Introduction to Time-Frequency Analysis and Wavelet Transforms
Prof. Arun K. Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture – 6.7
Cohen's class and Ambiguity Functions

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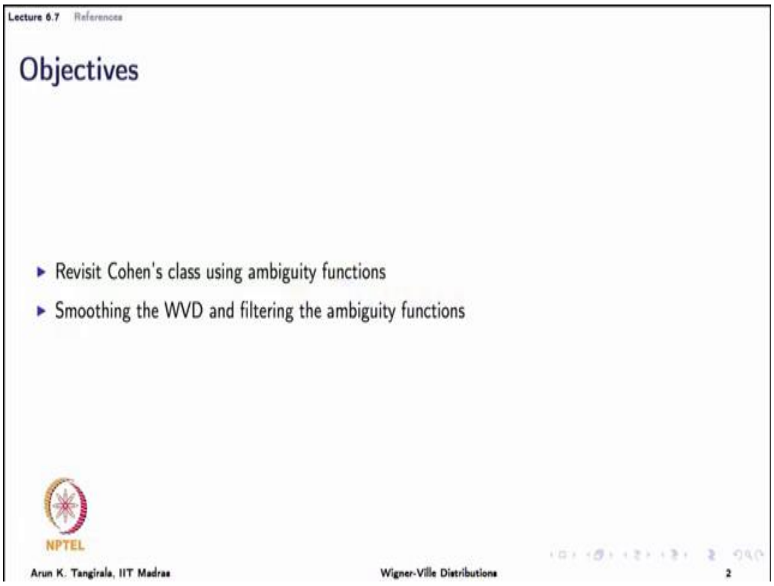

Welcome to lecture six point seven. This is actually a follow up lecture of lecture six point six where we introduced Cohen's class of distributions. What we are going to do in this lecture is to revisit Cohen's class using ambiguity functions. We talked about this,, but now it is time to look at the mathematical definition and essentially, we will ask how this smoothing with the Wigner-Ville distribution, what it means in the ambiguity domain. In fact, now we are going to work in the nu, s that is Doppler and delay domain instead of the time-frequency plane, that is the central idea here. And we will show, that is, smoothing the Wigner-Ville is nothing, but in fact, waive waiting it should read the filtering as actually waiting the ambiguity function appropriately.

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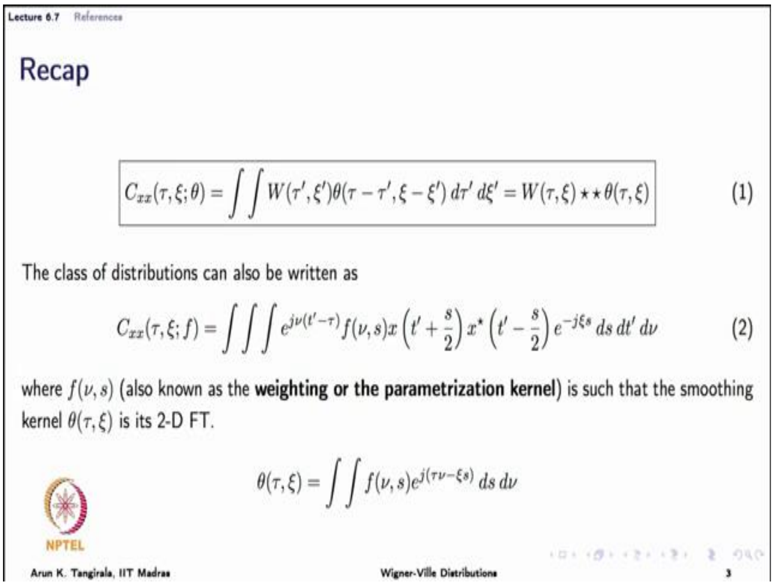

Just to recap the Cohen's class is, either a convolution in the time-frequency plane as I have given in equation 1 here or evaluating this, generalize Wigner-Ville distribution with the help of this weighting or the parametrization kernel f and this f is a function of (nu, s). And we said earlier, nu is the Doppler and s is the shift or the delay.

And what we are going to do in a next few slides is talk about this nu, s plane, where we will introduce the ambiguity function. And as usual, as we had before the relation between the smoothing kernel in time-frequency and the weighting kernel is through this two-dimensional Fourier transform where it is inverse with respect to one-dimensional and forward with respect to another dimension.

