Introduction to Time-Frequency Analysis and Wavelet Transforms
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Lecture - 6.4
Discrete WVD

Hello friends welcome to lecture 6.4 on the topic of Wigner-Ville distributions. In the previous three lectures we have looked at the, definitions of Wigner-Ville. Studied the theoretical properties, and looked at illustrations of Wigner-Ville on few standard signals. Where we also learn that, there are a few drawbacks with Wigner-Ville that we would like to address, but before we do that, it is useful to know how the Wigner-Ville itself is implemented in practice, and that is the subject of this lecture, where we are going to talk about discrete Wigner-Ville distribution.

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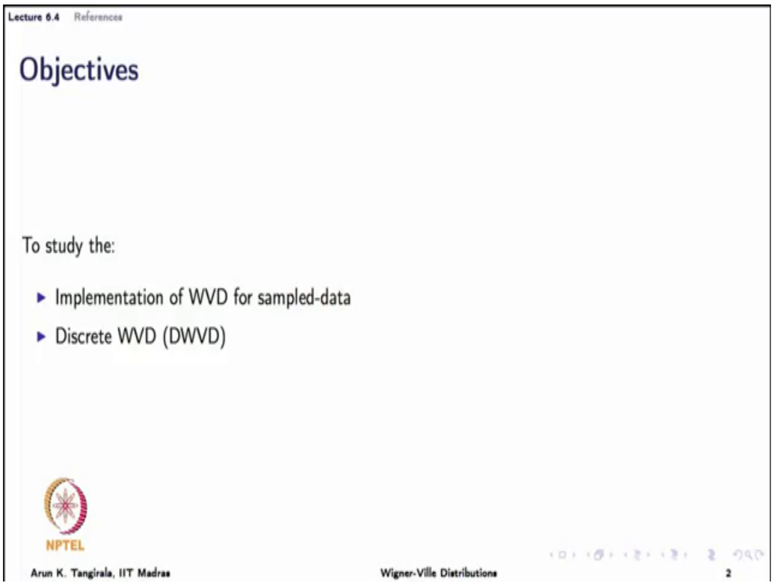

In particular, we look at implementation of Wigner-Ville, for sample data of course, and that is what we call as discrete Wigner-Ville. In fact, this terminologies analogs to, the discrete Fourier transform, where we compute the Fourier transform on finite length data, over a finite grid of frequencies. And the issues are more or less the same, but there's is an additional complexity associated with Wigner-Ville. Also we whatever we are going to discuss, although it is being discussed in a context of Wigner-Ville. It is equally applicable to the smooth versions, and the modified Wigner-Ville that we shall learn subsequently. Therefore, it is good to know, how this issues are addressed in the Wigner-Ville, because the theory also is, fairly easier to understand. Once we go to the smooth ones, then it is difficult to follow what is happening, there is a lot more math there.

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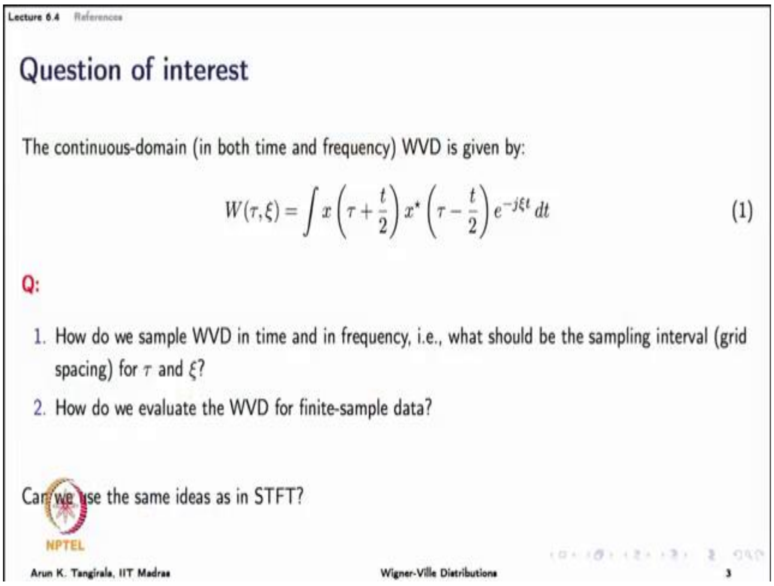

