Introduction Time-Frequency Analysis and Wavelet Transforms
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Lecture - 6.6
Cohen's Class and Smoothed WVD
Part 1/2

Hello friends. Welcome to lecture 6.6, where we shall discuss Cohen's class and smooth Wigner-Ville distributions. In the previous lecture, that is 6.5 we had introduced to the concept of smoothing for the purpose of enforcing positivity. In fact, we looked at one-dimensional smoothing in the form of the pseudo Wigner-Ville distribution and then, a two-dimensional smoothing for positivity.

Now, we will take that idea forward and study a general class of smooth Wigner-Ville distributions known as the Cohen's class of distribution which was introduced at least more than four decades ago by Cohen himself. Now, also let me tell you that smoothing can be done in a number of different ways and Cohen's class performs one kind of smoothing. In the next lecture, we will learn what is known as an affine class of smooth Wigner-Ville distributions.

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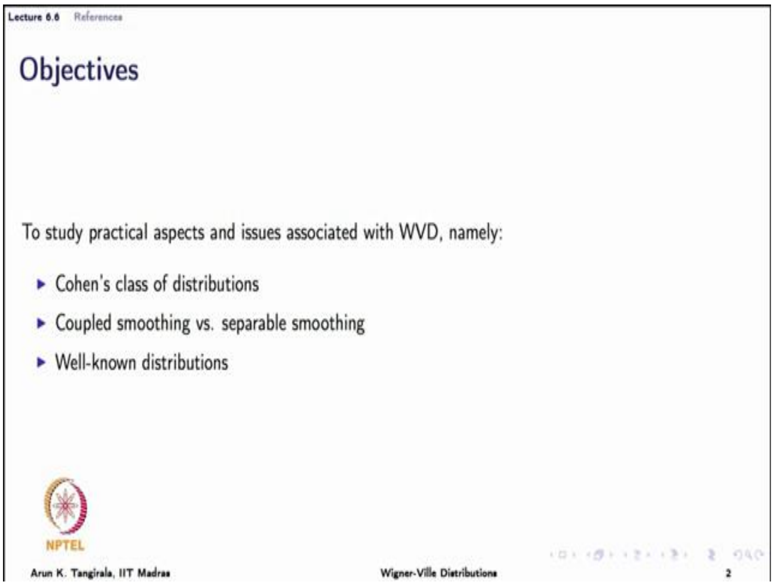

So, the objectives of this lecture is to primarily study the Cohen's class of distributions, and in this context, we will look at what is known as couples smoothing which is what is done by spectrogram versus separable smoothing. There are some advantages of separable smoothing and finally, just quickly have a glance at a few well-known distributions. The lecture is definitely going to be heavy on the math. There is no escape to that, but from time to time, we will have a few illustrations to lighten up and also to illustrate the concept themselves, ok.

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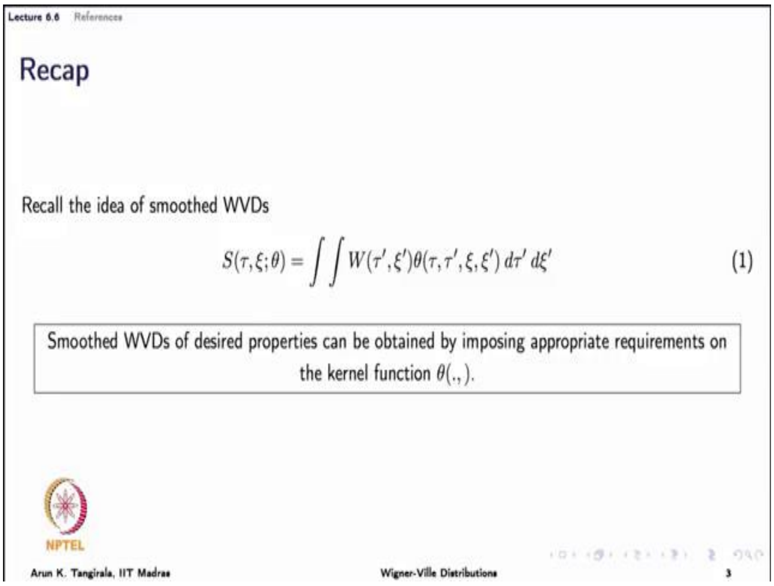

