 1 over root modulus s. So, this shows why this one over root s is necessary in front of in the expression for the child function.

Now, having cleared that point, now we generalize this idea to a, to a reference function, some reference function to center frequencies f naught. So, now, we started from sine wave and moving onto functions, which have center frequency F naught. So, therefore, they are not perfectly localized in frequency, they are localized in frequency,,, but not perfectly like the sinusoid and this localization of energy is centered around F naught, which is not equal to 0.

Further, we shall demand, that this reference function have finite width in time. Now, that is because we want to do, we want to get the local features of this signal in time. So, I am choosing a mother function, that that does not look like a sine wave,,, but as a finite width in time. In some special cases, this mother function may have this characteristic, that it does not have a finite width,,, but it rapidly decays in time like a, it is like a Gaussian signal and. So, on. Modulate waves and Mexican ((Refer Time: 19:38)) waves have these characteristics.

Regardless of this, again we have the child function of this scaled function as 1 over root s psi naught of t over s. So, in step 1, we have sine wave; in step 2, we have a general function of finite width, that with center frequency F naught, which is not equal to 0.

Now, because this new function, that I am looking at and its children have all finite width, I need a translation parameter. So, as to analyze the full length of the signal. In the case of Fourier analysis, I did not need these translation parameters because the sine wave exists for infinitely in time. So, I do not have to worry about moving the analyzing function across the length of the signal. But now, the analyzing function is of finite width, therefore, I have to move this manually like I do in short-time Fourier transform and that is achieved by introducing this translation parameter tau. As a result, I have now the. So, called wavelet atom,. So, to speak. We have psi subscript (tau, s).

Until step 2, we used only the scaling parameter of the subscripts,,, but we have introduced the translation parameter, which will allow me to move, travel along the length of a signal. So, I have now 1 over root s psi naught of t minus tau by s and as we have already mentioned we will restrict ourselves to positive scales and finite scales.

So, started with the sine wave, generated scale versions of it, though scale versions are no different from the sinusoids that we have used in Fourier analysis,,, but the two prime changes, that we have made are one to generalize that idea to a, to a function, which has center frequency f naught. It is not perfect sine wave and that it has a finite width, right.