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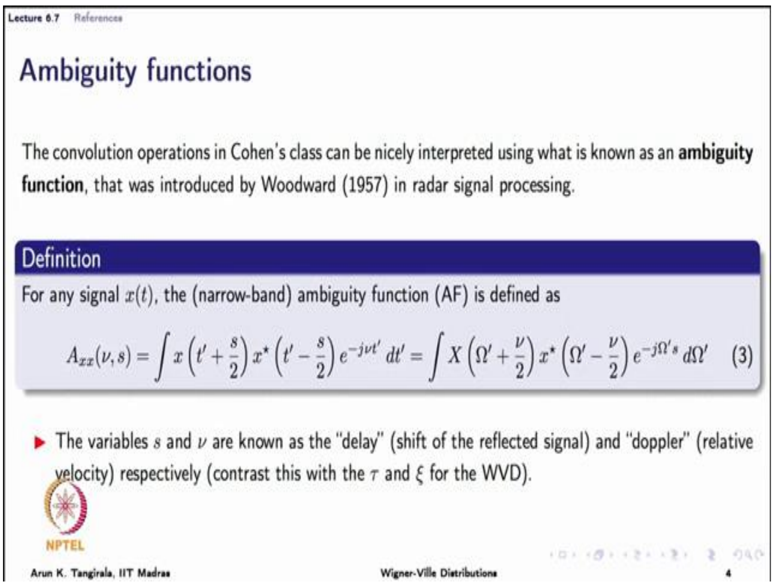

So, ambiguity function is a function that measures the time-frequency correlation. It was introduced long ago in radar signal processing by Woodward and is used extensively in both radar and sonar signal processing. It was pretty much like the Wigner-Ville gives a lot of information about the time-frequency correlation, that is, how the correlation, how the two signals are correlated in the time-frequency plane.

Now, the way it was introduced, it was completely different from the way Wigner-Ville was introduced. The context is radar signal processing where one is interested in knowing the delay in, determining the delay between the transmitted signal and the received reflected signal. So, there is a radar, which is trying to detect the location and the speed of the target.

It does this by transmitting a certain signal at a certain frequency and then it, there is a receiver seating in the radar, which receives the reflected signal. Once the signal hits the target, it is reflected back and assuming under ideal conditions, that is, there is no loss of the velocity and. So, on due to the medium, which is typically air in radar signal processing, then what should happen is, that the reflected one should be essentially a time shifted version of the transmitted signal, right. And in addition to that there will be a Doppler effect and I presume, that you know what is Doppler effect. It is the perceived frequency of a moving object, right. When a, when an object moves at a certain frequency, it is at a certain velocity, then you perceive the frequency in a different manner. So, this, this frequency depends on the velocity itself, that is, a classic example is the train coming on the platform and then leaving, that is a classic example of phenomenon where Doppler effect is experienced.

So, in radar signal processing the objective is to determine the location and a velocity and this is location is determined by first calculating the delay. Both are actually calculated, determined by calculating what is known as a delay, that is, a round trip delay and the. So, called Doppler, which is called the nu, that is, the Doppler is a relative velocity, relative to the speed of that frequency of that signal in the medium in which it is traveling.

So, the ambiguity function is essentially a correlation between the signal and of course,, a reflected version of itself. You can define ambiguity function between two different signals because we are talking of radar signal processing. What we are doing is, we are shifting. You can think of this x of t plus tau by 2 times x star of t minus tau by 2. In the same way, as we wrote in the previous lecture, can be written as x of t prime times x of t prime. Here, I have s here,. So, x of t prime time x star of t prime minus s. So, x star of t prime minus s is a, is a shifted, time shifted version and the star is essentially to care take care of the analytical representation of the signals.

So, you are looking at this. In this time-frequency plane how does this correlation change in the frequency, in the time-frequency plane and therefore, to evaluate that you take a two-dimension, you take in single integral with respect to the. So, called Doppler, right and that is what you have here. And ambiguity function is the function of this Doppler nu and the shift s.

The reasons for the name ambiguity is ideally, when the shifts match exactly. Then, you want the ambiguity function to be one and where the shifts do not match, that is, between the reflected and the transmitted signal, then you want it to be 0. Unfortunately, that is not going to be the case. The match between the shifted and the shifted, reflected and the transmitted signal is going to be non zero even though it, it does not exactly match the delay. And therefore, there is some ambiguity and loosely people have called it as ambiguity function,, but otherwise there is no ambiguity about the function value itself. So, you should be clear about that.

Coming now to the math, the ambiguity function is this, a integral here. It is single integral of the, it is a Fourier transform of some kind of an instantaneous or correlation unlike the Wigner-Ville where the integrating variable is s, here the integrating variable is t prime. So, that is a prime difference. You can rewrite this ambiguity function in terms of the Fourier transforms of x and x star in the fashion that I have given here where the frequency variable, now the dummy frequency variable is omega prime. Here, the dummy time variable is the t time prime.

As I mentioned earlier, s and nu are known as the delay and Doppler and you can read up any text book on radar signal processing, which will give you more insides into the historical development of this ambiguity function.