So, just to quickly recap as to what we studied in previous lecture, the idea smooth Wigner-Ville distribution was to use a kernel function to smooth the Wigner-Ville in the time and frequency axis respectively. Recall what smoothing means is convolution also, but smoothing is much more than that is. It includes a lot of other operations apart from convolution. In this lecture, it turns out that the Cohen's class is nothing, but convolution based smoothing of the Wigner-Ville distribution. Now, the important point is one, there are two important points. One, the purpose of smoothing is to improve upon the drawbacks of the Wigner-Ville, right. For example, positivity is one that is a very desirable property and we realize a spectrogram achieves that with the help of the convolution based smoothing kernel.

The second point is in general whatever desirable property that I want to have of the smooth Wigner-Ville, I am going to transfer those requirements on the kernel, and at an appropriate point in the lecture, we will see what the conditions are that I have to place on this kernel. Before we move on, let me also tell you that although we begin this smoothing or the Cohen’s class itself the smoothing in the time and frequency approach. There are a number of other ways in which Cohen’s class can be presented and it depends on the author, depends on the text. So, please do not think that this is the only way to introduce Cohen’s class. This is one of the natural ways to do that and also, there are going to be some differences in the notation and sometimes in the definition of certain kernel functions that you may use. You should be able to see through that as long as the text book is consistent with all the definition, you should be alright. So, please keep that in mind.

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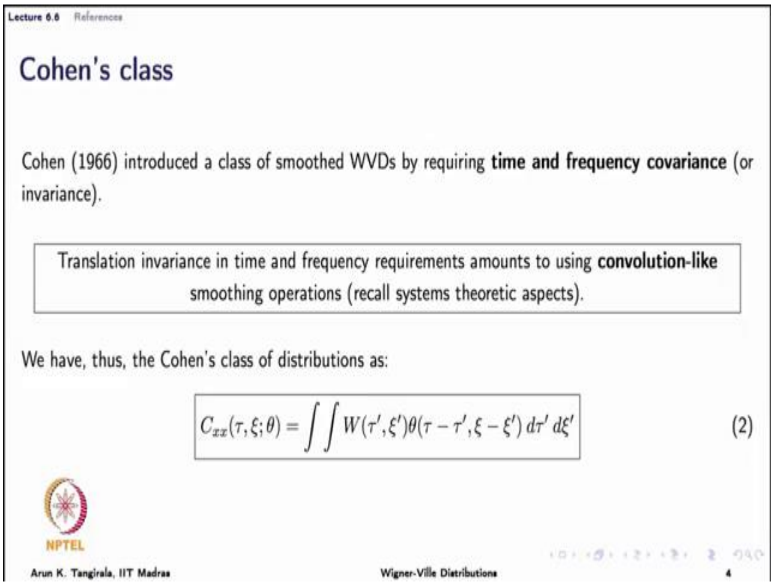

So, now, let us move on to Cohen’s class. Cohen basically required that the smooth Wigner-Ville distribution that you obtained; satisfies to key properties which is the time and in wave frequency covariance. Sometimes this is also called invariance, and you should be able to switch between these two. What this time and frequency covariance or invariance means is, if there are time shift in the signal, the energy distribution to reflect that and if there are frequency shift or modulation, the same should be reflected in the joint energy distribution as well.

Now, if you recall from linear system theory, although we are not discussing that in this course, if you have some familiarity with linear systems theory, then you should be able to recall that translation invariance in time. In a particular domain amounts to using convolution like operators and I will just quickly recap that for you from review that from linear systems theory.