So let us look at the prime question of interest in this lecture. The continuous time Wigner-Ville is given by this definition. This is again something that we have seen in the lecture 6.1. And the two questions that we want to ask this, how do we sample this Wigner-Ville in time and in frequency, because its theoretical definition, is define over a continues domain of time and frequency plane. Essentially, we want to know what should be the sampling interval for both tau and chi; that is the time and frequency axis. And is there any issue, other than any challenges in evaluating the Wigner-Ville for finite sample data. At this moment of course, the natural question that arrives is, why is this special questions to ask, or why does is deserve special attention. Cannot we use the same ideas, that we have used in short time Fourier transform, where we sample the time, in the same way as we sample time for the signal itself. So, the spacing in time for the short time Fourier transform, is exactly the same as for the discrete time signal.

There was nothing to be really worried about. The only concern was phasing in frequency, but then, because short time Fourier transform involves a Fourier transform of the segmented signal. We good simply borrow ideas from d f t, and therefore, there were no specific issues to be addressed. Why does Wigner-Ville distribution, or why does the sampling of Wigner-Ville distribution call for a special attention. Well the prime reason n is, that Wigner-Ville distribution is a non-linear function of the signal. In fact, just look at the equation one; it is fairly clear that it is a quadratic function of the signal, which means it is a non-linear function. Whereas, a short time Fourier transform is a linear function of the signal.

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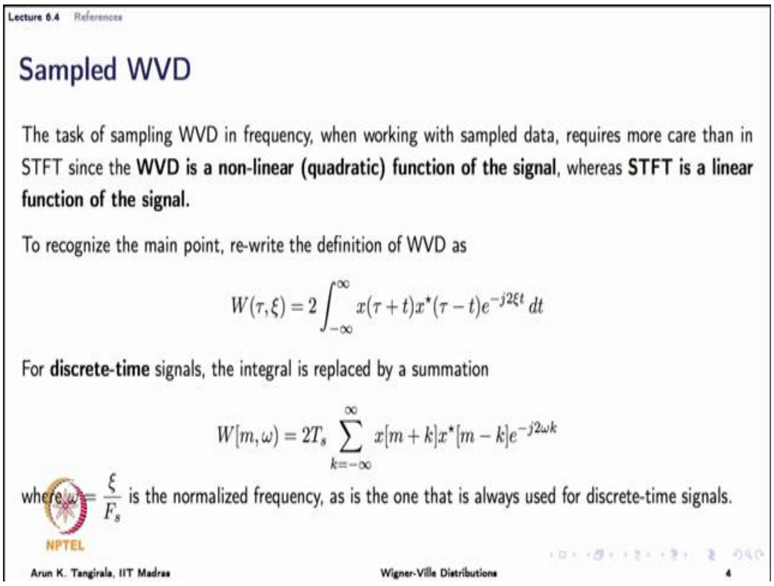

Whenever you are sampling linear functions of the signals, you are ok with using the sampling theorem that we used for, the signal itself, in a sense that whatever sampling frequency, or the sampling interval I choose for the continuous time signal. I can use that for the transform as well, but in the case of Wigner-Ville, that is not the case, because I have a non-linear function here, and will try to understand why this non-linearity presents a challenge, or an additional issue. First let us rewrite the definition of Wigner-Ville in this fashion, where I replace t by two with t, and therefore, I have a two appearing in front of the integral, and the z becomes two z here, that is the way you can look at it, but the rest of the expression looks the same. And when I write this for discrete time signal, the integral is a replace by summation, and now I have t s appearing, because d t is t s now, which is a sampling interval. And z is replaced by the normalize frequency, exactly the same way that we saw for discrete time signals, when we move from continuous time to discrete time. We move from continuous frequency, or the frequency for the continuous time signal to the normalize frequency. So, we move from z to omega, and or omega for the rest of the discussion, is the normalized frequency, which is z over f x. Omega has now the units of radians per sample.

