**Introduction to Time-Frequency Analysis and Wavelet Transforms**

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**Lecture - 1.2**
**Brief tour of wavelet transforms**

Hello friends, welcome to lecture 1.2 of the course on introduction to time frequency analysis and wavelet transforms. This is the second lecture in the introduction unit of this course. In the previous lecture that is lecture 1.1, we obtained an overview of the course of course, but mostly the multi scale analysis itself, and what is a multi scale process, why and how multi scale systems can be challenging in terms of analysis. Of course, multi scale systems are challenging in other respects too, particularly in terms of sampling or simulating where you run into the stiff behavior when it comes to numerical integration or sampling, because you have a range of scales from the micro or even lower nano to the macro. And choosing a sampling rate that will suit all scales is almost impossible.

You will have to choose of course, based on the finest scale, but then the other scales appear as if they are not changing at all, if you choose your sampling rate based on the finest time scale. So, there are lot of challenges there in sampling. In this course, of course we are more concerned about the analysis of the signals coming from multi scale systems.

Now, in the previous lecture we looked at how Fourier transforms are unsuitable or unsuited for the analysis of multi scale systems, why they are not suitable; and also obtained a quick overview of the techniques of short time Fourier transforms, particularly the spectrogram and Winger Ville distributions. In this lecture what we are going to do is take a brief tour of wavelet transforms; what wavelets are and how they allow us in analyzing multi scale systems in a very efficient manner.

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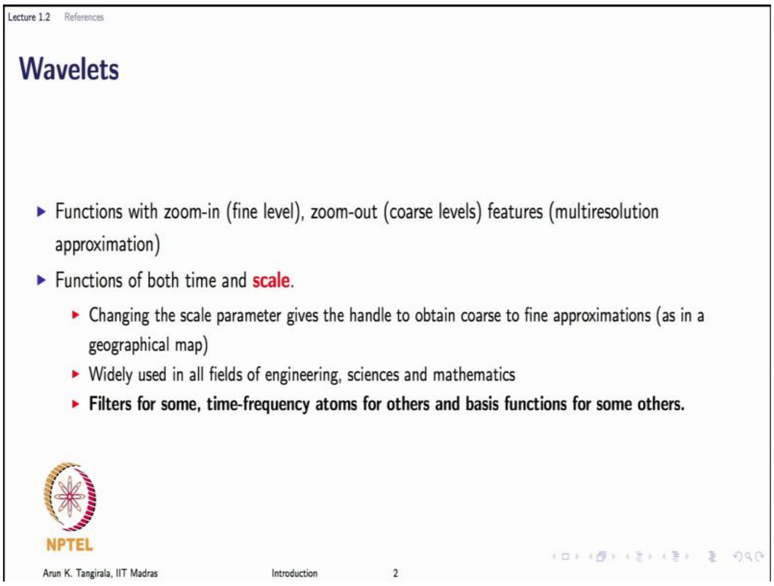


So, what are wavelets? Essentially, they are just another set of analyzing functions like the Fourier analyzing functions, but with certain key differences that make some very popular, and of course suitable for signals in the time frequency plane.

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So, just to compare when it comes to the Fourier domain, the analyzing functions are e to the j omega t. And if I where to draw this as a function of time, these analyzing functions exist all over the time. I am only drawing the real portion of this. Of course, it is impossible to draw the entire length of the signal, because they exist for ever. And that is the key point that we noted in the previous lecture. This global nature of the analyzing function does not get me the local properties of the signal in time.

Of course, the short time Fourier transform makes an improvement by clipping this global feature, say the global sine wave, maybe to a length such as this. So, it says now I am going to analyze signal only using this wide sine wave, but not the infinitely wide sine wave, right. And this finite width or the finite duration nature of the short time Fourier transforms analyzing function which is also known as the time frequency atom. It gives the ability to analyze the local properties of the signal.

In the Wigner Ville distribution there is no such analyzing function per say. It directly computes energy. Now, coming to wavelets, wavelets are also set of finite duration analyzing functions or we call them as time frequency atoms. But they have a different shape and they are characterized in a different way. The short time Fourier transform atom here is characterized by this width and the frequency that you are using to analyze the signal.