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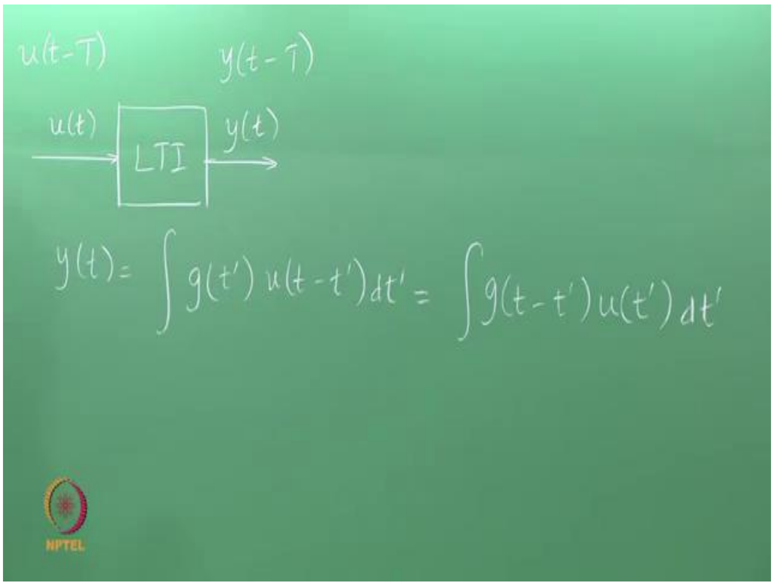

This is not something new. We have already seen that. If I have an LTI system, then we know that this LTI system input is u and output is y is described by this convolution equation which can also be written in this way, g of t minus t prime u of t prime d t prime. In both cases, the limits run from minus infinity to infinity. So, this is the governing equation for the continuous time linear time invariance system. In fact, I am not going to derive the convolution operation for you, but starting from the properties of the linearity and time invariance, you can derive the convolution equation. That is how it presented in all the text on linear systems theory. The message that I want to give you here is time invariance is usually characterized by convolution, although we use the notion of time here. In general you have to now look at this in a broader sense. It could be space, it could be frequency, whatever domain in whichever domain, you want invariance. Typically you will run into this convolution like operation.

Of course, there is linearity that is also playing a role here, but predominantly this convolution comes about due to the time invariance requirement. What is time invariance requirement? Again here is the same story. If I shift the input by a certain amount, big T, then y should also shift by the same amount, and that is what we mean by time invariance here. In other words, regardless of when I give the input, the output of the system should be the same for that and the same story here. I want this smooth Wigner-Ville distribution or modify Wigner-Ville distribution to have time and frequency invariance. If I shift the feature of the signal in time or if I shift the signal itself, the joint energy density should shift in time and vice versa and also, likewise for frequency as well.

So, now, what we are going to do is, we are going to assume that rather than going to work with convolution like kernels because I want invariance in time and frequency domains. Therefore, we have the Cohen's class of distribution in this way that for expression given in equation 2 as a set, earlier Cohen's class can introduce in number of different ways. I have followed the interpretation or the approach that is used in (()) book, in few other books, but if you take Cohen's book itself, you will see the Cohen does not introduce the Cohen's class in this way. There is another expression which you are going to come across which will be the starting expression, but this is a nice thing to begin with because we know convolution and smoothing or equivalent. Therefore, you can straight away write a generic expression for Cohen's class based on the primary requirement at time, and frequency invariance should be a property of smooth Wigner-Ville distribution.

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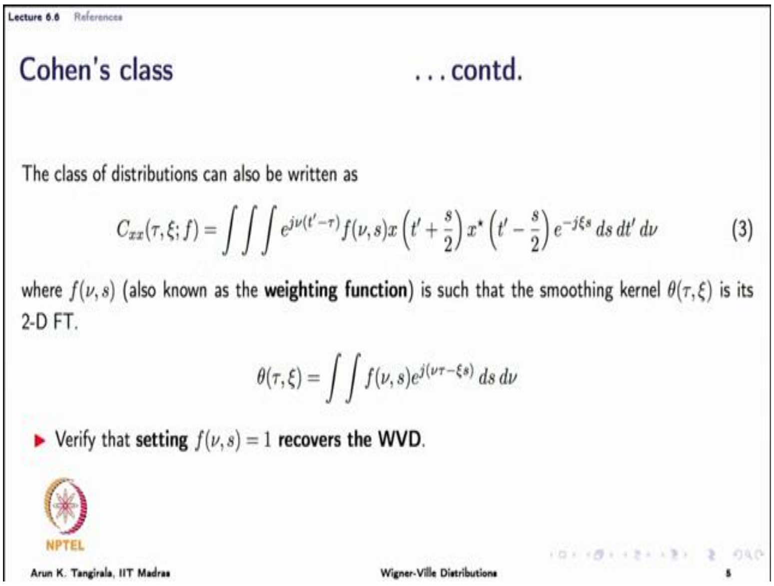