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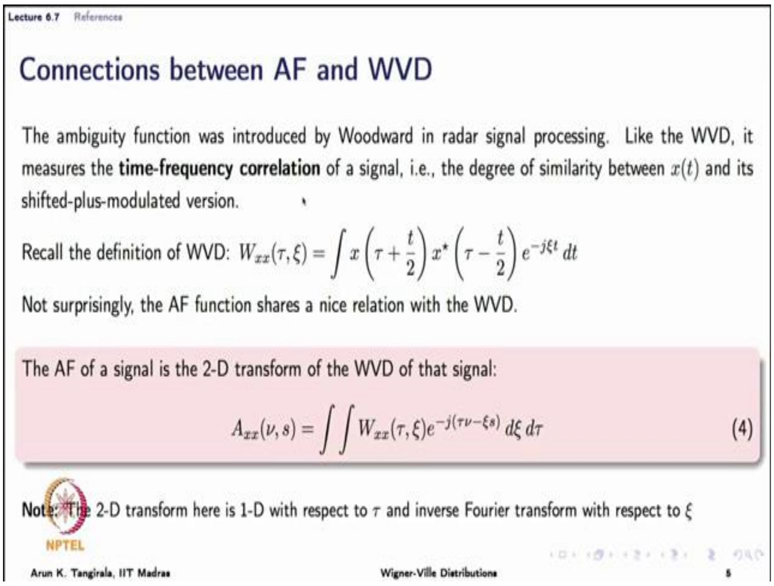

The prime purpose of introducing this ambiguity function is to reinterpret Cohen's class, as I mentioned early on in this lecture, and it gives us a fresh perspective. To compare the definition of ambiguity function with the Wigner-Ville distribution, I have given you the definition of Wigner-Ville. As I mentioned earlier, when I evaluate the Wigner-Ville, the integrating variable here is t, whereas, when I evaluate the ambiguity function, the integrating variable is the t prime. That means, if I were to write the Wigner-Ville for exactly the same notation here, I would integrate with respect to s. Therefore, you should expect some parallel between Wigner-Ville and the ambiguity function. In fact, that is what is the case.

The ambiguity function of a signal is the two-dimensional, again Fourier transform of the Wigner-Ville. These two-dimensional Fourier transform, again is the same case, that we mentioned earlier, inverse with respect to one and forward with respect to another variable. So, this is the relation, formal relation between the ambiguity function and the Wigner-Ville. Now, you can start relating this to what we mentioned in the previous lecture. The Cohen's class is a smooth Wigner-Ville or the convolution of the Wigner-Ville with the smoothing kernel theta. And I can write the smoothing operation in the Fourier domain as or the convolution operation in one domain as the product operation in the dual domain and that is exactly what we are going to do now.

We have already determined what is the Fourier transform of Wigner-Ville, which is ambiguity function and we know already, that the Fourier transform, two-dimensional Fourier transform of theta will give me f. So, you should expect now, that the Cohen's class can be written as a product of the kernel f and the ambiguity function. Then, of course,, an inverse Fourier transform of that will get, will get me the Cohen's class, it is not exactly the Cohen's class, right. So, that is the basic point here.

And also, I should caution you, there are certain texts, which will define ambiguity function in a slightly different manner and you have to watch out for. Then, all these expressions changing accordingly. And one change that you will find in a few text books is, instead of the negative sign here you would find a positive sign and that changes the way you write this transform. So, everything else has to be consistent. So, when you are reading a text book watch out for these differences. But I have used conventional definition of the ambiguity function and also notice, that we have switched the order, the ordering of variables in Wigner-Ville and ambiguity function in Wigner-Ville.

The first variable tau as units of time, t units of time and zee has units of frequency. There is ambiguity function as the variables ordered, where nu is the velocity Doppler velocity and s is the, s has units of time t.

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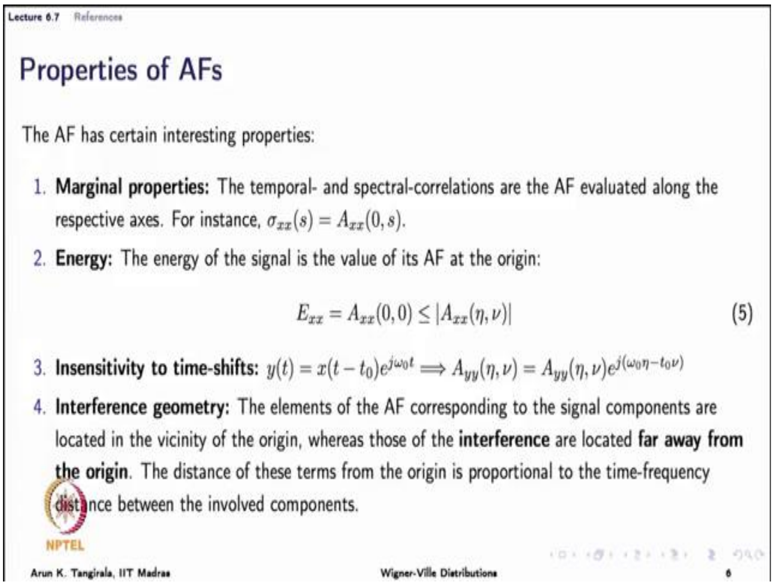

Before we proceed to reinterpret Cohen's class using ambiguity functions, it is interesting to note the certain properties of ambiguity functions, primarily because it helps us understand how interferences are removed in the ambiguity plane. We know already, smoothing in the Wigner-Ville removes interferences because here essentially performing some kind of averaging in time and frequency

When we rewrite Cohen's class in terms of the ambiguity functions, we will notice something very interesting about the interferences and that is related to one of the properties that I have listed here.