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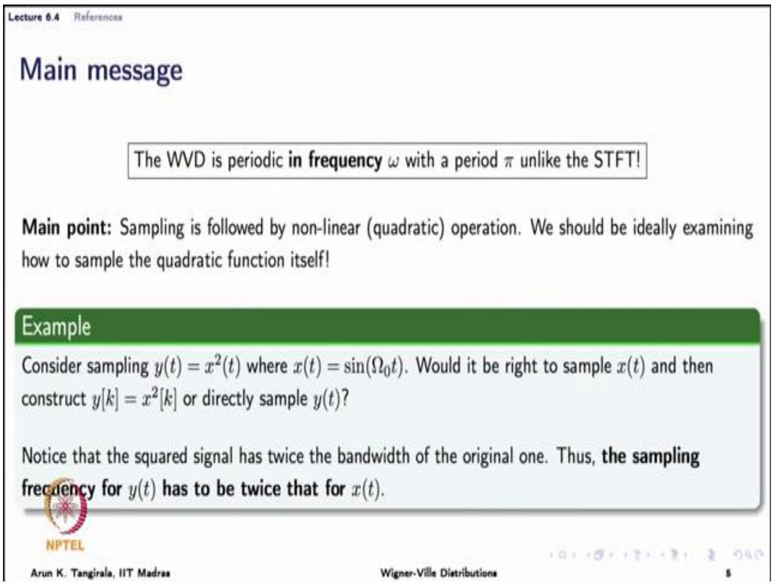

Now, the main point, is that the Wigner-Ville is periodic in frequency; that is in frequency omega with a period pi unlike the short time Fourier transform. The short time Fourier transform, has the same period as a Fourier transform itself. There is a discrete time Fourier transform, whereas, the Wigner-Ville has period of pi in frequency. Now how does it make a difference. Let us look at this example, and understand the consequences. In the example we have a signal x, and I have let us say a transform version of that, which is we call as y, and the transform is simply a quadratic. So, that this example is more consistent or closer to what we do in Wigner-Ville. For the purpose of discussion assume, that the signal x is a sine wave. So, what I am doing is, I am taking a sine wave and squaring it. And the question in hand is, whether it is right to sample x of t, and then construct the transform, or directly sample the transform itself. That is exactly the question also we have in Wigner-Ville. Should I sample the signal, should I decide the sampling for Wigner-Ville based on the sampling of the signal, or should I decide the sampling for y based on the Wigner-Ville distribution itself. Now to get a feel for the answer to be actual question. In this example, observe that is the square signal is, twice the bandwidth of the original signal. Well that is always the case, not only in this example, whenever he was squaring the signal, the bandwidth of the squared signal is going to be twice, that is because; look at x, x is a sine wave of frequency omega naught, and y would be sine square, and using trigonometry identities I can always right sine square as 1 minus cosine 2 omega t by 2; that is half of that.

Therefore, the frequency in y is double, and therefore, the bandwidth is also double. Consequently whatever sampling rate I am using for x, has to be double when it comes to sampling y, because sampling theorem is based on the frequency content of a continuous time signal, and the sampling, the maximum frequency in y. At least in this example is two omega naught, whereas, the maximum frequency next is omega naught. Therefore, whatever sampling rate that I choose for x may not be applicable to y, because I can I can choose for example three omega naught to sample x; that is not enough to sample y and avoid aliasing. So, the bottom line is, to avoid aliasing of, in sampling y; that is a transform x or the quadratic transformed x. I need to choose a sampling rate for y it is twice the sampling rate that I choose for x, and that is the same story for Wigner-Ville as well. Of course, I am arriving at this result in an intuitive way with the help of an example. There are formal proofs available in the literature, and I will give you reference to look up this formal proofs.