So, we have know convolution in time and frequency, I could rewrite the convolution as a product, but you may not able to see that straight away right now. In fact, all of this equivalence between different expressions for Cohen's class will become clearer partly in this lecture and partly in the next lecture, where we will talk about what are known as ambiguity function. So, the class of distributions, that is the Cohen's class of distribution can also be written in this way, where I have now a triple integral. Earlier I had a double integral because I am performing smoothing in time and frequency. Now, to replace the Wigner-Ville with that is definition and the kernel that is the theta kernel with the equivalent kernel in the time frequency, that in same domain as the signal time frequency plane, then you will have a triple integral. In fact, you should expect that because I have convolution in the frequency domain time frequency plane here, right.

Now, if I recalled the property of Fourier transform that you have studied in one-dimension, but if you extend that two-dimension, then the same properties hold convolution becomes a product, that is convolution in one domain is product in another domain. So, for the moment assume that I was to take Fourier transform of this c, right. I am going to take a Fourier transform of this convolution. So, what happens when I take the Fourier transforms of convolution is that then it becomes a product in the dual domain. So, it requires a bit of imagination here, but as long as you hold on to that property of convolution becoming product, then you are (()).

So, imagine that if I am taking Fourier transform of the double integral in one, then I get a product in the dual domain. We will talk about what a dual domain is. All I have to do is take an inverse of that product to recover my c. If you understand that point, then this equation 3 will become clear to you as to why that expression. Of course, the way it has been written does not really give you that feeling, but gradually you will understand that this is nothing, but the inverse two-dimensional Fourier transform of what you have in equation 1. So, with that the intimidation that you get by looking at an equation 3 should come down.

So, now instead of theta, we will have f. What is this f? Well, if you look at the relation between theta and f, theta is nothing, but the two-dimensional Fourier transformation of f. Now, you have to be careful here. In fact, I have written it as e to j, nu tau minus g s. So, there are a set of new variable's that have arrived here in this slide. Earlier we had only tau and g keeping track of the points in the time frequency plane. Now, I have nu and s. These are the nu variables that have appeared rightfully. So, that is why I have variable called nu. What is the relation between f and theta? So, this is nu and s is a new plane which we call later on when we talk of ambiguity functions. We will call this nu as the doppler and s as the shift. This is a terminology that is used in radar signal processing.

So, you can get familiar with this terminology upfront itself. Nu is the doppler and s is the shift here. You can think of theta being the two-dimensional transform of f, or tool inverse to two-dimensional fourier transform whichever we way look at it, but this two-dimensional fourier transform that you see is not a straight forward extension of the one-dimension. In one-dimension Fourier transform, you would have an e to the minus j. Here I have e to the minus j on one factor and I have e to the j on another factor, right. If you look at the equation carefully and therefore, you cannot strictly call it as the two-dimensional Fourier transforms, but the moment you move to multi-dimensional Fourier transform, there are number of combinations. You can have a minus on both sides, both dual variables, you can have a negative sign on one of them and positive sign on another and so on.

So, this is one of those classes and you should get comfortable with this form of two-dimensional Fourier transform because this will continue throughout this lecture and next lecture, and probably even in the closing lecture as well. So, this is the two-dimensional Fourier transform that we are going to work with all. You have to be sure is to keep track of in which variable it is the inverse Fourier transform and in which another variable, it is a forward fourier transform. Here theta is the two-dimensional Fourier transform of f or you can say f is the two-dimensional Fourier transformed theta depending on which plane variable you are considering as an inverse and forward.