The first property is that of the ambiguity function is that when I evaluate the ambiguity function along one-dimension, I recover the respective correlation. For example, if I want to recover, let us say, the auto correlation of the signal, then all I have to do is, I have to evaluate the ambiguity function along the s axis itself. That means, I set the Doppler to 0. These are called ambiguity functions with Doppler cuts and shift cuts or delay cuts. There is a Mat lab function also in one of the tool boxes, not in the regular tool boxes, which commercial tool boxes, which computes the ambiguity function and there you will see the term called Doppler cuts and delay cuts and. So, on, I think, it is called phase array tool box or something like that. I will, I can give you the exact answer in the forum.

So, likewise, when I evaluate the ambiguity function along the Doppler axis, then I recover the spectral auto correlation, right, of the signal, that is, auto correlation in the frequency domain. In fact, that is something else that is not listed here I will talk about, let me just finish. There is another property that is interesting.

Once I say finish discussing this properties, the energy of the signal, the interesting fact is, energy of the signal is the value of the ambiguity function at the origin. This is unlike what you see in Wigner-Ville, right. Wigner-Ville preserves the energy you have to integrate along the entire ((Refer Time: 12:39)) plane to recover the energy. Here, I have to evaluate the ambiguity function at a point, then that point is the origin and that exactly ((Refer Time: 12:48)), which means now you should expect the energy of the signal to be concentrated around the origin. That is one of the key things that you should start noticing right and

Obviously, it reaches the maximum at the origin compared to the ambiguity function reaches a maximum and origin, compare to its magnitude at all other points in the nu s plane. There is a small mistake here, it should read as nu, s and will correct at in both this expression in equation 5 as well as one below.

The ambiguity function, unlike the Wigner-Ville is insensitive to time shifts and therefore, the magnitude is useful in radar signal processing. In Wigner-Ville, we wanted time and frequency to be sensitive to time and frequency shifts,, but the ambiguity function is not. In general, ambiguity function is a complex value dump quantity

When we say it is insensitive, the magnitude is insensitive. As you can see from the expression here, y is a time shifted and frequency modulated version of x and the ambiguity function of y is, again here it should be ambiguity function of x times the e to the j something. When I take the magnitudes on both sides they should match, which means, it is insensitive to, insensitive to time and frequency with time.

And now, coming to the most important one. Well, one important point was the energy, that energy of the signal is concentrated at the origin. Therefore, you should expect the signal components to be concentrated around the origin in the ambiguity function plane. The interferences, how do they map onto the ambiguity function plane or the nu, s plane, it turns out be, that the interference geometry unlike in Wigner-Ville, they are located far away from the origin in the Wigner-Ville plane, that is, the tau, z time-frequency plane. The interference terms occurred exactly midway in the time and frequency based on the time and frequency centers of the signal components. Now, the interferences are going to be located far away from the origin. Let me explain that to you with the help of an example.

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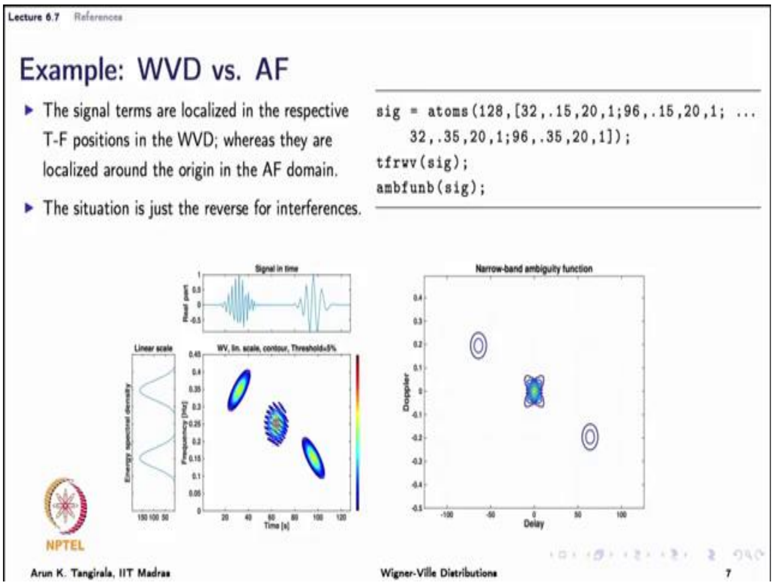

So, here I have for comparison the Wigner-Ville and ambiguity function on a signal that we have seen earlier. There are two atoms, one of high frequency and one of low frequency located at different time centers. This is the familiar Wigner-Ville we have seen earlier. They both are, because of the amplitude modulation these atoms are of finite width.

What is of important to us is the presence of this interference term. The interference term is actually located midway in the time-frequency plane with respect not in the time-frequency plane,, but midway with respect to the time and frequency centers of the atoms. Whereas, in the ambiguity function plane, that is, the Doppler and delay plane notice, that I have delay or the s along the x-axis and Doppler or nu along y-axis.

Now, for the ambiguity function. The center here in the contour that you see in the center of the delay, Doppler plane is essentially the signal components, that is, both the atoms, the signal components here, which are these two atoms have mapped to the center at the, around the origin. We should expect that because a signals energy is the value of the ambiguity function at the origin. And what you see as circles away from this origin are the interference terms. So, what has happened is the ambiguity function has taken these two atoms in the time-frequency plane and put them at the origin and taken interference term and split them far away from the signals.