Whereas wavelets, you take a wavelet which will denote by psi, is a function of 2 parameters. This, if the short time Fourier transform is also the function of 2 parameters which is the center frequency, sorry, the center time, that is the center around which it existing in time; and a center frequency that is you can say in this case the frequency of this sine wave itself. When it comes to wavelets you have 2 parameters which we denote by tau and s, where tau is the center in time once again of the wavelet, I will draw the wavelet for you; and s is a scale that we are introducing right now.

Here the short time Fourier transform is the function of frequency, whereas, the wavelet is the function of scale. And in the previous lecture, we noted the connection between scale and frequency. We said qualitatively they are the inverses of each other; quantitative relationship depends on the wavelet and so on. So, how does a wavelet look like? Let us say I am looking at scale 1; that means, what we mean my scaling in general is, scaling in time, the stretching or compressing.

A typical wavelet would look like this. This we say is centered at the origin. So, tau is 0 here, and this scale is said to be 1. If this scale is said to 1 we call this as a mother wave. And all the other wavelets which are the children of this mother wave are essentially arrived at by either stretching or compressing this signal. And that is why they are called wavelets. So, here we have wavelets which are the children of this mother wave.

So, what is the difference between this and this is, something that we will learn in this course. But very quickly let me tell you, that from a filtering view point, as you must have read many times in the literature as well, both wavelets, the wavelet atoms or wavelets themselves and the short time Fourier transform atoms act like filters or band pass filters.

The only difference being, and which is a key difference is that this filters that come out of band pass filters that come out of short time Fourier transform have a constant band width; that means, whether they look at the high frequency components of the signal or the low frequency component, they are going to filter in the same manner. Whereas, wavelets will perform the filtering of the signal in a completely different manner and which is very intuitive; and we will talk about it very soon.

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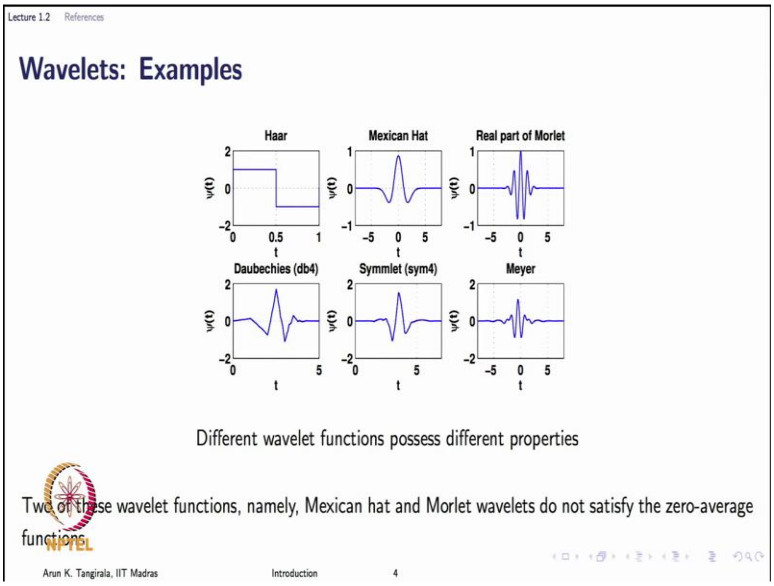


So, to get a feel of the different wavelets that we may have, we will come back to that slide shown. You can have numerous wavelets. I have just drawn 1 on the board for you, but on the slide I show you different wavelets that you can have. And again, this is not exhausted. On the top left you have what is known as Haar wavelet which is supposed to be the original wavelet conceived by Haar in 1910. And then you have the remaining wavelets; some of them being the Daubechies, and so on.

All of them have a common feature which is that they exist for a short time, that is a key requirement for us to get the local properties of the signal. Of course, each wavelet is suited to a particular class of signals. For example, Haar wavelet is suited for analyzing discontinuities in the signal. You must notice that the Haar wavelet itself is discontinuous, and therefore, it is suited for analyzing discontinuities. This is a key property or this is a key point that you should remember.

Like basis, like information, which means whatever features the basis are your atom analyzing function has, such will be the information that you will be able to extract. So, if I want to extract the regularity of a signal I should choose a wavelet that is a regular and so on. So, these form the guidelines for choosing a wavelet. We will learn all of that later on.

Mathematically, a wavelet as I said is generated by translating and scaling a mother wave. So, this is a mother wave situated at the origin waiting to be translated and scaled. And if I were to translate this without scaling, then it would essentially shift to the right or to the left. Why is this translation required? Because, what I am going to do is I am going to take this wavelet and I am going to carry it with me, travel along the length of the signal, march in time, and that will require translation.