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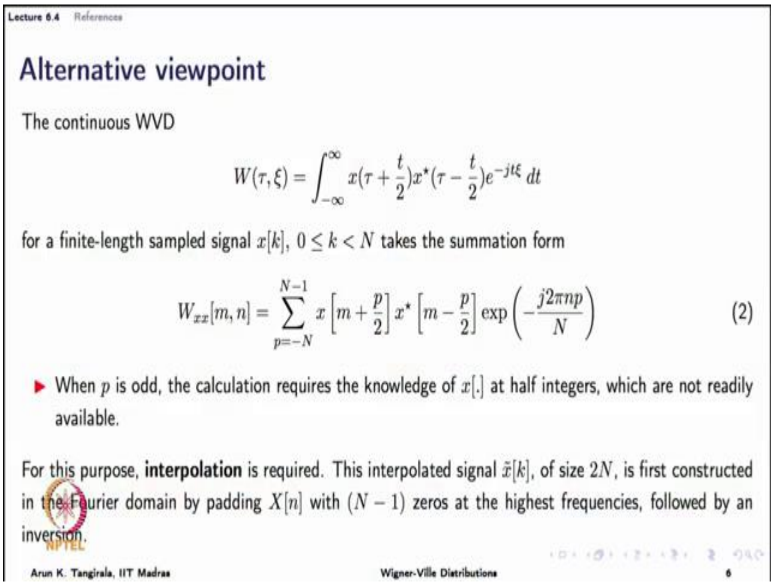

Before we talk about the further remedies or other possible remedies, let us take an alternative view point, which kind of hints the same thing that we learnt just now, that I need to sample the quadratically transform signal twice, as that of the signal itself. So, again begin from the continuous time or continuous Wigner-Ville distribution. Its continuous in both time and frequency, and right this integral for a finite length sample signal, assume that you have n observations, and that you have obtain this discrete time signal at a sampling frequency f s. The integral now takes the form of the summation, where this summation index runs from minus n to n minus 1. And, importantly observe that to compute the discrete Wigner-Ville. Now it is both discrete in time and frequency, like the d f d. To compute that I need the values of x at fractional instance. Exactly half the mid wave between the instance. Particularly when p is odd I require that right. So, for example, I would need values of the signal at 0.5, 1.5, 2.5 and so on which I do not have, because I am only have the values of discrete time signal at 0 1 2 and so on. So, how do I generate these values. Well there are two possibilities; one is interpolation, where I do not have access to the continuous time signal; therefore, I can go and re sample at a faster way.

So, I would interpolate, and how is interpolation typically done. Will you take the discrete Fourier transform of the sample signal, the finite length sample signal, pad it with the required number of zeros at the highest frequencies. Why highest frequencies, because the new signal which we call us x tilde, which will be of size two n, where it will also have values at exactly mid wave between the sampling instance, is going to be of higher frequency. I have now samples more frequently than the original one. Therefore, the x tilde as going to be of higher frequency content; therefore, I have to actually concentrate on moderating the, or modifying the high frequency content of x, which is the original signal that I have. So, I pad this d f d of x with the requisite number of zero's at higher frequencies, perform Fourier inversion, and then get my x tilde; that is one way of interpolating. Of course, if I have access to the continuous time signal itself, I would go back and sample x twice as that, that I did previously which is at f x. I would go and sample x 2 f x. And then I would get the values at the intermediate instant as well. So, both methods are telling me that I need to acquire the value of the signal at twice the sampling rate as I have chosen, which is the same massage that we learnt just now, where we took different perspective. So, both perspectives are giving me the same answer

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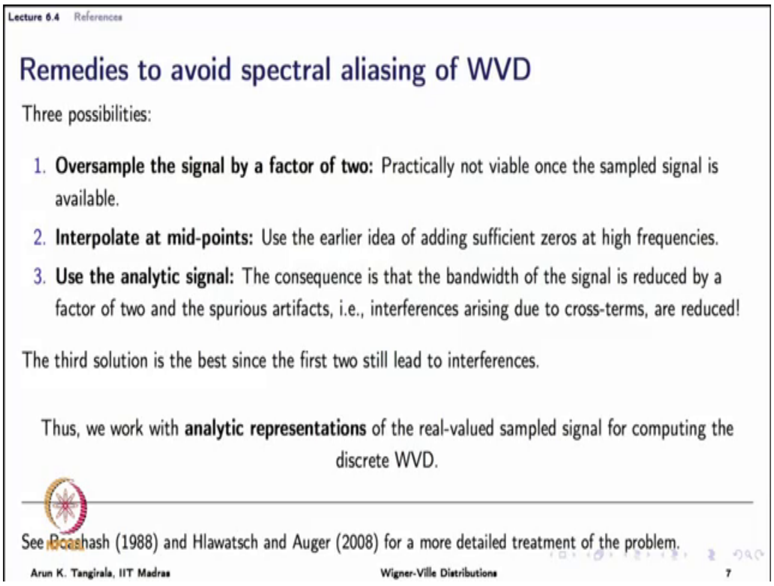