So, it is exactly the reverse of what is happening of what you have seen Wigner-Ville and that is because of the dual nature of the ambiguity function and the Wigner-Ville, right. We have also seen this kind of duality between time and frequency in the classical Fourier domain. This is what is happening here. All the signal energies concentrated at the origin, around origin and the interference systems are far apart. You may ask what is the advantage. Now, that there is a big advantage because I know theoretically, this is the property of the ambiguity function.

If I want to remove interferences what should I do? A common sense approach would be to apply some kind of waiting or windowing to the ambiguity function such that the window will only pick values of the ambiguity function within the vicinity of the origin and leave out the rest and then perform an inverse Fourier transform. It will get me the Wigner-Ville and that is it. So, I would have obtained the smooth Wigner-Ville. This is the basis for reinterpreting Cohen's class using ambiguity functions.

In the smooth Wigner-Ville approach we said these interferences, that I see in the Wigner-Ville can be gotten with by performing at time and frequency averaging using a particular kernel, that now is equivalent to apply in some kind of a waiting here in the ambiguity function of the delay-Doppler plane. So, that this kernel, now in the ambiguity function plane, will only pick values, give importance to values of the ambiguity function within the vicinity of the origin and leave out the rest. And then, I perform the inverse Fourier transform, two-dimensional Fourier transform of that, I will get a smooth Wigner-Ville that is the basic idea.

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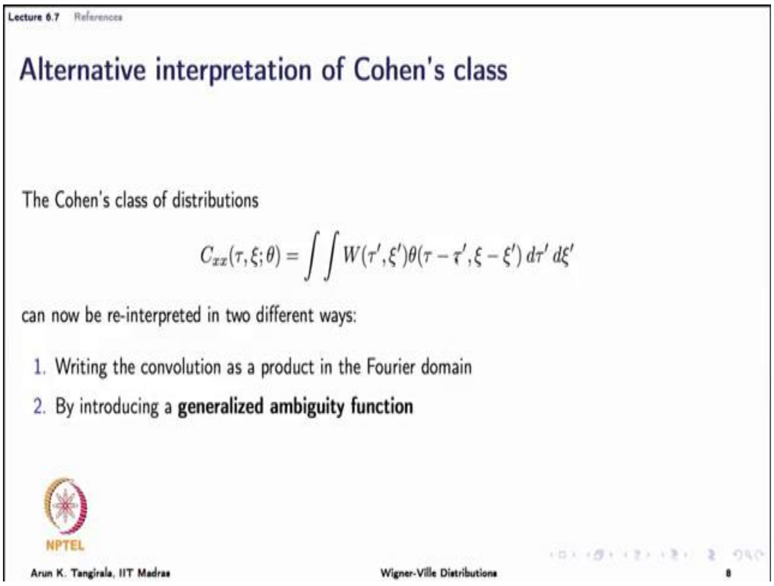

So, with this idea we can reinterpret Cohen's class in two different ways, both lead to the same interpretation. The first one is the one that we mentioned earlier in the previous lecture where I could write the convolution in the time and frequency as the product in the Fourier domain. We know, that the two-dimensional Fourier transform of Wigner-Ville is ambiguity and we also know, that the two-dimensional of theta is f, right.

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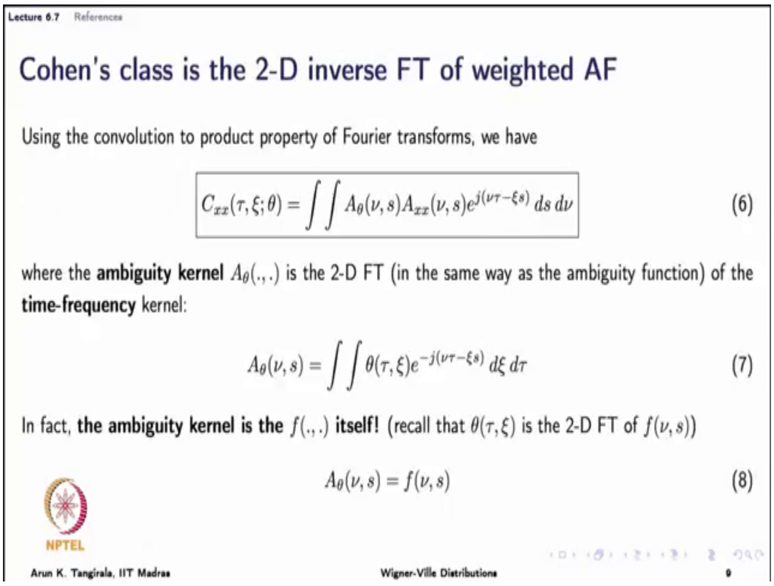

Now, all I need to do is to reinterpret. Let us look at first one and then talk about the second one. All I need to do that is to rewrite the Cohen's class as an inverse Fourier transform of the, Fourier transform of the convolution. So, the first step is Fourier, Fourier transform of the convolution. What does that fetch me? I know Fourier transform of convolution is a product of what? Fourier transforms of the respective ones, the Fourier transform of w is ambiguity function, the Fourier transform of theta is f, right, which we denote as a theta here,, but this A theta is nothing, but f itself. And then, I have to perform an inverse Fourier transform and that fetches me the Cohen's class.