The signal is going to be fixed there for me in the analysis and I am going to march ahead with this analyzing function. To keep track of this marching in time we have this parameter tau. This scaling will help me analyze the high frequency and the low frequency features of the signal. How does it do that? I will show you with an example shortly.

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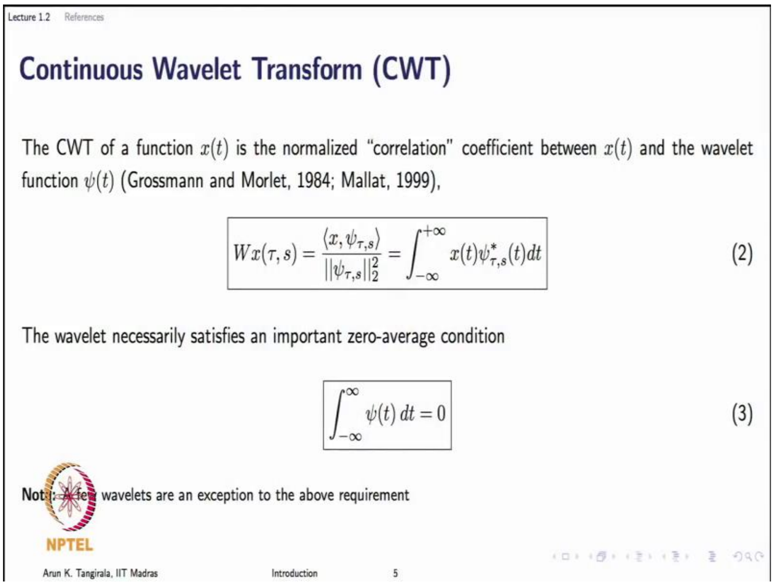

But first let us define, not really define, but let us quickly look at the continuous wavelet transform, as to how it is defined. We are not going to go into the math, but this is just for you to contrast between the definitions of the short time Fourier transform that you have seen, the Wigner Ville and the wavelet transform. We will, of course, look at this very carefully and closely in each unit separately.

So, the definition of a continuous wavelet transform is along the same lines as any other transform. If you look at the integral that I have here, on the top, it is essentially the wavelet transform is an integral of x (t) with the complex conjugate of the wavelet, right. So, I use slightly different parameters there which is the convention that we will follow in the course also, the a and b, that you have there; sorry, the tau and s that you have here is the same, but in the course we will use a different notation.

And let us not go into what is integral means right now and how it is derived and what this time, what this will get me mathematically, but what is important to note is that essentially evaluating the transform gets me the correlation of the signal with the atom that I am using. So, when I analyze the signal with this wavelet I will get the correlation, and the same applies to Fourier transform as well as short time Fourier transform.

And you notice some angular brackets there; they stand for inner products, alright. And we will talk about inner products later on. If you are not familiar with inner products, there is a unit, there is a lecture that gives you a quick review of inner products. Most, now, one of the key requirements of a wavelet is that it should have 0 average, and you should have verified that. Any wavelet before you use should have a 0 average.

Of course, this is just a qualitative picture, but I showed you a few wavelets before, and you should indeed at least verify visually that they have 0 average. In fact, you can check for the Haar wavelet, it has a 0 average. And all the other wavelets also, except for a couple which do not, couple of wavelets which do not have exactly 0 average, but nearly 0 average. So, strictly speaking they are not wavelets, but they more or less are approximate, act as wavelets.

Now, why is this 0 average required? Well, again, it is to get the local details of the signal. Whenever I want to get the local details of the signal then I have to have the analyzing function to be of 0 average. Its connection with filtering we will know later on, but this is one of the key requirements. There is no such requirement on any of these atoms. So, that also makes big difference.

Let us understand a point that we made earlier, that when I scale this wavelet and analyze the signal then I will essentially filter the signal, right. And in choosing the scaling values I can choose anywhere between 0 and infinity. So, these could be the values for the scale. And typically, we draw the line, we draw the partition for the scale at s equals 1, that is it is a reference; you say, the mother wave is at scale 1; it is just a reference point; there is no strict definition for it.

So, we say, the mother wave is at scale 1, and whenever I choose values of s less than 1, what I am going to do is I am going to actually compress the signal, right. I am going to generate a high frequency. When I compress you can imagine that it would look like this qualitatively, not exactly. So, let us say, I compressed it and translated it, then it would look like this; for some scaling value it would look like this.