Here you can say that it is inverse with respect to nu and forward with respect to s, right or I can say it is the two-dimensional inverse Fourier transform with respect to s and f whichever way we can look at, it does not matter. Essentially there is a two-dimensional Fourier relation between theta and f, and this is what we said earlier when I take the Fourier transform of the convolution operation. Here I will end up with a product in the dual domain. So, I have a product of f and also, the usual terms as you have seen Wigner-Ville and so on. This equation will be clearer to you when we introduce ambiguity function and so on. So, let us proceed. This is the starting equation presented in many texts on Cohen’s class.

Now, the other interpretation that you can give to Cohen's class is that is the generalized Wigner-Ville distribution, right. Why is generalized? It is because if I said f equals 1 in both dimension, then I will recover the Cohen's class. In fact, I leave that as an exercise to you. Be careful when you do that. When you say f equals 1, you will have to use the familiar result that we have integral e to the minus j x t d t will be dragon x, right. So, you should use that relation to be able to simplify the Cohen's class to the regular Wigner-Ville in that sense because when I said f equals 1, I recover the Wigner-Ville. I can call the Cohen's class as the generalized Wigner-Ville distribution. It is also customary to rewrite Cohen's class in number of other kernels. The purpose is to be able to interpret in different domains.

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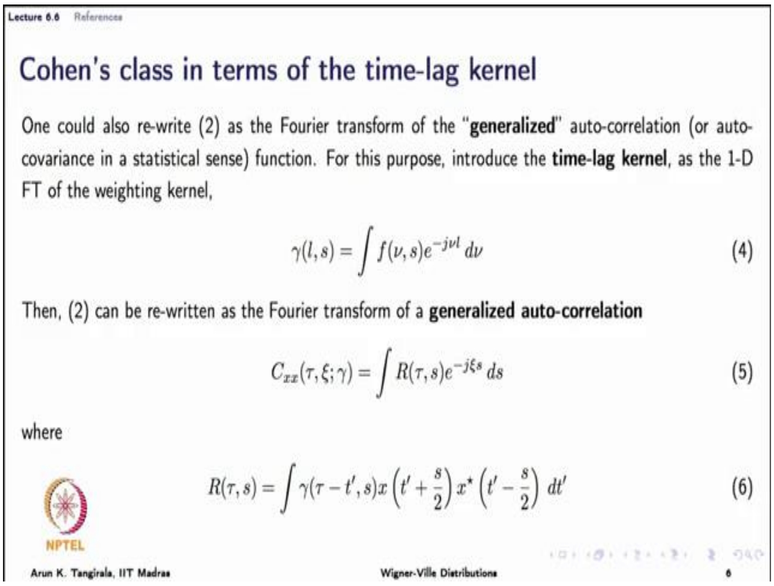

Each domain will cater to certain application depending on the way you are looking at it. So, it is useful to know how Cohen's class can be rewritten in terms of different kernels. So, we have now come across the theta which is the time frequency. You can call that as the smoothing kernel. Then, f is called as the parameterizations of the waiting function kernel.

Now, we have what is known as a time lag kernel. What is the interpretation? What is the approach here? Well, it stamp from the interpretation that the Wigner-Ville when we wrote the Wigner-Ville, we said it is a Fourier transform of an instantaneous auto correlation, right. The instantaneous auto-correlation being x of t plus tau by 2 times, sorry tau plus t by 2 times x star of tau minus t by 2, that is the instantaneous correlation auto correlation.

Now, you can interpret Cohen's class in the same way because it is a generalized Wigner-Ville. We can think of Cohen's class being the Fourier transform of the generalized auto-correlation and to realize this first introduce the time lag kernel. So, what we are going do is, we are going to take a one-dimensional Fourier transform along the new axis. So, I am going to have a function which is function of this lag l, and the shift s that we talk about the nu is going to be eliminated because I am integrating in. I am actually smoothing along the nu axis. The moment I do that, I get what is known as a generalized correlation. Remember the regular auto-correlation that we talk about for stationary class of the signals is the function of the single variable lag l.