So, that is a very straight forward, interpret, reinterpretation of the Cohen's class where now we call this f as the ambiguity kernel. This is just as a straight forward extension of the definition of ambiguity function itself. So, now f assumes this interpretation of A being the ambiguity kernel. In fact, that is also what is the term that is used ((Refer Time: 19:57)) book and few other text.

So, to summarize in this approach the Cohen's class is nothing, but the two-dimensional inverse Fourier transform of the weighted ambiguity function. And now, I have better feel for what role the f plays. Earlier we called this f as the weighting or the parameterization kernel. Now, it justifies, that terminology it weights the ambiguity function ((Refer Time: 20:24)). So, obviously, now I, if I want to choose f as a Gaussian one, it should have more weighting in the origin and less weighting. So, it, the values of f should fade away as you move away from the origin in the delay-Doppler plane.

Now, you also understand why we use the same variables to characterize f. We had used (nu, s) and the same variables are used to characterize A. So, all of that now becomes a lot clearer.

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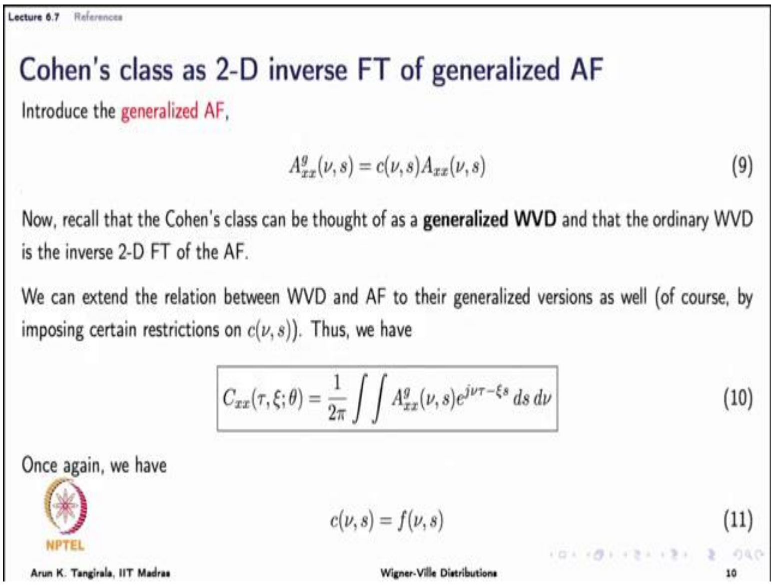

The second interpretation is to introduce what is known as a generalized ambiguity function. It is the same thing. All we are saying is, first recall, that the Cohen's class is a generalized Wigner-Ville, right. Now, if I take the Fourier transform of the Wigner-Ville, I get the ambiguity function. So, if I take the Fourier transform of a generalized Wigner-Ville, I should get generalized ambiguity. So, what I do is, I construct generalized ambiguity and take the inverse two-dimensional inverse Fourier transform. That is exactly what I am doing here.

I modify the ambiguity function with another function, two-dimensional function called C and called that as a generalized ambiguity function. This C cannot be anything arbitrary. Again, I have to impose certain restrictions on C if I want certain properties on the Cohen's class. In fact, this C is nothing, but f. You should have guessed that by now. This C, sorry for the confusion, this C and the same c, this is a small c, this small c is nothing, but the f.

Now, the Cohen's class is nothing, but the inverse two-dimensional Fourier transform of this generalized ambiguity function. So, Wigner-Ville will give me ambiguity function, that is, they are duals of each other in a two-dimensional Fourier sense, generalized Wigner-Ville. And generalized ambiguity functions are also duals of each other in the sense of two-dimensional Fourier transform. So, this small c that we have introduced to modify the ambiguity function is nothing, but the kernel f itself. So, again this is another interpretation,, but it is more or less is same interpretation that we have acquired earlier, which is that f plays a role of a weighting on the ambiguity function.

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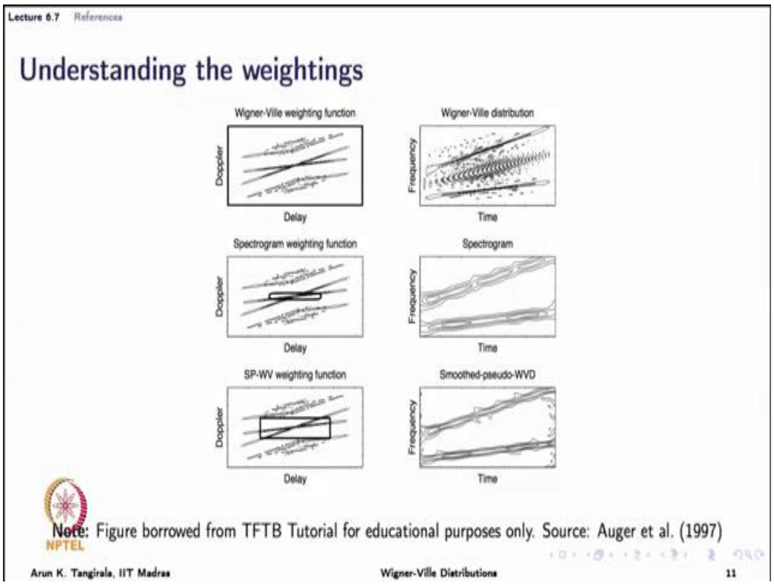