So, now, compare; this has a larger width while this has a smaller width, and it changes more rapidly than the mother wave. This is obtained for scale less than 1. And when I stretch this, that is when I choose values of s greater than 1, then I am dilating it or stretching it, then in that case, for a different value of translation I would have a wide wavelet. So, again, just to qualitatively illustrate, it is not exactly drawn, I would have something like this.

So, this is centered right now at tau. So, this is for a value of s greater than 1. Of course, as I said these are all qualitative; ideally, you should scale in such a way that all of them have the same energy. And that is why you have 1 over root s that you saw earlier in the definition of wavelet. This is much wider than the mother wave itself. And that means, this is also changing slowly, so this will help me get the, extract the low frequency content, while this frequency atom will help me extract the high frequency content.

So, now, you understand the inverse relation that you have between scale and frequency; s less than 1, will get me the high frequency content. And the wavelets at scales greater than 1 will get me the low frequency content of the sigma; and that is exactly what I show you in the slide for you. Of course, we will only discuss this figure that you see some what superficially right now, just drive home the point. The exact mathematics of it will become clear towards later on.

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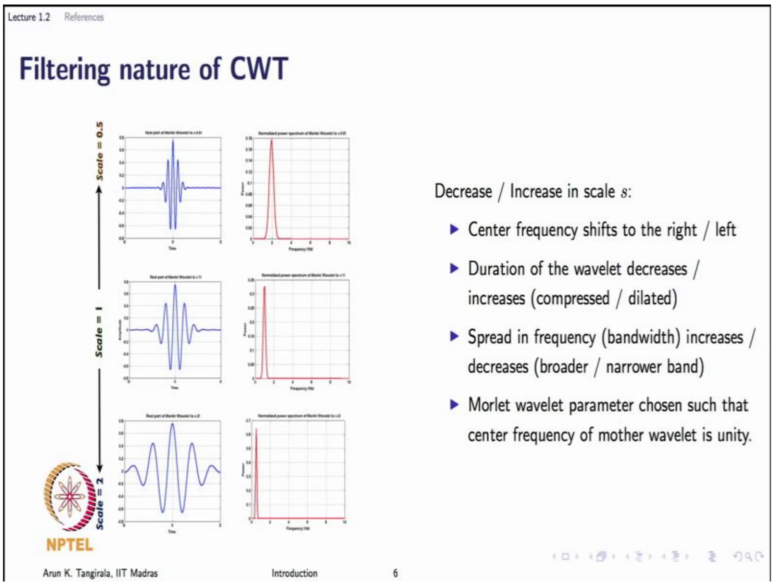


So, what we have here is the mother wave in the center which we say is at scale 1. And then on the top we have the compressed wave or we call as the high frequency wavelet which is at scales less than 1. And then you have the dilated wavelet at the bottom which is for values of s greater than 1. What you see on the right here? On the right you see essentially the spectral band width or the band width of the corresponding wavelet. That is, in the frequency domain.

The way you should interpret this is, the band width here will determine what frequency content of the signal I am going to extract from this side, right. So, if I use the mother wave I will be able to extract these frequencies in the signal very effectively. When I use the high frequency wavelet on the top then I am going to extract this band of frequencies from the signal, and likewise for the low frequency wavelet, right.

So, what is the difference between the high frequency wavelet and the low frequency wavelet? The high frequency wavelet has a larger band width, but of course, the center frequency has shifted to the right. What we mean by center frequency, is where this band width centered around or the way this curve in the right is centered around. And for the low frequency wavelet the band width is smaller that is narrower, but shifted to the low; obviously, because it is a low frequency wavelet.

So, whenever I use the high frequency wavelet to analyze the signal or scales less than 1, I am going to extract the high frequency content of the signal, but the band width is going to be larger; that means, I am going to get more frequencies surrounding the center frequency of interest than when I use the low frequency which has a very narrow band width; the larger the band width, more frequencies I am going to extract.

So, cannot I actually have a narrow band width for the high frequency one? Unfortunately not, and that is what is the duration band width principle that we very briefly mentioned in lecture 1.1. The high frequency wavelet here exists for a very short time. And the duration bandwidth principle essentially says, whenever a signal is a fraught duration it will have a wide bandwidth. And you can apply the vice versa condition here to the low frequency wavelet.