Therefore we have these three remedies, to avoid spectral earlier aliasing of Wigner-Ville. Once I generate this 2 n observation of x from n observation, or once I have the sample x at faster rate 2 f x, then the frequency grid, the choice goes along the same lines as d f t itself, I would choose one over 2 m as a frequency grid. So, coming back to the remedies now, based on a discussion just now. To avoid spectral aliasing, either I should over sample the signal by factor of two, which is only possible if I have access to the continuous time signal, and that may not be true for most situations. Then the second remedies to interpolate at mid points, which is fine, it fixers the problem, but there is an additional problem, which is speculate to Wigner-Ville, which is a problem of interferences. And the third remedy is to use analytical signal. Now this is a bright idea, because the moment I construct the analytical associated of a signal, I would be reducing the bandwidth by factor of two. Recall that whenever I construct the manual take associate of a given signal, I am giving two zero out all the frequencies in the negative frequencies, and that immediately brings on the bandwidth by a factor of two.

And because I am bring now the bandwidth of the signal by factor of two, multiplying that by 2. Now for the analytic signal, gives me the same sampling frequency at which I have already obtained the data. Therefore, I do not have do any re-sampling or over-sampling and so on or interpolation, I am fine. So, the analytic signal will reduce a bandwidth. The previous two basically, not touching the bandwidth, but trying to address the sampling frequency itself. The positive side effect of using the analytic signal is that, I can reduce a number of interferences. Recall the example that we had in the previous lecture, where we had a comparison of Wigner-Ville on real representation of the signal, and analytical representation of the signal. In the real representation the number of cross terms, is going to be higher, because any real valued signal can be expressed as a sum of two complex exponential with negative frequencies. And I know from the property of Wigner-Ville that whenever I have a sum of two frequencies, I am going to have interferences.

Therefore, if I user real value representation, the interferences are going to be more, because a number crossed on is going to be more; whereas, with an analytic version I do not have that issue. So, looking at all days, the recommendation that comes down to as, is that we should work with analytical representation, which will take care of both spectral aliasing, as well as the detection in the interferences. As I said, we have arrived at this recommendation based on intuitive arguments; of course, theoretical findings, but we are not formally proved anything. So, if you want to see formal proofs, you can refer to this short note or short corresponding by Boashash in 1988, and Boashash is the prominent name in time frequency analysis, or refer to the book edited by this two gentleman, which gives you very nice proof and insights into the implementation of discrete Wigner-Ville.

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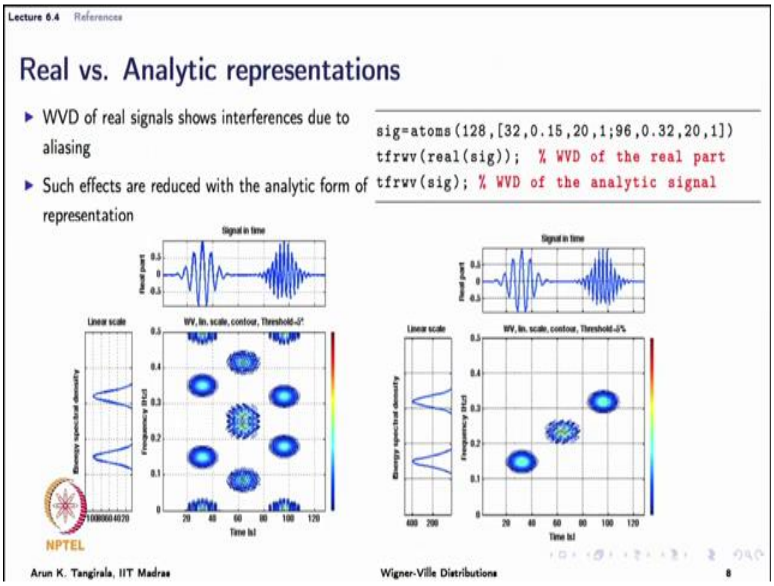