Now, just to give you a feel of how things look like, I have borrowed this figure from the time-frequency tool box tutorial, which gives you a fantastic overview of all the time-frequency tools that you have.

These figures are generated for three different distributions that we are familiar with: the Wigner-Ville, the smooth pseudo, the spectrogram and the smooth pseudo Wigner-Ville. On the left, sorry, on the left what you have? They are the weighting functions, that is, the values of f in the delay-Doppler plane and on the right you have the smooth Wigner-Ville.

So, what you can get from this is, let us start with Wigner-Ville. We know that to generate the Wigner-Ville, f has to be one along the Doppler plane in the delay-Doppler plane. We have seen that earlier. And what this, what is shown here is not f alone, it, what is also shown is the Wigner-Ville of some signal, right. The middle term here are the interferences, sorry, the middle, what, what is shown here all though it is, says, Wigner-Ville is weighting function, what is shown here is ambiguity function.

The middle one is the signals components and the far off ones are the interferences. We have seen that, right, because we are working in the delay-Doppler plane, we are looking at the ambiguity function. Wigner-Ville does not perform any weighting on the ambiguity function. Therefore, there is no modification done simply and inverse transform of this ambiguity function will recover the Wigner-Ville on the right. And you see, this presence of huge interferences and these two dark lines are the signals components ((Refer Time: 24:26)). Do not worry about what is the signal, right now you can imagine what the signal. Signal has essentially two kinds of chirps,, but there are interferences there. They are located differently.

Now, the spectrogram performs some kind of weighting on the ambiguity function. Recall, we said earlier in the previous lecture, that the weighting or the values of f that will generate the spectrogram is the ambiguity function of the window, that I use in the spectrogram itself. Remember, spectrogram is generated from short-time Fourier transform. There is the window function.

We said, in terms of smooth Wigner-Ville the kernel is the Wigner-Ville of the window in terms of the, in the ambiguity function plane. We say, that the weighting kernel, which is the f is nothing, but the ambiguity function of the window itself. So, that is exactly what is happening if f, for a particular choice of window, the f looks like this So, what it is doing is it is picking values of the signals energy and leaving out the interferences, as a result of which the interferences have vanished in the Wigner-Ville. But what has happened is, with respect to the original Wigner-Ville, I have some kind of, I have some kind of smearing of the energy. This is the sacrifice that I am making and the weighting here tells you why that smearing occurs.

Lastly, we have the smooth pseudo Wigner-Ville, which performs different kind of weighting. What is the prime difference between the spectrogram weighting and the smooth pseudo Wigner-Ville? Well, please do not tell me, that it is a size of the window, right. Of course, that is a qualitative, but more importantly it is, it can take some interferences also with it when it is varying the ambiguity function, whereas, the spectrogram is the weighting function. Spectrogram only concentrates on the signal components that is what we mentioned earlier when we discussed about Cohen's class.

The smooth signal, the smooth pseudo Wigner-Ville can produce interferences eventually depending on how you choose your widths, right. If you choose the certain width of the windows in time and frequency that you have, it is a smooth pseudo Wigner-Ville. Therefore, it is a separable one. I have two windows to clear around with and depending on how I choose the widths, you can end up taking the interferences or not. For a particular choice you will get spectrogram,, but there exist a lot of other choices and I am only showing you here the window, that almost goes up to the interference, which is what is done in the time-frequency tutorial and therefore, some interferences, very minor interferences can come in back. Mostly, there is none. In fact, if you slightly readjust this, fine tune this, then you will be able to get better energy distribution, better localized energy distributions than the spectrogram and still be devoid of interferences.

So, there is again a reiteration of what we had seen with respect to another example previously. This way you can imagine the weighting functions for all the other distributions. In fact, just go through this exercise, take the kernels that we have listed in the table in the previous lecture. For some well-known distributions take a signal plot, the signals ambiguity function and on the same plot, plot the weighting function also and see how the weighting function performs with respect to removal of interferences. If the weighting function is able to remove interferences you are guaranteed, that you will get a positive value distribution by virtue of Wigner’s theorem once again.

So, to summarize what we have done in this lecture is reinterpreted Cohen's class using what are known as ambiguity functions that were introduced in signal processing, whereas, Wigner-Ville distribution originated from a combination of approaches in physics and signal processing. But this shows, that there exist a lot of parallels between the tools, that originate from different domains, which is what we are also mentioned in the introductory lecture with respect to wavelets as well, where wavelets were introduced in different domains and different waves,, but the ideas are more or less same. The nice thing that we have seen here is the duality between the ambiguity function and the Wigner-Ville and it gave as a fresh interpretation of the weighting function.