It is of longer duration, and therefore its bandwidth is going to be narrow. This is what essentially the wavelets do for you. They will get you the high frequency features that are short lift and low frequency features that are long lift. If a signal has features different from these, ones that we have just described then wavelets are not probably the best tools to analyze. Fortunately a lot of signals that we encounter do have high frequency features living for a shorter period of a time compared to low frequency features which persists for a longer time.

This is the zooming and zoom out feature you can say, or this is the key property of wavelet transforms which automatically adjust themselves according to the duration bandwidth principle and get to the features or the best possible localization of the energy in time and frequency which the short time Fourier transform does not have. So, the length, the width of the wavelet in time and the width of the corresponding filter in frequency are nicely tide together in wavelet analysis which is not the case in short time Fourier transform; that is a key difference. Otherwise, they are all analyzing functions for you in the time frequency plane.

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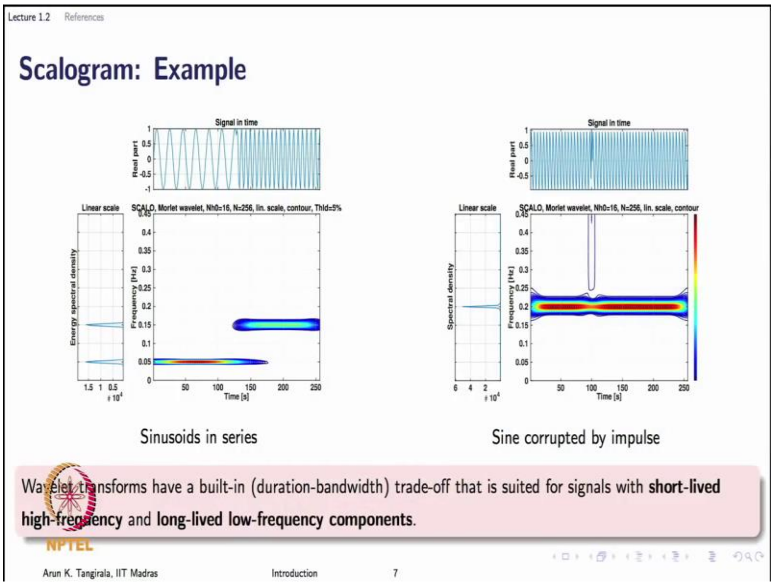


So, just to give you an idea of how you would use wavelets; this is what is known as a scalogram that you see; pretty much like a spectrogram. So, you compute the wavelet transform, and then compute the squared magnitude like you do in spectral density, a spectrum and spectrogram and so on. Just to show you how effective these wavelets are in distinguishing the features, local features of the signal; and these are the same examples that we have studied earlier in the case of spectrogram, Fourier transforms, and Wigner Ville and so on.

So, clearly, the scalogram is able to distinguish the low and high frequency components of the signal. So, let us look at an example of what is known as a scalogram. This is analogues to spectrogram and spectrum and so on, where we compute the transform and take the magnitude square. So, scalogram is also obtained by first computing the wavelet transform and then taking the squared magnitude.

And the examples, signals that we have taken are the same that we have taken before in the case of Fourier transforms and short time Fourier transform. In example 1 here, I have a signal which has low frequencies in the beginning for a while fix frequency and then high frequency later on. And the scalogram is able to correctly pick those features. And what you observe in the scalogram is consistent with what we discussed just now.

That is, when I am looking at the low frequency content of the signal I have very nice localization of the energy in the frequency domain, but it cannot looses out on the duration that is localization of the energy in time. It is unable to exactly tell you, at least the scalogram is unable to visually tell you when this low frequency components existed, and likewise for high frequency as well. For a high frequency component you notice that the time localization is nice of the energy here; we are not taking about time localization is signal, you have to be careful; we are talking about time localization of the energy.

From this, how you infer signal duration is, there is a mathematical process to it, but visually what people do is normally look at the scalogram and try to infer the signal properties just by visual examination; that is not really correct; it is only good qualitatively. But a more appropriate way is to evaluate, treat these as densities and then look at the moments carefully and then compute the duration and bandwidth and so on.

But visually if you look at it and which is also true, the energy has not been localized very well in frequency for the high frequency component; whereas it is in the case for the low frequency and vice versa for time localization of the energy; that is exactly what we discussed earlier. When it comes to extracting high frequency components, wavelets are very good at time localization of the energy, but not frequency localization, and then vice versa for low frequencies.