Now, the generalized auto-correlation is the two-dimensional auto-correlation. You have to keep track of two things right because now we are dealing with time varying signal or frequency content that is varying the time. So, single lag is not going to help me characterize the signal. I would require two variables, one is a lag and other is a shift. Here we will understand the shift better in the next lecture when we introduce ambiguity functions. So, coming back to the discussion, we have the time lag kernel which is fine as one-dimensional Fourier transform. The traditional one-dimensional Fourier transform of this free f kernel or the waiting kernel with help of this, I can rewrite the Cohen's class as one-dimensional fourier transform of a generalized auto-correlation.

Notice that this big R is not the instantaneous auto-correlation. It is actually the generalized auto-correlation, right. So, there are different terminologies here. Instantaneous auto-correlation is simply the product of x of tau minus t by 2 times x star of tau minus t by 2. When I integrate the instantaneous auto-correlation, I get the regular auto-correlation, right and to see that let us just workout the couple of derivations. Just assume expressions and it will be clear what instantaneous auto correlation is. What is auto correlation? What is generalized auto-correlation?

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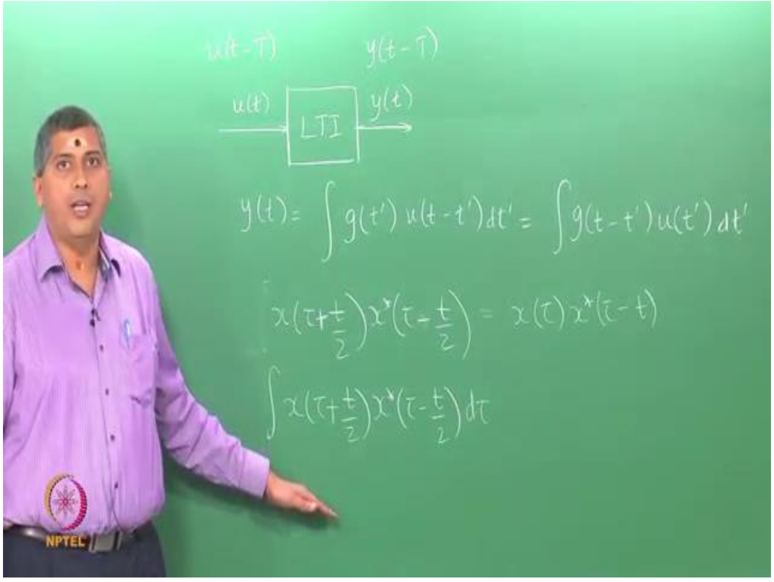

So, the instantaneous auto-correlation is x of tau minus t by 2 times x star. So, this 1 of tau plus t by 2 times x of x star of the tau minus t by 2. In fact, rather than looking at this as tau plus t by 2 and tau minus t by 2 with the change of variable, you could also rewrite this as x of tau times x of x star of tau minus t, right. In fact, this t is the dummy variable here. This is only telling me what the distance between these two observations is. That is in terms of sampling instance. You can think of t s lag l also. Now, the general, the regular auto-correlation is an integral of this instantaneous auto-correlation, right and the integral being here with respect to tau itself that is a regular auto-correlation and then, you will have the generalized auto-correlation. So, instantaneous and the regular auto-correlation which is based on the definition of the regular auto-correlation, we have seen early in unit 2.

Now, you have the generalized auto-correlation. So, you could interpret the Wigner-Ville and Cohen's class in terms of all of this, but the most convenient one of the most convenient ways is to interpret the Cohen's class in terms of the generalized auto-correlation. I have given an expression for the generalized auto-correlation for you in the bottom most equation, where I have in addition to this a function. Another function that has come up which is gamma, right and let us talk about it.

So, this gamma is what gives the generalized nature to the auto-correlation. When I said gamma equals 1, then I recover the gen regular auto-correlation, alright and that is how things proceed. So, the message now is Cohen’s class is the Fourier transform of a generalized auto-correlation. Wigner-Ville distribution is the Fourier transform of the instantaneous auto-correlation, right. That is how it is going to be. Well, the generalization I have, we have also seen the generalization in the previous slide.