So, hopefully you enjoyed the lecture. We will see again in the next lecture, which will close the topic on Wigner-Ville where we will look at, briefly look at the. So, called affine class of transforms. Here, Cohen's class insist that the smooth Wigner-Ville satisfied the time and frequency invariants property. In the affine class we will insist, that it satisfies the time and dilation are the scaling invariants property. Then, we will briefly talk about what is known as reassignment, which is more of a modern development and close the topic on Wigner-Ville distributions.

Thank you.

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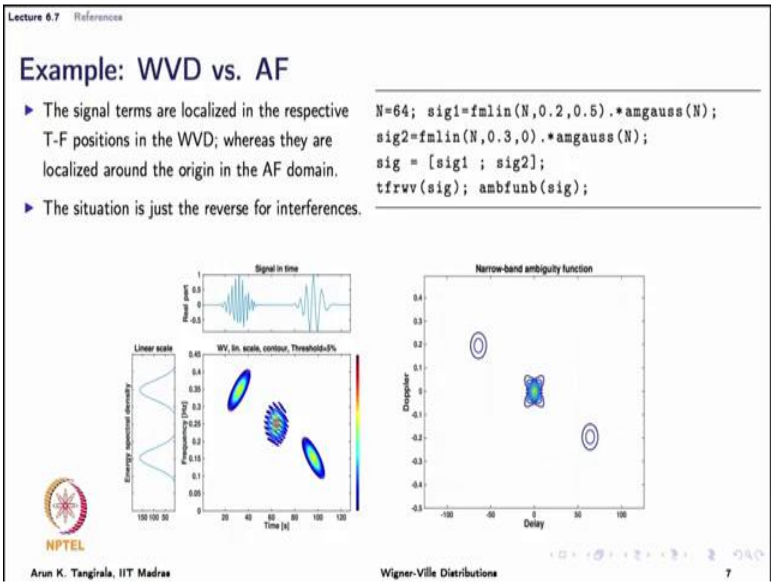

((Refer Time: 29:43)) The example that we take up here for illustrating the difference between Wigner-Ville and ambiguity function. In fact, rather, rather than difference in duality is one that is given in the tutorial, time-frequency toolbox tutorial. It consists of 64 observations of a signal, which has modulated chirps spaced apart in time and you can read of the range of frequencies here that are given. In fact, they are the same chirp is being modulated. Sorry, the one chirp is being modulated within Gaussian amplitude modulation and there is another chirp, which is being modulated with another amplitude, Gaussian modulated amplitude.

So, now what we do is, to generate the signal we concatenate these two signals and the signal here is shown in the top, as usual, with the spectrum here on the left panel of the figure. Let us come to the point. The point, that we made earlier is, that the signal's components in the Wigner-Ville plane are located exactly at the point, respective points in the time-frequency plane, that is, that the, at the respective time-frequency centers and that is what we see here in the figure. And the interference terms has now earlier, are concentrated midway between the time-frequency centers of the respective components of the signal.

When we move to the ambiguity function plane, which is essentially the delay-Doppler plane, the reverse happens. All the signal components are concentrated at the center at the origin, in fact, of the delay-Doppler. This is an absolute origin here and of course,, what has happened is both the atoms, that is, both the signal components are modeled up here. They are all nicely and cozily settled. We know, that the value of the ambiguity function and the origin gives me the energy of the signal.

Now, the other important point that we learned is, that these interference terms move away from the origin and that is exactly what has happened. The interference term, that occurs midway in the tau-zee plane or the time-frequency plane is, is now split apart and it is located far away from the signals components in the delay-Doppler plane and they are symmetrical. They are not really located at some arbitrary portions. There is also a certain geometry to this interference terms in the delay-Doppler plane.

One of the basic idea of constructing the smooth Wigner-Ville distribution is as follows, based on which we can now set the tone for reinterpreting the Cohen's class. The basic idea is to apply a weighting function in the ambiguity function plane, that is, to pick only certain values and to leave out certain other values that is a role of the weighting function. And what are the values that we are interested in? Obviously, the signal. So, I apply the weighting function in such a way that it picks the values of the signal components, which is, which are centered around the origin and leaves out the values of the ambiguity function, which are far away from the origin. Of course, what is far and what is near is what is determined by the shape of the weighting functions itself. But once I do that then, what I have done is, I capture the signal components in the delay-Doppler plane.

All I need to do to construct the smooth Wigner-Ville is to perform a two-dimensional inverse Fourier transform and then I will get the smooth Wigner-Ville. Earlier we did this by directly working with the Wigner-Ville distributions itself by subjecting it to a smoothing operation using a convolution kernel. The convolution now becomes a product. All I have do is now multiply the ambiguity function with another weighting function and this weighting function is none other than the f of (nu, s), that we have seen earlier when we introduced Cohen's class. And we said at that time, that we will acquire a fresh interpretation of the weighting function and that is what we are going to do with these basic ideas established. We can now reinterpret the Cohen's class.