We will conclude the lecture with an example, to reinforce the point that we are just discussed, where we will show how the analytical representation solves both issues. And this is in fact, an example that we are seen earlier, where we have two amplitude modulated sine waves of low and high frequencies. In fact, of frequencies 0.15, center frequency 0.15, and center frequency 0.32 as you can see from the comment here. Now the amplitude modulation is a Gaussian amplitude modulation. If I use a real representation for the signal, and I look at the Wigner-Ville distribution, I can see these four atoms here in the time frequencies plane. In addition to that I have these interferences, coming out of the interactions of this four frequencies or center frequency atoms. Now why do I have four, when I have only two frequencies in the signal. The main reason is, let us look at the first one. The center frequency of the first atom here is 0.15, but because I am using the real representation. Remember a sine of frequency 0.15. Underline is a sine wave, it is only that you see an amplitude modulation. So, sine of 0.15 can be written as e to the j 0.15. Let us e to minus j 0.15, and e to the minus j 0.1 0.5 will manifest as e to the j 0.35 with respect to the 0.5 limit that you have. And also mind you that I am only showing you the positive frequencies; in fact, you will see the same thing happening in the negative frequencies as well.

So, the same argument can be applied to the high frequency modulated atom, whose center frequencies 0.32, so you have 0.32 here, and therefore, you have 0.18 appearing as a reflection about 0.5, and you would see a minus 0.18 also, appearing in the negative frequencies. So, the real representation produces this spurious frequencies, which are essentially alias reflections around 0.5, and then you have interferences arising out of this. Whereas, analytic representation avoids all of this spurious artifacts, and consequentially the interferences are reduced. You have only a single interfering term here, which as we have learnt earlier, lays mid wave in the time frequency plane between these two atoms. So, this hopefully has now help you appreciate the need for analytical representation when you are working with Wigner-Ville distributions. Earlier we had seen this in the context of reducing interferences, but now it has reduced spectral aliasing and interferences, this is for wavelet (()).

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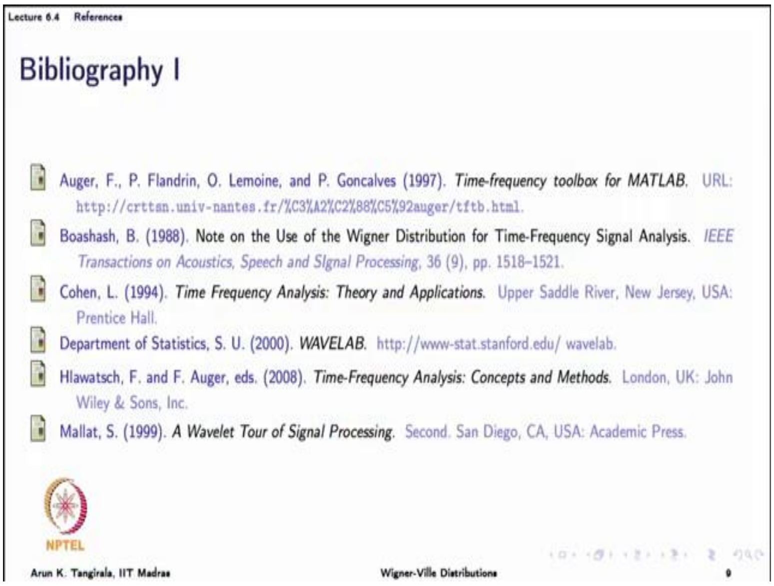

With that example we draw or lecture to a close. I suggest that you read the two references that I have mentioned earlier, to get a more formal insight, and look at the formal derivation, for the recommendations that we have given in this lecture. And the other point that I would like to read write is, we have discussed this in a context of Wigner-Ville, but the same applies to other modified and smooth Wigner-Ville distributions as well, which we will start looking at in the next lecture.

Thank you.